

Linear elasticity problems solved by using the fictitious domain method and Total - FETI domain decomposition

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Abstract—The main goal of this paper is to show a possibility, how to solve numerically elliptic boundary value problems arising in 2D linear elasticity by using the fictitious domain method (FDM) and the Total-FETI domain decomposition method. We briefly mention the theoretical background of these methods and demonstrate their performance on a benchmark.

Keywords—linear elasticity, fictitious domain method, Total-FETI, domain decomposition, saddle-point system

I. INTRODUCTION

THE paper deals with solving method for 2D linear elasticity problems based on the use of the fictitious domain method [5], [6], together with the Total-FETI domain decomposition [2], [7]. The main idea of the fictitious domain method applied on elliptic boundary value problems is to embed the original domain ω with possibly complicated geometry into a new simple shaped domain called the fictitious one. The original problem is reformulated to a new one defined on the fictitious domain. The advantage of this method is in using a special partition of the fictitious domain, which enables us to apply effective solvers to compute the solution of the resulting algebraic system. The classical approach of the fictitious domain method is based on the use of Lagrange multipliers, defined on the original boundary, to enforce boundary conditions on that boundary. This approach leads to a singularity concentrated on the original boundary. To get better convergence rate we use the modified (smooth) fictitious domain approach [6], based on the introduction of a new auxiliary boundary located outside of the original domain. On this boundary we define new control variable, which enforces the boundary conditions prescribed on the original boundary, instead of Lagrange multipliers. The solution still has a singularity but now located on the auxiliary boundary instead of the original boundary.

For increasing efficiency we also use the Total-FETI domain decomposition based on the decomposing of the fictitious domain to non-overlapping subdomains. These subdomains are glued together again by Lagrange multipliers. This method is use to find a solution in parallel. After the finite element discretization of the fictitious domain formulation we get a linear system of algebraic equations called the generalized

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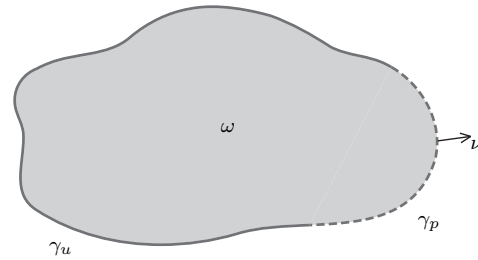


Fig. 1: Elastic body ω

saddle-point system. This system is solved by the projected biorthogonal conjugate gradient method for non-symmetric operators with preconditioning (ProjBiCGSTAB) [6] and for the numerical solution we use the MatSol library. [8]

II. FORMULATION OF THE LINEAR ELASTICITY PROBLEM

Let us consider an elastic body which is represented by the domain $\omega \subset \mathbb{R}^2$ with a smooth boundary γ , see Fig. 1. This boundary is divided into two disjoint parts γ_u and γ_p , where $\gamma = \overline{\gamma_u} \cup \overline{\gamma_p}$. On γ_u and γ_p we impose Dirichlet and Neumann conditions, respectively. Let us formulate equilibrium equation together with the standard boundary conditions:

$$\left. \begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} \quad \text{in } \omega, \\ \mathbf{u} &= \mathbf{o} \quad \text{on } \gamma_u, \\ \boldsymbol{\sigma}(\mathbf{u})\boldsymbol{\nu} &= \mathbf{p} \quad \text{on } \gamma_p, \end{aligned} \right\} \quad (1)$$

where $\boldsymbol{\sigma}(\mathbf{u})$ is the stress tensor in ω , $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is the unit outward normal vector to γ and $\mathbf{u} = (u_1, u_2)$ is the unknown displacement.

