Some Clopen sets in the Uniform Topology on BCI-algebras

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***Abstract*—** In this paper some properties of the uniformity topology on a BCI-algebras are discussed.

***Keywords*—**(Fuzzy) ideal, (Fuzzy) subalgebra, Uniformity, clopen sets.

I. Introduction

In 1966, K. Iseki introduced the concept of BCI-algebra [4]. In 1965, L.A. Zadeh [6] defined the concept of a fuzzy set, as a function from a non-empty set to $[0,1]$. In [1], B. Ahmad, apply this notion to BCI-algebra.
In this paper we will discuss some properties of the uniform topology on a BCI-algebra.

II. Preliminaries

Definition 2.1. By a BCI-algebra we mean an algebra $(X;*,0)$ of type (2,0) satisfying the axioms:

BCI-1) $((x*y)*(x*z))*(z*y)=0$,

BCI-2) $(x*(x*y))*y=0$,

BCI-3) $x*x=0$,

BCI-4) $x*y=y*x=0$ implies $x=y$,

BCI-5) $x*0=0$ implies $x=0$.

For all x,y and z in X.

From now on $X=(X;*,0)$ is a BCI-algebra.

Definition 2.2 [3]. A subset B of X is called:

i) an ideal if for any x,y in X.

(1) $0\in B$

(2) $x*y$, $y\in B$ imply $x\in B$.

ii) a subalgebra if for any x,y in B, $x*y\in B$.

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Definition 2.3 [1]. A fuzzy subset μ of X is called:

i) a fuzzy ideal of X if for any $x,y\in X$, we have

(1) $\mu(0)\geq\mu(x)$, for all x in X,

(2) $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}$.

ii) a fuzzy subalgebra of X if for any $x,y\in X$

$$\mu(x*y)\geq\min\{\mu(x),\mu(y)\}.$$

Notation 2.4. The set of all (non-zero fuzzy) ideal of X is denoted by $(FI(X))I(X)$.

Note that by non-zero fuzzy set of X we mean, there is $x\in X$ such that $\mu(x)>0$.

Lemma 2.5. (i) $A\in I(X)$ iff $\chi_A\in FI(X)$, where χ_A is the characteristic function of A.

(ii) If $\mu,\eta\in FI(X)$, then $\mu\cap\eta\in FI(X)$, where $\mu\cap\eta$ is a fuzzy subset of X which is defined by $\mu\cap\eta(x)=\min\{\mu(x),\eta(x)\}$, for all $x\in X$.

(iii) If $\mu\in FI(X)$, then

$$\mu(x*y)\geq\min\{\mu(x*z),\mu(z*y)\},\quad \forall x,y,z\in X.$$

(iv) If $\mu\in FI(X)$, then $\mu(0)>0$.

(v) A is a subalgebra of X if and only if χ_A is a fuzzy subalgebra.

Proof. The proofs of (i), (ii), (iv) and (v), are easy, and the proof of (iii) follows from BCI-1.

Remark 2.6. (i). By BCI-5, $\{0\}\in I(X)$ and hence $\chi_{\{0\}}\in FI(X)$.

(ii) For all $x\in X$, $A\in I(X)$, $\chi_A(x*x)=1$, by BCI-3.

Definition 2.7 [2]. A BCI-algebra X is called medial if

$$(x*y)*(z*u)=(x*z)*(y*u),\quad \forall x,y,z,u\in X.$$

Definition 2.8 [1]. A BCI-algebra X is called quasi right alternate if

$$x*(y*y)=(x*y)*y,\quad \forall x,y\in X.$$

Definition 2.8. Let $\mu\in FI(X)$. We define the relation $\sim_\mu$ on X as follows:

$x \sim_\mu y$ if and only if $\min\{\mu(x*y), \mu(y*x)\} > 0$.

Proposition 2.9. The relation $\sim_\mu$ is an equivalence relation on X.

Notations. Let X be a non-empty set and U, V be subsets of $X \times X$. We let

i) $U \circ V = \{(x,y) \in X \times X \mid \exists z \in X$ such that $(x,z) \in V$ and $(z,y) \in U\}$;

ii) $U^{-1} = \{(x,y) \in X \times X \mid (y,x) \in U\}$;

iii) $\Delta = \{(x,x) \in X \times X \mid x \in X\}$.

