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The Banzhaf-Owen value for fuzzy games with a coalition structure

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Abstract—In this paper, a generalized form of the Banzhaf-Owen value for cooperative fuzzy games with a coalition structure is proposed. Its axiomatic system is given by extending crisp case. In order to better understand the Banzhaf-Owen value for fuzzy games with a coalition structure, we briefly introduce the Banzhaf-Owen values for two special kinds of fuzzy games with a coalition structure, and give their explicit forms.

Keywords—cooperative fuzzy game, Banzhaf-Owen value, multilinear extension, Choquet integral

I. Introduction

As an important kind of games, games with coalition structures proposed by Owen [1] have been researched by many scholars and experts. The payoff indices for games with coalition structures are researched by Owen [1, 2] and Alonso-Meijide and Fiestras-Janeiro [3]. The axiomatic systems of the payoff indices for games with coalition structures are considered in [1-8]. But all above are only introduced for crisp games.

As Aubin [9] pointed, there are some situations where players do not fully participate in a coalition, but take action only according to the rate of participation. Generally speaking, it is difficult to identify a characteristic function of fuzzy games in practice. Hence, the characteristic function of fuzzy games is often constructed on the basis of the characteristic function of the original crisp games. Owen [10] defined a kind of fuzzy games, which is called fuzzy games with multilinear extension form. Tsurumi et al. [11] defined a kind of fuzzy games with Choquet integral form, and the Shapley function defined on this class of games is given. Butnariu [12] defined a class of fuzzy games with proportional value, and gave the expression of the Shapley function on this limited class of games. Recently, Butnariu and Kroupa [13] expanded the fuzzy games with proportional value to fuzzy games with weighted function, and gave the corresponding Shapley function. Meng and Zhang [14] studied a class of fuzzy games with multilinear form. More researches about fuzzy games can be seen in [15-22].

The purpose of this paper is to study the Banzhaf-Owen value for fuzzy games with coalition structures. We first study a simplified expression of the Banzhaf-Owen value for fuzzy games with coalition structures by giving the corresponding axioms. In order to better understand the given value for fuzzy games with coalition structures, we further research the Banzhaf-Owen value for two special kinds of fuzzy games with coalition structures, which are proposed by Owen [10] and Tsurumi et al. [11].

In section 2, we recall some notations and basic definitions, which will be used in the following. In section 3, an axiomatic definition of the Banzhaf-Owen value for fuzzy games with coalition structures is offered, and its explicit form is given. In section 4, we briefly discuss the Banzhaf-Owen values for two special kinds of fuzzy games with coalition structures, and give and investigate the explicit forms of the given Banzhaf-Owen values.

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II. Crisp games with coalition structures

Let $N = \{1, 2, \ldots, n\}$ be the player set, and $P(N)$ be the set of all crisp subsets in N. The coalitions in $P(N)$ are denoted by $S_0, T_0, \ldots$. For any $S_0 \in P(N)$, the cardinality of S_0 is denoted by the corresponding lower case s. A function$v_0 : P(N) \to \Re_+$, satisfying $v_0(\emptyset) = 0$, is called a crisp game. Let $G_0(N)$ denote the set of all crisp games in N.

A crisp coalition structure $\Gamma = \{B_0^1, B_0^2, \ldots, B_0^m\}$ in is a partition of N, i.e.,$\cup_{1\leq j\leq m} B_0^j = N$ and $B_0^i \cap B_0^j = \emptyset$ for any $i \neq j$, where $i, j \in M = \{1, 2, \ldots, m\}$. A crisp coalition structure in N is denoted by (N, Γ). For any S_0, $S_0 \in P(N, \Gamma)$ is called a feasible coalition, where $P(N, \Gamma)$ denotes the set of all feasible coalitions in (N, Γ).

Example 1: Let $N = \{1, 2, 3, 4\}$ and $\Gamma = \{B_0^1, B_0^2\}$, where $B_0^1 = \{1, 2\}$ and $B_0^2 = \{3, 4\}$, then $P(N, \Gamma) = \{\emptyset, \{i\}(i), \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}$.