The strain which is caused by the displacement \mathbf{u} is characterized by the symmetric strain tensor given as

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u}).$$

The stress tensor is related to the strain tensor by the linearized Hooke law for linear isotropic materials, written as:

$$\boldsymbol{\sigma}(\mathbf{u}) := c_1 \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + 2c_2 \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \omega,$$

where "tr" denotes the trace of matrices defined as

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}, \quad \mathbf{A} \in \mathbb{R}^{n \times n},$$

$\mathbf{I} \in \mathbb{R}^{2 \times 2}$ is the identity matrix and $c_1, c_2 > 0$ are the Lamè constants defined as follows:

$$c_1 = \frac{Ev}{(1+\nu)(1-2\nu)}, \quad c_2 = \frac{E}{2(1+\nu)},$$

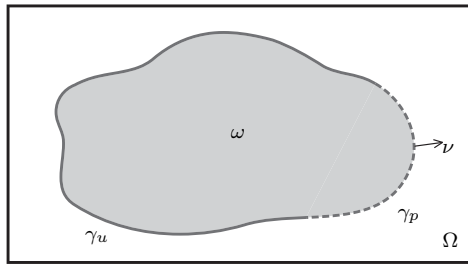


Fig. 2: Fictitious domain method

with $E > 0$ being the Young modulus and $\nu \in (0, 1/2)$ the Poisson ratio. The stress tensor can be written as follows:

$$\sigma(\mathbf{u}) = \begin{pmatrix} (c_1 + 2c_2) \frac{\partial u_1}{\partial x_1} + c_1 \frac{\partial u_2}{\partial x_2} & c_2 \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\ c_2 \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & c_1 \frac{\partial u_1}{\partial x_1} + (c_1 + 2c_2) \frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

Above we formulated the linear elasticity problem and in the following chapter we introduce the first ingredient which is used to solve it efficiently. This ingredient is based on the use of a fictitious domain containing the elastic body ω .

III. FICTITIOUS DOMAIN METHOD

The fictitious domain method represents an efficient tool for the numerical solution of complicated problems arising in physics and industry. The main reason for its popularity is that it allows us to transform the original problem defined in a domain ω with a possibly complicated geometry to a new one, solved in a simple shape domain Ω containing original domain ω . The advantage of the fictitious domain method (sometimes called imbedding method) is that we can use fairly structured meshes in Ω making possible to apply fast effective solvers for the numerical solution of the resulting algebraic system and special preconditioning techniques. There are several ways how to associate the new problem in Ω with the original one defined in ω . For example we can use Lagrange multiplier technique, optimal control approach, penalty approach etc.

Let us describe mentioned principles in more details. Let ω be a bounded domain in \mathbb{R}^2 with the Lipschitz boundary $\partial\omega$. On this domain we define an elliptic boundary value problem. The main idea of this method is to embed the real domain ω of the original problem with possibly complicated geometry to a new simple shaped domain Ω called the fictitious one, see Fig. 2. The original problem is reformulated to a new one defined in the fictitious domain Ω in such a way that its solution when restricted to ω matches with the solution of the original problem.

In classical approach, a possible way to formulate the new problem is based on the use of Lagrange multipliers. The imposed boundary conditions on γ can be viewed as a constraint. This constraint is enforced by Lagrange multipliers defined on γ . Thus the new formulation in Ω involves two unknowns introduced as the primal variable $u \in V$ and the corresponding Lagrange multiplier $\lambda \in \Lambda$ enforcing prescribed boundary conditions on γ which leads to the singularity of u concentrated on γ , that can result in an intrinsic error of the

computed solution. For more details, see [4]. To improve the convergence rate a modified (smooth) approach was proposed [6]. This modification is based on introduction of a new control variable instead of the Lagrange multiplier defined on the other auxiliary boundary Γ located outside of the domain $\bar{\omega}$, see Fig. 4. The boundary Γ satisfies the condition $\delta = \text{dist}(\Gamma, \gamma) > 0$. This new control variable enforces the original boundary condition on γ . Because the singularity is moved from $\bar{\omega}$, the solution is smoother in ω , see Fig. 3.