Definition 2.10 [5]. By a uniformity on X we shall mean a non-empty collection K of subsets of $X \times X$ which satisfies the following conditions:

(U_1) $\Delta \subseteq U$, for any $U \in K$;

(U_2) If $U \in K$, then $U^{-1} \in K$;

(U_3) If $U \in K$, then there exist a $V \in K$, such that $V \circ V \subseteq U$;

(U_4) If $U, V \in K$, then $U \bigcap V \in K$;

(U_5) If $U \in K$, and $U \subseteq V \subseteq X \times X$, then $V \in K$.

Theorem 2.11. Let $\mu \in FI(X)$ and

$$U_\mu = \{(x,y) \in X \times X \mid x \sim_\mu y\}.$$

If

$$K^* = \{U_\mu \mid \mu \in FI(X)\},$$

then K^* satisfies the conditions (U_1)- (U_4).

Theorem 2.12. Let $K = \{U \subseteq X \times X \mid U_\mu \subseteq U$, for some $\mu \in FI(X)\}$.
Then K satisfies a uniformly on X and the pair (X, K) is a uniform structure.

Notation. Let $x \in X$, and $U \in K$, we define

$$U[x] := \{y \in X \mid (x,y) \in U\}.$$

Theorem 2.13. Let $\boldsymbol{u} = \{G \subseteq X \mid \forall x \in G, \exists U \in K, U[x] \subseteq G\}$. The $\boldsymbol{u}$ is a topology on X.

Remark 2.14. Note that for any x in X, $U[x]$ is an open neighborhood of x.

Definition 2.15. Let (X, K) be a uniform space. Then the topology $\boldsymbol{u}$ is called the uniform topology on X induced by K.

III. Main Results

Proposition 3.1. Every ideal I of X is a clopen set in $(X, \boldsymbol{u})$.

Proof. Let I be an ideal of X. To prove that I is closed, we shall show that $I^C = \bigcup_{x \notin I} U_{\chi_I}[x]$. Indeed, assume $y \in I^C$, then from $y \in U_{\chi_I}[y]$ it follows that $y \in \bigcup_{x \notin I} U_{\chi_I}[x]$. Hence

$$I^C \subseteq \bigcup_{x \notin I} U_{\chi_I}[x]. \qquad (1)$$

Conversely, let $y \in \bigcup_{x \notin I} U_{\chi_I}[x]$. Then there is $z \in I^C$ such that $y \in U_{\chi_I}[z]$. Hence $y*z$ and $z*y \in I$. Now we show that $y \notin I$. On the contrary, let $y \in I$. Then from $z*y \in I$, we get that $z \in I$, which is contradiction. Therefore

$$\bigcup_{x \notin I} U_{\chi_I}[x] \subseteq I^C \qquad (2)$$

consequently from (1) and (2) we obtain that I is closed. To prove that I is open we show that

$$I = \bigcup_{x \in I} U_{\chi_I}[x]. \qquad (3)$$

Clearly $y \in U_{\chi_I}[y]$, $\forall y \in X$. Hence, $I \subseteq \bigcup_{x \in I} U_{\chi_I}[x]$.
On the other hand, let $y \in \bigcup_{x \in I} U_{\chi_I}[x]$, then there is $z \in I$ such that $y \in U_{\chi_I}[z]$. Thus $y*z \in I$ and $z*y \in I$. Now by BCI-2 we have

$$(y*(y*z))*z = 0 \in I.$$

Since $z \in I$ and $z*y \in I$ we get that $y \in I$. Thus

$$\bigcup_{x \in I} U_{\chi_I}[x] \subseteq I.$$

Therefore (3) holds, and hence I is open.

Theorem 3.2. Each $U_\mu[x]$ is a clopen set for all $\mu \in FI(X)$.