Owen [10] proposed the Banzhaf-Owen value for crisp games with a coalition structure, which can be expressed by

$$\varphi_i(N, v_0, \Gamma) = \sum_{R\subseteq M\setminus k} \sum_{T_0\subseteq B_0^k\setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{b_k-1}} (v_0(Q_0 \cup T_0 \cup i) - v_0(Q_0 \cup T_0)) \qquad \forall i \in N, \qquad (1)$$

where $Q_0 = \cup_{q\in R} B_0^q$. $B_0^k \in \Gamma$ is an union such that $i \in B_0^k$.

III. The Banzhaf-Owen value for fuzzy games with a coalition structure

Let $L(N)$ denote the set of all fuzzy coalitions in N. The coalitions in $L(N)$ are denoted by $S, T, \ldots$. For a fuzzy coalition S and player i, $S(i)$ indicates the membership grade of i in S, i.e., the rate of the ith player's participation in S. For any $S \in L(N)$, the support is denoted by Supp$S = \{i \in N | S(i) > 0\}$, and the cardinality is denoted by $|\text{Supp}S|$. We use the notation $S \subseteq T$ if and only if $S(i) = T(i)$ or $S(i) = 0$ for any $i \in N$. A function $v : L(N) \to \Re_+$,

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satisfying $v(\emptyset) = 0$, is called a fuzzy game. Let $G(N)$ denote the set of all fuzzy games in $L(N)$. For any $S \in L(N)$, we will use $S = \{S(i)\}_{i \in N}$ to denote it, and we will omit braces for singletons, e.g., by writing $v(S), S \vee (\wedge) T, S \setminus T, U(i)$ instead of $v(S), \{S\} \vee (\wedge)\{T\}, \{S\} \setminus \{T\}, \{U(i)\}$ for any $v \in G(N)$ and any $\{U(i)\}, \{S\}, \{T\} \in L(N)$. In this paper, union and intersection of two fuzzy coalitions are defined as usual, i.e., $(S \vee T)(i) = S(i) \vee T(i)$ and $(S \wedge T)(i) = S(i) \wedge T(i)$.

Similar to crisp case, a fuzzy coalition structure $\Gamma_F = \{B_1, B_2, \ldots, B_m\}$ in is a partition of U, i.e., $\cup_{1 \leq j \leq m} \mathrm{Supp} B_j = \mathrm{Supp} U$, and $\mathrm{Supp} B_i \cap \mathrm{Supp} B_j = \emptyset$ for any $i \neq j (i, j \in M = \{1, 2, \ldots, m\})$, where $B_i \subseteq U$. The fuzzy coalition structure Γ_F in U is denoted by (U, Γ_F). For any $S \in L(U, \Gamma_F)$, S is called a feasible fuzzy coalition, where $L(U, \Gamma_F)$ denotes the set of all feasible fuzzy coalitions in (U, Γ_F).

Example 2: Let $N = \{1, 2, 3, 4\}$, and $U = \{U(i)\}_{i \in N}$ be a fuzzy coalition in $L(N)$ such that $U(i) > 0$ for any $i \in N$. $\Gamma_F = \{B_1, B_2\}$ is a fuzzy coalition structure in U, where $B_1 = \{U(1), U(2)\}$ and $B_2 = \{U(3), U(4)\}$, then

$$L(U, \Gamma_F) = \{\emptyset, \{U(i)\}(i \in N), \{U(1), U(2)\}, \{U(3), U(4)\}, \{U(1), U(2), U(3)\}, \{U(1), U(2), U(4)\}, \{U(1), U(3), U(4)\}, \{U(2), U(3), U(4)\}, U\}.$$

Remark 1: In this paper, if there is no special explanation $\Gamma_F = \{B_1, B_2, \ldots, B_m\}$ and $M = \{1, 2, \ldots, m\}$ always hold. Moreover, we use (N, v_0, Γ) and (U, v, Γ_F) to denote crisp game $v_0 \in G_0(N)$ in coalition structure (N, Γ) and fuzzy game $v \in G_(N)$ in coalition structure (U, Γ_F), respectively.

Definition 1: Let $v \in G_(N)$, v^B is called a quotient game on Γ_F, if it satisfies $v^B(R) = v(\vee_{l \in R} B_l)$ for all $R \in M$.

Similar to Lehrer [23], we give the concept of the "reduced" fuzzy game as follows:

For any $U(i), U(j) \in U$, put $U(p) = \{U(i), U(j)\}$ and consider the "reduced" fuzzy game v_p (with $(U \setminus U(p)) \vee \{U(p)\}$ as the set of players) defined by $v_p(S) = v(S)$ and $v_p(S \vee U(p)) = v(S \vee U(p))$ for any $S \subseteq U \setminus U(p)$.