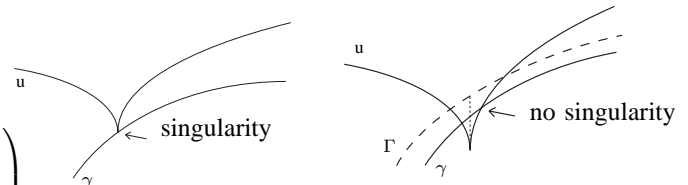


Fig. 3: Singularity on γ

To explain the modified fictitious domain method we define the space

$$V(\omega) = \{v \in (H^1(\omega))^2 \mid v = \mathbf{0} \text{ on } \gamma_u\}$$

Then the weak formulation of (1) reads as follows:

$$\left. \begin{array}{l} \text{Find } \mathbf{u} \in V(\omega) \text{ such that} \\ a_\omega(\mathbf{u}, \mathbf{v}) = \int_\omega \mathbf{f} \cdot \mathbf{v} \, dx + (\mathbf{p}, \mathbf{v})_{\gamma_p} \quad \forall \mathbf{v} \in V(\omega), \end{array} \right\} \quad (2)$$

where

$$a_\omega(\mathbf{u}, \mathbf{v}) = \int_\omega \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx$$

and $(\cdot, \cdot)_{\gamma_p}$ is the scalar product in $(L^2(\gamma_p))^2$.

Further we define the fictitious domain Ω such that $\bar{\omega} \subset \Omega$ and auxiliary boundary Γ surrounding the original domain ω , also we define space

$$V(\Omega) = (H_0^1(\Omega))^2,$$

with

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}.$$

Now we can introduce the modified fictitious domain formulation of the original problem, which is given as:

$$\left. \begin{array}{l} \text{Find } (\hat{\mathbf{u}}, \boldsymbol{\lambda}) \in V(\Omega) \times \Lambda(\Gamma) \text{ such that} \\ a_\Omega(\hat{\mathbf{u}}, \mathbf{v}) + \langle \mathbf{v}, \boldsymbol{\lambda} \rangle_\Gamma = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \quad \forall \mathbf{v} \in V(\Omega), \\ \langle \boldsymbol{\mu}_u, \hat{\mathbf{u}} \rangle_{\gamma_u} = 0 \quad \forall \boldsymbol{\mu}_u \in \Lambda(\gamma_u), \\ \langle \boldsymbol{\mu}_p, \boldsymbol{\sigma}(\hat{\mathbf{u}})\boldsymbol{\nu} \rangle_{\gamma_p} = \langle \boldsymbol{\mu}_p, \mathbf{p} \rangle_{\gamma_p} \quad \forall \boldsymbol{\mu}_p \in \Lambda(\gamma_p), \end{array} \right\} \quad (3)$$

where $\Lambda(\Gamma) = (H^{-1/2}(\Gamma))^2$, $\Lambda(\gamma_u) = (H^{-1/2}(\gamma_u))^2$, $\Lambda(\gamma_p) = (H^{-1/2}(\gamma_p))^2$ and $\langle \cdot, \cdot \rangle_{\gamma_u}$, and $\langle \cdot, \cdot \rangle_{\gamma_p}$ stand for the duality pairings between $H^{1/2}(\gamma_u)$ and $H^{-1/2}(\gamma_u)$ and $H^{1/2}(\gamma_p)$ and $H^{-1/2}(\gamma_p)$, respectively. Finally, $a_\Omega : V(\Omega) \times V(\Omega) \rightarrow \mathbb{R}$ and $\langle \mathbf{v}, \boldsymbol{\lambda} \rangle_\Gamma : V(\Omega) \times \Lambda(\Gamma) \rightarrow \mathbb{R}$ are two bounded bilinear forms. Here, the second component $\boldsymbol{\lambda}$ can be viewed as a control variable working on Γ to enforce boundary conditions imposed on the original boundary.