Proof. Let $\mu \in FI(X)$, $x \in X$. We want to show that $U_\mu[x]$ is a closed subset of X. Let $y \in (U_\mu[x])^c$. We claim that for the given element y we have

$$U_\mu[y] \subseteq (U_\mu[x])^c. \qquad (4)$$

Let $z \in U_\mu[y]$, then $\mu(z*y) > 0$ and $\mu(y*z) > 0$. If $z \in U_\mu[x]$, then $\mu(x*z) > 0$ and $\mu(z*x) > 0$. By Lemma 2.5 (iii), $\mu(x*y) > 0$ and $\mu(y*x) > 0$. It follows that $y \in U_\mu[x]$, which is a contradiction. Hence

$z \in (U_\mu[x])^c$, and (4) holds. Therefore $(U_\mu[x])^c$ is open, that is $U_\mu[x]$ is closed.

Theorem 3.3 [1]. In a quasi right alternate BCI-algebra, fuzzy ideals and fuzzy subalgebra coincide.

Corollary 3.4. Let X be a quasi right alternate BCI-algebra, then

i) Every subalgebra of X is clopen set in $(X, \boldsymbol{u})$.

ii) If μ is a fuzzy subalgebra of X, then $U_\mu[x]$ is a clopen set in $(X, \boldsymbol{u})$.

Proof. The proof follows from Theorems 3.12, 3.13 and Proposition 3.11.

Proposition 3.1. K is a discrete topology.

Proof. Let x be an arbitrary element of X. Then

$\{x\} = \{y \in X \mid y = x\}$

$= \{y \in X \mid x * y = 0,\ y * x = 0\}$

$= \{y \in X \quad \chi_{\{0\}}(x * y) > 0 \quad \chi_{\{0\}}(y * x) > 0\}$

$= U_{\chi_{\{0\}}}[x]$.

Now, the proof follows from Theorem 3.2.

Remark. Clearly $(X \times X,\ \otimes\ ,(0,0))$ is a BCK-algebra, where

$$\otimes : (X \times X) \times (X \times X) \to X \times X$$
$$((x, y), (x', y')) \mapsto (x * x', y * y').$$

Now, by $\boldsymbol{u}_{X \times X}$ and $\boldsymbol{u}_X$ we mean the uniform Topology on $X \times X$ and X respectively.

Theorem 3.6. Let X be a medial BCI-algebra. Then the operation $* : X \times X \to X$ is continuous.

Proof. Let $f : X \times X \to X$ be defined by

$f(x, y) = x * y$, $\forall x, y \in X$, $G \in \boldsymbol{u}_X$ and

$(x, y) \in f^{-1}(G)$.

Then there is $U \in K_X$ such that $U[x * y] \subseteq G$. Hence $U_\mu \subseteq U$, for some $\mu \in FI(X)$. Now we define fuzzy subset η of $X \times X$ by

$$\eta(x, y) = \mu(x * y).$$

we show that $\eta \in FI(X \times X)$.

$\eta(0,0) = \mu(0 * 0) = \mu(0) \geq \mu(x * y) = \eta(x, y)$, for all $x, y \in X$. On the other hand

$$\min\{\eta((x, y) * (z, u)), \eta(z, u)\} =$$

$\min\{\mu((x * z) * (y * u)), \mu(z * u)\}$

$= \min\{\mu((x * y) * (z, u)), \mu(z, u)\}$

$\leq \mu(x * y)$

$= \eta(x, y), \quad \forall x, y, z, u \in X$.

Therefore $\eta \in FI(X \times X)$. Now consider U_η in $K^*_{X \times X}$. We show that $U_\eta[(x, y)] \subseteq f^{-1}(G)$. Let

$$(z, u) \in U_\eta[(x, y)],$$

then

$$\min\{\eta((x, y) \otimes (z, u)), \eta((z, u) \otimes (x, y))\} > 0.$$

So,

$$\min\{\eta(x * z, y * u), \eta(z * x, u * y)\} > 0.$$

In other words,

$$\min\{\mu((x * z) * (y * u)), \mu((z * x) * (u * y))\} > 0.$$

Hence

$$\mu((x * y) * (z * u)) > 0$$

and

$$\mu((z * u), (x * y)) > 0.$$

.It follows that,

$(x * y, z * u) \in U_\mu \subseteq U$ and so $z * u \in U[x * y] \subseteq G$.

It means that $(z * u) = f(z, u) \in G$ or $(z, u) \in f^{-1}(G)$.

Consequently, $f^{-1}(G) \in \boldsymbol{u}_{X \times X}$.

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