From above, we know the "reduced" fuzzy game v_p has index $|\mathrm{Supp} U| - 1$.

Furthermore, we give the concept of the "reduced" game for quotient game v^B on(U, Γ_F). For any different indices $k, q \in M$, put $g = \{k, q\}$ and consider the "reduced" game v_g^B (with $(M \setminus g) \cup \{g\}$ as the set of players) defined by $v_g^B(R) = v^B(R)$ and $v_g^B(R \cup \{g\}) = v^B(R \cup g)$ for any $R \in M \setminus g$.

From above, we know the "reduced" quotient game has index $m - 1$.

Similar to Alonso-Meijide et al. [3], we give an axiomatic definition of the Banzhaf-Owen value for fuzzy games with a coalition structure as follows:

Definition 2: Let $v \in G_(N)$. The function $f : (U, v, \Gamma_F) \to \Re_+^{L(U)}$ is called a Banzhaf-Owen value in (U, Γ_F) if it satisfies

Axiom 1 (null property) Let $i \in \mathrm{Supp} U$, if we have $v(S \vee U(i)) = v(S)$ for any $S \in L(U, \Gamma_F)$ such that $i \notin \mathrm{Supp} S$, then $f_i(U, v, \Gamma_F) = 0$;

Axiom 2 (additivity) Let $v, w \in G(N)$, if we have $(v + w)(S) = v(S) + w(S)$ for any $S \in L(U, \Gamma_F)$, then $f(U, v + w, \Gamma_F) = f(U, v, \Gamma_F) + f(U, w, \Gamma_F)$;

Axiom 3 (symmetric in the fuzzy unions) Let $B_k \in \Gamma_F$ and all $i, j \in \mathrm{Supp} B_k$ with $U(i) = U(j)$, if we have $v(S \vee U(i)) = v(S \vee U(j))$ for any $S \in L(U, \Gamma_F)$ such that $i, j \notin \mathrm{Supp} S$, then

$$f_i(U, v, \Gamma_F) = f_j(U, v, \Gamma_F);$$

Axiom 4 (2-efficiency in the fuzzy unions) Let $B_k \in \Gamma_F$ and $i, j \in \mathrm{Supp} B_k$, we have

$$f_i(U, v, \Gamma_F) + f_j(U, v, \Gamma_F) = f_p(U, v_p, \Gamma_F),$$

where $p = \{i, j\}$.

Axiom 5 (1-quotient game property) Let $B_k \in \Gamma_F$, if we have $B_k = \{U(i)\}$, then

$$f_i(U, v, \Gamma_F) = \sum_{R \subseteq M \setminus k} \frac{1}{2^{m-1}} (v^B(R \cup k) - v^B(R))$$

.

Axiom 6 (neutrality under individual desertion) Let $B_k \in \Gamma_F$ and $i, j \in \mathrm{Supp} B_k$, we have

$$f_i(U, v, \Gamma_F) = f_i(U, v, \Gamma_F^{-j}),$$

where

$$\Gamma_F^{-j} = \{B_1, \ldots, B_{k-1}, B_k \backslash U(j), B_{k+1}, \ldots, B_m, \{U(j)\}\}.$$

Theorem 1: Let $v \in G_(N)$, the function $\beta : (U, v, \Gamma_F) \to \Re_+^{L(U)}$ is defined by

$$\beta_i(U, v, \Gamma_F) = \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{U(i)\}} \frac{1}{2^{m-1}} \frac{1}{2^{|\mathrm{Supp} B_k| - 1}} (v(S \vee Q \vee U(i)) - v(S \vee Q)) \quad \forall i \in \mathrm{Supp} U, \quad (2)$$

where $Q = \vee_{p \in R} B_p$. m and r denote the cardinalities of M and R, respectively. Then β is the unique Banzhaf-Owen value in (U, v, Γ_F).