Let $V_h \subset (H_0^1(\Omega))^2$, $L_H^{\gamma_u} \subset (H^{-1/2}(\gamma_u))^2$, $L_H^{\gamma_p} \subset (H^{-1/2}(\gamma_p))^2$, $L_H^\Gamma \subset (H^{-1/2}(\Gamma))^2$, $h, H > 0$ be finite

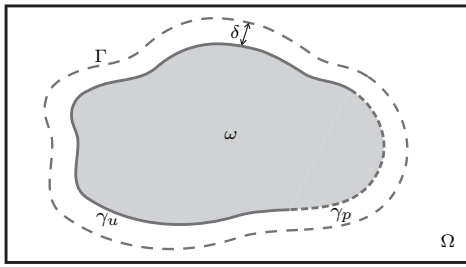


Fig. 4: Modified fictitious domain method

dimensional subspaces of the indicated spaces. Let $\dim L_H^{\gamma_u} = 2m_u$, $\dim L_H^{\gamma_p} = 2m_p$, and $\dim L_H^{\Gamma} = 2m$, where $m_u + m_p = m$. When we have finite dimensional subspaces, we can discretize variational problem (3) and then the discrete algebraic form, based on the mixed finite element method, leads to the following generalized algebraic saddle point system:

$$\left. \begin{aligned} &\text{Find } (\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^{2n} \times \mathbb{R}^{2m} \text{ such that} \\ &\left(\begin{array}{c|c} \mathbf{K} & \mathbf{B}_{\Gamma}^T \\ \mathbf{B}_{\gamma_u} & \mathbf{0} \\ \mathbf{C}_{\gamma_p} & \mathbf{0} \end{array} \right) \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{p} \end{pmatrix}, \end{aligned} \right\} \quad (4)$$

where $\mathbf{K} \in \mathbb{R}^{2n \times 2n}$ is the stiffness matrix. Matrices $\mathbf{B}_{\Gamma} \in \mathbb{R}^{2m \times 2n}$ and $\mathbf{B}_{\gamma_u} \in \mathbb{R}^{2m_u \times 2n}$ are the Dirichlet trace matrices on Γ and γ_u , respectively. The matrix $\mathbf{C}_{\gamma_p} \in \mathbb{R}^{2m_p \times 2n}$ is the Neumann trace matrix, they are constructed to be full row-rank and highly sparse matrices. Vectors \mathbf{f} and \mathbf{p} are of order $2n$ and $2m_p$, respectively.

As follows we focus our attention to an efficient method based on the domain decomposition which will be used to solve the generalized algebraic saddle point system (4).

IV. TOTAL-FETI DOMAIN DECOMPOSITION

This section deals with the second powerful ingredient based on the using of the Total-FETI decomposition. The FETI (Finite Element Tearing and Interconnecting) was introduced by Farhat and Roux [3] and became to be one of the most powerful method for parallel solution of ill-conditioned systems of linear equations. The main idea of this method is decomposing of the computational domain into non-overlapping subdomains. These subdomains are glued together again by Lagrange multipliers.

We focus our attention to a new variant of FETI called Total-FETI (TFETI), introduced by Dostal, Horak and Kucera [2]. The main idea is to simplify work with pseudoinversions of the subdomain stiffness matrices by using Lagrange multipliers not only for gluing interconnected subdomains but also for enforcing the Dirichlet boundary conditions, see Fig. 5. The important property of the Total-FETI method is that we know kernels of subdomain stiffness matrices which can be formed directly. For more details about FETI and TFETI methods see [2], [3], [7].

