

Applications of Trigonometric Measures of Fuzzy Entropy to Geometry

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Abstract—In the literature of fuzzy measures, there exist many well known parametric and non-parametric measures, each with its own merits and limitations. But our main emphasis is on applications of these measures to a variety of disciplines. To extend the scope of applications of these fuzzy measures to geometry, we need some special fuzzy measures. In this communication, we have introduced two new fuzzy measures involving trigonometric functions and simultaneously provided their applications to obtain the basic results already existing in the literature of geometry.

Keywords—Entropy, Uncertainty, Fuzzy Entropy, Concavity, Symmetry.

I. INTRODUCTION

IN real life situation, uncertainty arises in decision-making problem either due to lack of knowledge or due to inherent vagueness. Such types of problems can be solved using probability theory and fuzzy set theory respectively. Fuzziness, a feature of imperfect information, results from the lack of crisp distinction between the elements belonging and not belonging to a set. A measure of fuzziness which is often used and cited in the literature of fuzzy information is an entropy first mentioned by Zadeh [17]. However, the two functions measure fundamentally different types of uncertainty. Basically, the Shannon's [15] entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment.

De Luca and Termini [2] introduced some requirements which capture our intuitive comprehension of the degree of fuzziness and consequently developed a measure of fuzzy entropy which corresponds to Shannon's [17] probabilistic entropy, given by:

$$H(A) = -\sum_{i=1}^n [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))] \quad (1)$$

Kapur [7] took the following measure of fuzzy entropy corresponding to Havrda and Charvat's [5] probabilistic entropy:

$$H^\alpha(A) = (1 - \alpha)^{-1} \sum_{i=1}^n [\{\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha\} - 1]; \alpha \neq 1, \alpha > 0 \quad (2)$$

Bhandari and Pal [1] developed the following measure of

fuzzy entropy corresponding to Renyi's [15] probabilistic entropy:

$$H_{\alpha}(A) = \frac{1}{1 - \alpha} \sum_{i=1}^n \log [\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha]; \alpha \neq 1, \alpha > 0 \quad (3)$$

Parkash [9] introduced a new generalized measure of fuzzy entropy involving two real parameters, given by:

$$H_{\alpha}^{\beta}(A) = [(1 - \alpha) \beta]^{-1} \sum_{i=1}^n \left[\left\{ \mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha} \right\}^{\beta} - 1 \right] \quad (4)$$

; $\alpha > 0, \alpha \neq 1, \alpha \neq 1$

And called it $(\alpha - \beta)$ fuzzy entropy which includes some well known fuzzy entropies.

Rudas [16] discussed elementary entropy function based fuzzy entropy construction and fuzzy operations. The author defined new generalized operations as minimum and maximum entropy operations. Guo and Xin [2] studied some new generalized entropy formulas for fuzzy sets and proposed new idea for further development of fuzzy information. Parkash and Sharma [12] provided the applications of fuzzy measures to prove some noiseless coding theorems. Parkash and Gandhi [11] used fuzzy divergence measures to generate some new measures of fuzzy entropy. Parkash, Sharma and Mahajan [12, 13] delivered the applications of weighted measures of fuzzy entropy for the study of maximum entropy principles.

Some other interesting findings related with theoretical measures of fuzzy entropy and their applications have been investigated by Zadeh [17], Kapur [6], Gurdial, Petry and Beaubouef [3], Pal and Bezdek [8], Hu and Yu [5] etc. In section 2, we have introduced two new trigonometric measures of fuzzy entropy and provided their applications to the field of geometry.

II. TWO NEW TRIGONOMETRIC FUZZY MEASURES AND THEIR APPLICATIONS

Since this paper deals with applications of fuzzy measures to geometry, we need some special fuzzy measures to extend the scope of their applications towards geometry. To provide the applications of these fuzzy measures, we introduce two new measures involving trigonometric functions and simultaneously provide their applications to obtain the basic results already existing in the literature of geometry.

I. We, first propose a new measure of fuzzy entropy, given by

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$H_1(A) = -\sum_{i=1}^n \left[\sin \frac{\pi}{2n\mu_A(x_i)} + \sin \frac{\pi}{2n(1-\mu_A(x_i))} - n^2 \sin \frac{\pi}{2n} \right]$ inscribed by n sides of a convex polygon at the centre of circle so t

$$B_1 + B_2 + B_3 + \dots + B_n = 2\pi \quad (6)$$

Differentiating equation (5) w.r.t. $\mu_A(x_i)$, we get

$$\frac{\partial H_1(A)}{\partial \mu_A(x_i)} = \frac{\pi}{2n} \left[\frac{1}{\mu_A^2(x_i)} \cos \frac{\pi}{2n\mu_A(x_i)} - \frac{1}{(1-\mu_A(x_i))^2} \cos \frac{\pi}{2n(1-\mu_A(x_i))} \right]$$