proof: Existence. From Eq.(2), we easily get Axioms 1, 2.
Axiom 3: From Eq.(2), we have

$$\begin{aligned}
&\beta_i(U, v, \Gamma_F) \\
&= \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{U(i)\}} \tfrac{1}{2^{m-1}} \tfrac{1}{2^{|\mathrm{Supp} B_k| - 1}} (v(S \vee Q \vee U(i)) \\
&\quad - v(S \vee Q)) \\
&= \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{U(i), U(j)\}} \tfrac{1}{2^{m-1}} \tfrac{1}{2^{|\mathrm{Supp} B_k| - 1}} (v(S \vee Q \vee U(i) \\
&\quad \vee U(j)) - v(S \vee Q \vee U(j)) + v(S \vee Q \vee U(i)) - \\
&\quad v(S \vee Q)) \\
&= \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{U(i), U(j)\}} \tfrac{1}{2^{m-1}} \tfrac{1}{2^{|\mathrm{Supp} B_k| - 1}} (v(S \vee Q \vee U(i) \\
&\quad \vee U(j)) - v(S \vee Q \vee U(i)) + v(S \vee Q \vee U(j)) \\
&\quad - v(S \vee Q)) \\
&= \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{U(j)\}} \tfrac{1}{2^{m-1}} \tfrac{1}{2^{|\mathrm{Supp} B_k| - 1}} (v(S \vee Q \vee U(j)) - \\
&\quad v(S \vee Q)) \\
&= \beta_j(U, v, \Gamma_F);
\end{aligned}$$

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Axiom 4: From Eq.(2), we have

$$\begin{aligned}&\beta_i(U,v,\Gamma_F)+\beta_j(U,v,\Gamma_F)\\&=\sum_{R\subseteq M\setminus k}\sum_{S\subseteq B_k\setminus\{U(i)\}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(v(S\vee Q\vee U(i))-\right.\\&\quad\left.v(S\vee Q)\right)+\sum_{R\subseteq M\setminus k}\sum_{S\subseteq B_k\setminus\{U(j)\}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(v(S\vee\right.\\&\quad\left.Q\vee U(j))-v(S\vee Q)\right)\\&=\sum_{R\subseteq M\setminus k}\sum_{S\subseteq B_k\setminus\{U(i),U(j)\}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(v(S\vee Q\vee U(i)\right.\\&\quad\left.\vee U(j))-v(S\vee Q)\right)\\&=\sum_{R\subseteq M\setminus k}\sum_{S\subseteq B_k\setminus\{U(p)\}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(v(S\vee Q\vee U(p))-\right.\\&\quad\left.v(S\vee Q)\right)\\&=\beta_p(U,v_p,\Gamma_F);\end{aligned}$$

Axiom 5: From Eq.(2), we get

$$\begin{aligned}&\beta_i(U,v,\Gamma_F)\\&=\sum_{R\subseteq M\setminus k}\sum_{S\subseteq B_k\setminus\{U(i)\}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(v(S\vee Q\vee U(i))\right.\\&\quad\left.-\,v(S\vee Q)\right)\\&=\sum_{R\subseteq M\setminus k}\tfrac{1}{2^{m-1}}\left(v(Q\vee U(i))-v(Q)\right)\\&=\sum_{R\subseteq M\setminus k}\tfrac{1}{2^{m-1}}\left(v(Q\vee B_k)-v(Q)\right)\\&=\sum_{R\subseteq M\setminus k}\tfrac{1}{2^{m-1}}\left(v^B(R\cup k)-v^B(R)\right)\\&=\beta_k(U,v^B,\Gamma_F);\end{aligned}$$

Axiom 6: From Eq.(2), we obtain

$$\begin{aligned}&\beta_i(U,v,\Gamma_F)\\&=\sum_{R\subseteq M\setminus k}\sum_{\substack{S\subseteq B_k,\\ j,i\notin\mathrm{Supp}S}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(v(S\vee Q\vee U(i))-\right.\\&\quad\left.v(S\vee Q)\right)+\sum_{R\subseteq M\setminus k}\sum_{\substack{S\subseteq B_k,\\ j,i\notin\mathrm{Supp}S}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(v(S\vee\right.\\&\quad\left.U(i)\vee U(j)\vee Q)-v(S\vee U(j)\vee Q)\right).\end{aligned}$$