We simplify the generalized algebraic saddle point system (4) as

$$\left(\begin{array}{c|c} \mathbf{K} & \mathbf{B}_{\Gamma}^T \\ \mathbf{B}_{\gamma} & \mathbf{0} \end{array} \right) \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}, \quad (5)$$

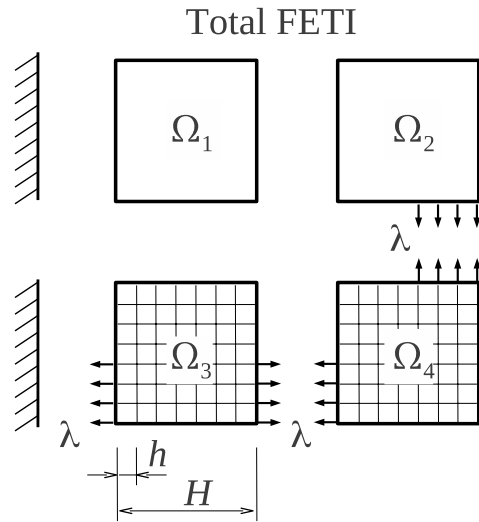


Fig. 5: Decomposition and discretization for Total FETI

where $\mathbf{B}_{\gamma} = (\mathbf{B}_{\gamma_u}^T, \mathbf{C}_{\gamma_p}^T)^T \in \mathbb{R}^{2m \times 2n}$, $\mathbf{f} \in \mathbb{R}^{2n}$ and $\mathbf{g} = (\mathbf{0}^T, \mathbf{p}^T)^T \in \mathbb{R}^{2m}$.

We decompose given domain Ω into s subdomains Ω^p , $p = 1, \dots, s$ and from that reason matrix $\mathbf{K} = \text{diag}(\mathbf{K}_1, \dots, \mathbf{K}_s)$. The diagonal blocks \mathbf{K}_p that correspond to the subdomains Ω^p are positive semidefinite sparse stiffness matrices with a-priori known kernels and $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_s) \in \mathbb{R}^{2n}$.

We introduce $(2m \times 2n)$ full row rank matrix \mathbf{B}_G and vector \mathbf{c} of order $2m$, where the matrix \mathbf{B}_G with its rows \mathbf{b}_i and the vector \mathbf{c} with the entries c_i enforce the prescribed displacements on the part of the boundary with imposed Dirichlet boundary conditions and the continuity of the displacements across the auxiliary interfaces. The continuity requires that $\mathbf{b}_i \mathbf{u} = c_i = 0$, where \mathbf{b}_i are vectors of the order $2n$ with zero entries except 1 and -1 at appropriate positions. The matrix \mathbf{B}_G is called gluing matrix. Using the notation $m = m + M$, $\mathbf{B}_{\gamma} = (\mathbf{B}_G^T, \mathbf{B}_{\gamma}^T)^T \in \mathbb{R}^{2m \times 2n}$, $\mathbf{B}_{\Gamma} = (\mathbf{B}_G^T, \mathbf{B}_{\Gamma}^T)^T \in \mathbb{R}^{2m \times 2n}$ and $\mathbf{g} = (\mathbf{c}^T, \mathbf{g}^T)^T \in \mathbb{R}^{2m}$ we get problem of the same form as (5) again.

For solving the whole algebraic saddle-point system (5) we use the method based on the Schur complement reduction. As the stiffness matrix \mathbf{K} is singular, the first component \mathbf{u} of (5) cannot be completely eliminated. It follows that the Schur complement reduction leads to another algebraic system with two unknowns. The first unknown $\boldsymbol{\lambda}$ is from the previous saddle point system and new unknown $\boldsymbol{\alpha}$, which corresponds to the null space of \mathbf{K} . We can formulate this new algebraic system with unknowns $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ as

$$\begin{pmatrix} \mathbf{B}_{\gamma} \mathbf{K}^{\dagger} \mathbf{B}_{\Gamma}^T & -\mathbf{B}_{\gamma} \mathbf{R} \\ -\mathbf{R}^T \mathbf{B}_{\Gamma}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{\gamma} \mathbf{K}^{\dagger} \mathbf{f} - \mathbf{g} \\ -\mathbf{R}^T \mathbf{f} \end{pmatrix}$$

and the first unknown \mathbf{u} of the algebraic system (5) is given as

$$\mathbf{u} = \mathbf{K}^{\dagger} (\mathbf{f} - \mathbf{B}_{\Gamma}^T \boldsymbol{\lambda}) + \mathbf{R} \boldsymbol{\alpha}.$$