Also

$$\frac{\partial^2 H_1(A)}{\partial \mu_A^2(x_i)} =$$

$$\frac{\pi^2}{4n^2} \left[\frac{1}{\mu_A^4(x_i)} \sin \frac{\pi}{2n\mu_A(x_i)} + \frac{1}{(1-\mu_A(x_i))^4} \sin \frac{\pi}{2n(1-\mu_A(x_i))} - \frac{2n}{\pi} \left\{ \frac{1}{\mu_A^3(x_i)} \cos \frac{\pi}{2n\mu_A(x_i)} + \frac{1}{(1-\mu_A(x_i))^3} \cos \frac{\pi}{2n(1-\mu_A(x_i))} \right\} \right]$$

Using the relation, $\mu_A(x_i) = 1 - \mu_A(x_i)$ we get:

$$\frac{\partial^2 H_1(A)}{\partial \mu_A^2(x_i)} = -\frac{\pi^2}{2n^2} \left[\frac{2n}{\pi \mu_A^3(x_i)} \cos \frac{\pi}{2n\mu_A(x_i)} - \frac{1}{\mu_A^4(x_i)} \sin \frac{\pi}{2n\mu_A(x_i)} \right]$$

$< 0 \quad n > 3$

This shows that $H_1(A)$ is a concave function of $\mu_A(x_i)$ and its maximum value arises when $\mu_A(x_i) = 1/2 \forall i$.

Thus, we see that $H_1(A)$ satisfies the following properties:

- (i) $H_1(A)$ is a concave function of $\mu_A(x_i)$.
- (ii) $H_1(A)$ is an increasing function of $\mu_A(x_i)$ when

$$0 \leq \mu_A(x_i) \leq \frac{1}{2}$$

- (iii) $H_1(A)$ is a decreasing function of $\mu_A(x_i)$ when

$$\frac{1}{2} \leq \mu_A(x_i) \leq 1$$

- (iv) $H_1(A)$ does not change when $\mu_A(x_i)$ is changed to $1 - \mu_A(x_i)$.

- (v) $H_1(A) = 0$ when $\mu_A(x_i) = 0$ or 1 .

Under the above five conditions, the measure proposed in equation (5) is a valid trigonometric measure of fuzzy entropy.

Next, we apply the trigonometric measure introduced in equation (5) to the field of geometry.

Let $A_1 A_2 A_3 \dots A_n$ be n sided convex polygon inscribed in a circle of radius a and let $B_1, B_2, B_3, \dots, B_n$ be angles

The lengths of the sides of the polygon are given by

$$2a \sin \frac{B_1}{2}, 2a \sin \frac{B_2}{2}, \dots, 2a \sin \frac{B_n}{2}$$

Let us define the fuzzy distribution given by

$$\mu_A(x_i) = \frac{\pi}{nB_i} \quad (7)$$

Thus, the perimeter of the polygon is given by

$$P = 2a \left\{ \sin \frac{\pi}{n\mu_A(x_1)} + \sin \frac{\pi}{n\mu_A(x_2)} + \dots + \sin \frac{\pi}{n\mu_A(x_n)} \right\} \quad (8)$$

Since, μ_A is a measure of fuzzy entropy, its maximum value arises when the distribution is most fuzzy, that is, when $\mu_A(x_i) = 1 - \mu_A(x_i) = 1/2 \forall i$

Thus, we have

$$H_1(A) \leq H_1 \left\{ \frac{1}{2} \right\}$$

or

$$-\sum_{i=1}^n \left[\sin \frac{\pi}{2n\mu_A(x_i)} + \sin \frac{\pi}{2n(1-\mu_A(x_i))} - n^2 \sin \frac{\pi}{2n} \right] \leq -\sum_{i=1}^n \left[\sin \frac{\pi}{n} + \sin \frac{\pi}{n} - n^2 \sin \frac{\pi}{2n} \right]$$

Using the fact that $\mu_A(x_i) = 1 - \mu_A(x_i)$, the above equation reduces to

$$\sum_{i=1}^n \sin \frac{\pi}{2n\mu_A(x_i)} \geq \sum_{i=1}^n \sin \frac{\pi}{n} = n \sin \frac{\pi}{n} \quad (9)$$

The equality sign in equation (9) holds only if

$$B_1 = B_2 = B_3 = \dots = B_n = \frac{2\pi}{n}$$

From equations (8) and (9), we conclude that minimum perimeter of a polygon of n sides is $2na \sin \frac{\pi}{n}$ and this arises when the polygon is regular. Thus, out of all convex polygons of n sides inscribed in a circle of radius a, the minimum perimeter is that of a regular polygon and the minimum perimeter is

$$[P]_{\min} = 2a n \sin \frac{\pi}{n}$$

and as $n \rightarrow \infty$, this value reduces to $2a\pi$, the circumference of the circle.