Columns of the matrix \mathbf{R} span the kernel of the matrix \mathbf{K} and \mathbf{K}^{\dagger} is arbitrary generalized inverse of the matrix \mathbf{K} which

satisfies

$$KK^\dagger K = K.$$

We simplify this algebraic system to the following reduced one

$$\begin{pmatrix} F & G_1^T \\ G_2 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \alpha \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix}, \quad (6)$$

where

$$F := B_\gamma K^\dagger B_\Gamma^T, \quad G_1 := -R^T B_\gamma^T, \quad G_2 := -R^T B_\Gamma^T, \\ d := B_\gamma K^\dagger f - g \quad e := -R^T f.$$

For effective solving of (6), we can use orthogonal projectors. We define the orthogonal projectors P_i onto kernels of G_i as

$$P_i = I - G_i^T (G_i G_i^T)^{-1} G_i, \quad i = 1, 2.$$

The first projector splits the saddle-point algebraic structure of the reduced system and the second projector decomposes the unknown $\lambda \in \mathbb{R}^{2m}$ into two components λ_{Ker} and λ_{Im} as

$$\lambda := \lambda_{Ker} + \lambda_{Im},$$

where λ_{Im} belongs to the image of G_2 and λ_{Ker} belongs to the kernel of G_2 . Then λ is the first component of the solution to the algebraic system (6) if

$$\lambda_{Im} = G_2^T (G_2 G_2^T)^{-1} e$$

and λ_{Ker} satisfies the following equation

$$P_1 F \lambda_{Ker} = P_1 (d - F \lambda_{Im}).$$

The component λ_{Ker} is solved by a projected Krylov subspace method for non-symmetric operators with preconditioning, e.g. projBICGSTAB [6]. Finally the second component of algebraic system (6) is given by

$$\alpha = (G_1 G_1^T)^{-1} G_1 (d - F \lambda).$$

For the numerical solution we use the MatSol library [8].

V. NUMERICAL EXPERIMENT

Let us consider an elastic body which is represented by the domain $\omega \subset \mathbb{R}^2$ with a smooth boundary γ_u . Let us formulate equilibrium equation together with the Dirichlet boundary conditions:

$$\left. \begin{aligned} -\operatorname{div} \sigma(\mathbf{u}) &= \mathbf{f} \text{ in } \omega, \\ \mathbf{u} &= \mathbf{c} \text{ on } \gamma_u, \end{aligned} \right\} \quad (7)$$

The domain ω is defined as interior of the circle

$$\omega = \{(x, y) \in \mathbb{R}^2 \mid (x - 0.5)^2 + (y - 0.5)^2 < 0.25^2\},$$

which is embedded into the fictitious domain $\Omega = (0, 1) \times (0, 1)$ (see Fig. 6). The righthand sides of (1) are $\mathbf{f} = -\operatorname{div} \sigma(\hat{\mathbf{u}})$, $\mathbf{c} = \hat{\mathbf{u}}|_{\gamma_u}$, where $\hat{\mathbf{u}}(x, y) = (0.1xy, 0.1xy)$, $(x, y) \in \mathbb{R}^2$. The auxiliary boundary Γ is constructed by shifting γ_u $6h$ in the direction of outward normal vector ν . The problem is solved in parallel by using 8 processors.

In Fig. 7 we can see original and deformed geometry of ω for decomposition into 36 square subdomains of size H .

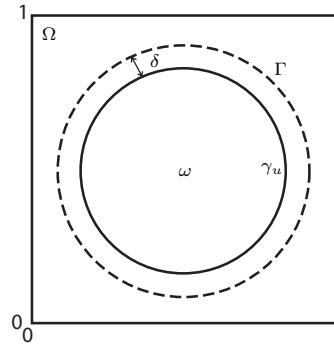


Fig. 6: Geometry of ω

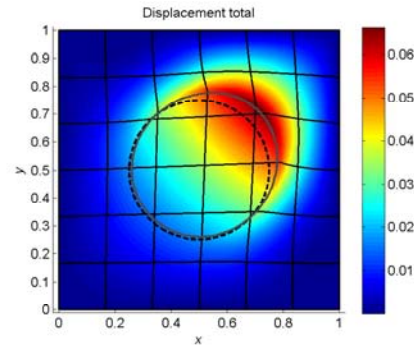


Fig. 7: Original and deformed geometry

Table I shows computed results for the fixed number of elements and increasing number of subdomains. We can see the number of subdomains, decomposition parameter h , number of primal and control variables, number of matrix multiplications, computational time and see relative errors of approximate solution $\hat{\mathbf{u}}_h$ to exact solution $\hat{\mathbf{u}}$ in these norms:

$$E_{rel}(\omega) = \frac{\|\hat{\mathbf{u}}_h - \hat{\mathbf{u}}\|_\omega}{\|\hat{\mathbf{u}}\|_\omega}, \quad E_{rel}(\gamma) = \frac{\|\hat{\mathbf{u}}_h - \hat{\mathbf{u}}\|_\gamma}{\|\hat{\mathbf{u}}\|_\gamma}.$$

TABLE I: COMPUTATIONAL RESULTS

N	64	144	196	256
h	1/256	1/384	1/448	1/512
Primal var.	139393	313632	426888	557568
Control var.	9442	20394	27428	35504
Matrix mult.	139	167	165	205
Time(s)	65,5	200	283	388
$E_{rel}(\gamma)$	2.9230e-004	1.6940e-004	1.1154e-004	6.8340e-005
$E_{rel}(\omega)$	1.0339e-004	5.9920e-005	3.3490e-005	2.0810e-005

VI. CONCLUSION

The goal of this paper was to illustrate the efficient way for the numerical solution of the linear elasticity problems based on the modified fictitious domain approach and the Total-FETI domain decomposition. We briefly explained those methods and illustrated their performance on a numerical example.

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REFERENCES

- [1] Dostal Z.: *Optimal quadratic programming algorithms: with applications to variational inequalities*, Springer: New Your, 2009.
- [2] Dostal, Z.; Horak, D.; Kucera, R.; *Total FETI - an easier implementable variant of the FETI method for numerical solution of elliptic PDE*, Communications in Numerical Methods in Engineering, 22(12), 2006, 1155-1162.
- [3] Farhat, C; Roux, FX; *A method of finite element tearing and inter-connecting and its parallel solution algorithm*, International Journal for Numerical Methods in Engineering 1991; 32:12051227.
- [4] Glowinski, R.; Pan, Tsorn-Whay; Periaux, J.; *A fictitious domain method for Dirichlet problem and applications*, Computing, Volume 84, Numbers 1-2, 2008, 69-96.
- [5] Haslinger, J.; Mkinen, R. A. E.; *Introduction to shape optimization: theory, approximation, and computation*, SIAM, 2003
- [6] Haslinger, J.; Kozubek, T.; Kucera, R.; Peichl, G.; *Projected the Schur complement method for solving non-symmetric systems arising from a smooth fictitious domain approach.* Lin. Algebra Appl., 14(2007), 713-739.
- [7] Kozubek, T.; Vondrak, V.; Mensik, M.; Horak, D.; Dostal, Z.; Hapla, V.; Kabelikova, P.; Cermak, M.; *Total FETI domain decomposition method and its massively parallel implementation*, Computers and Structures, submitted
- [8] Kozubek T, Markopoulos A, Brzobohaty T, Kucera R, Vondrak V, Dostal Z. *MatSol - MATLAB efficient solvers for problems in engineering.* <http://matsol.vsb.cz>.