- II. Next, we propose another measure of fuzzy entropy, given by

$$H_2(A) = -\sum_{i=1}^n \left[\tan \frac{\pi}{2n\mu_A(x_i)} + \tan \frac{\pi}{2n(1-\mu_A(x_i))} - n^2 \tan \frac{\pi}{2n} \right] \quad (10)$$

Thus, we have

$$\frac{\partial H_2(A)}{\partial \mu_A(x_i)} = \frac{\pi}{2n\mu_A^2(x_i)} \sec^2 \frac{\pi}{2n\mu_A(x_i)} - \frac{\pi}{2n(1-\mu_A(x_i))^2} \sec^2 \frac{\pi}{2n(1-\mu_A(x_i))} + \sum_{i=1}^n \left[\tan \frac{\pi}{2n\mu_A(x_i)} + \tan \frac{\pi}{2n(1-\mu_A(x_i))} \right]$$

Also

$$\frac{\partial^2 H_2(A)}{\partial \mu_A^2(x_i)} = -\frac{2\pi}{n} \left[\frac{\sec^2 \frac{\pi}{2n\mu_A(x_i)}}{\mu_A^3(x_i)} + \frac{\pi \sec^2 \frac{\pi}{2n\mu_A(x_i)} \tan \frac{\pi}{2n\mu_A(x_i)}}{n\mu_A^4(x_i)} \right]$$

$$< 0 \quad \forall n > 3$$

Hence, $H_2(A)$ is a concave function of $\mu_A(x_i)$ and its maximum value arises when $\mu_A(x_i) = 1/2 \quad \forall i$

Thus, $H_2(A)$ introduced above satisfies the following properties:

(i) $H_2(A)$ is a concave function of $\mu_A(x_i)$.

(ii) $H_2(A)$ is an increasing function of $\mu_A(x_i)$ when

$$0 \leq \mu_A(x_i) \leq \frac{1}{2}$$

(iii) $H_2(A)$ is a decreasing function of $\mu_A(x_i)$ when

$$\frac{1}{2} \leq \mu_A(x_i) \leq 1$$

(iv) $H_2(A)$ does not change when $\mu_A(x_i)$ is changed to $1 - \mu_A(x_i)$.

(v) $H_2(A) = 0$ when $\mu_A(x_i) = 0$ or 1

Under the above five conditions, the measure proposed in equation (10) is a valid trigonometric measure of fuzzy entropy.

Next, we apply the trigonometric measure introduced in equation (10) to the field of geometry.

Now, if we take a polygon circumscribing a circle of radius a , we get P , the perimeter of polygon as

$$P = 2a \left[\tan \frac{B_1}{2} + \tan \frac{B_2}{2} + \dots + \tan \frac{B_n}{2} \right] \quad (11)$$

and area A of the polygon is given by

$$A = a^2 \left[\tan \frac{B_1}{2} + \tan \frac{B_2}{2} + \dots + \tan \frac{B_n}{2} \right] \quad (12)$$

We know that $H_2(A)$ is maximum at $\mu_A(x_i) = 1/2$, we have

$$H_2(A) \leq H_2 \left\{ \frac{1}{2} \right\}$$

This gives the following mathematical expression:

$$\begin{aligned} & - \sum_{i=1}^n \left[\tan \frac{\pi}{2n\mu_A(x_i)} + \tan \frac{\pi}{2n(1-\mu_A(x_i))} - n^2 \tan \frac{\pi}{2n} \right] \\ & \leq - \sum_{i=1}^n \left[\tan \frac{\pi}{n} + \tan \frac{\pi}{n} - n^2 \tan \frac{\pi}{2n} \right] \end{aligned}$$

or

Using the fact that $\mu_A(x_i) = 1 - \mu_A(x_i)$, we have

$$\sum_{i=1}^n \tan \frac{\pi}{2n\mu_A(x_i)} \geq n \tan \frac{\pi}{n} \quad (13)$$

Multiplying both sides equation (13) by $2a$ and using (11), we get

$$P \geq 2an \tan \frac{\pi}{n} \quad (14)$$

Similarly, multiplying both sides equation (13) by a^2 and using (14), we get

$$A \geq a^2 n \tan \frac{\pi}{n} \quad (15)$$

III. CONCLUSION

From equation (14), we conclude that the minimum perimeter of the polygon of n sides is $2an \tan \frac{\pi}{n}$ and this arises when the polygon is regular. Also

$$[P]_{\min} = 2an \tan \frac{\pi}{n} \rightarrow 2a\pi \text{ as } n \rightarrow \infty \text{ which is also}$$

the circumference of the circle of radius a .

Also, from equation (15), we conclude that the minimum area of the polygon of n sides is given by

$$[A]_{\min} = a^2 n \tan \frac{\pi}{n} \rightarrow a^2 \pi \text{ as } n \rightarrow \infty \text{ which is also the}$$

area of circle of radius a .

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