Let $\Gamma_F^{-j}=\{B_1^{'},B_2^{'},\ldots,B_{m+1}^{'}\}$, where $B_{m+1}^{'}=\{U(j)\}$, $B_k=B_k^{'}\vee U(j)$ and $B_l=B_l^{'}$ for any $l\in\{1,2,\ldots,k-1,k+1,\ldots,m\}$. Thus,

$$\begin{aligned}&\beta_i(U,v,\Gamma_F^{-j})\\&=\sum_{R\subseteq M'\setminus k}\sum_{S\subseteq B_k^{'},i\notin\mathrm{Supp}S}\tfrac{1}{2^{m}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-2}}\left(v(S\vee U(i)\vee Q)-\right.\\&\quad\left.v(S\vee Q)\right)\\&=\sum_{R\subseteq M\setminus k}\sum_{\substack{S\subseteq B_k,\\ i,j\notin\mathrm{Supp}S}}\tfrac{1}{2^{m}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-2}}\left(v(S\vee U(i)\vee Q)-\right.\\&\quad\left.v(S\vee Q)\right)+\sum_{R\subseteq M\setminus k}\sum_{\substack{S\subseteq B_k,\\ i,j\notin\mathrm{Supp}S}}\tfrac{1}{2^{m}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-2}}\left(v(S\vee\right.\\&\quad\left.U(i)\vee U(j)\vee Q)-v(S\vee U(j)\vee Q)\right).\end{aligned}$$

Since, $\frac{1}{2^{m-1}}\frac{1}{2^{|\mathrm{Supp}B_k|-1}}=\frac{1}{2^{m}}\frac{1}{2^{|\mathrm{Supp}B_k|-2}}$, we have

$$\beta_i(U,v,\Gamma_F)=\beta_i(U,v,\Gamma_F^{-j}).$$

Uniqueness. Since, $v\in G_(N)$ limited in (U,Γ_F) can be uniquely expressed by $v=\sum_{\emptyset\neq T\in L(U,\Gamma_F)}c_Tu_T$, where $c_T=\sum_{\substack{H=H'\cup k,\\ H'\subseteq R\setminus k}}(-1)^{r-h}(\sum_{A\subseteq S}(-1)^{|\mathrm{Supp}S|-|\mathrm{Supp}A|}v(A\cup Q))$, r and h denote the cardinalities of R and H, respectively. $Q=\cup_{l\in H'}B_l$ and $T=S\vee_{l\in R\setminus k}B_l$ such that$\emptyset\neq S\subseteq B_k$.

$u_T(R)=\begin{cases}1 & T\subseteq R\\ 0 & \text{otherwise}\end{cases}$.

From Axiom 2, we only need to show the uniqueness of Eq.(2) on the unanimity games. For any $\emptyset\neq T\in L(U,\Gamma_F)$, let

$$M'=\{k\in M:\mathrm{Supp}B_k\cap\mathrm{Supp}T\neq\emptyset\}$$

and

$$\mathrm{Supp}B_k^{'}=\mathrm{Supp}B_k\cap\mathrm{Supp}T\qquad\forall k\in M.$$

Define the unanimity quotient game as follows:

$$u_T^B(R)=\begin{cases}1 & M'\subseteq R\\ 0 & M'\not\subseteq R\end{cases},$$

where $R\subseteq M$.

Let f be a solution in (U,u_T,Γ_F) that satisfies the mentioned axioms in Definition 2. From Axiom 1, we have$f_i(U,u_T,\Gamma_F)=0$ for any $i\notin\mathrm{Supp}T$.

Case (i): When $m'=1$, where m' denote the cardinality of M'. From Axioms 1, 2, 3, 4, we get

$$f_i(U,u_T,\Gamma_F)=\begin{cases}\frac{1}{2^{|\mathrm{Supp}B_k^{'}|-1}} & i\in\mathrm{Supp}B_k^{'},k\in M'\\ 0 & \text{otherwise}\end{cases}$$

From Eq.(2), we have $\beta(U,u_T,\Gamma_F)=f(U,u_T,\Gamma_F)$.

Case (ii): When$m'>1$, and $\mathrm{Supp}B_k^{'}=\{i\}$. From Axiom 5, we get

$$\begin{aligned}&f_i(U,u_T,\Gamma_F)\\&=\sum_{R\subseteq M\setminus k}\tfrac{1}{2^{m-1}}(u_T^B(R\cup k)-u_T^B(R))\\&=\sum_{M'\setminus k\subseteq R\subseteq M\setminus k}\tfrac{1}{2^{m-1}}\\&=\tfrac{1}{2^{m'-1}}.\end{aligned}$$

On the other hand, from Eq.(2), we have

$$\begin{aligned}&\beta_i(U,u_T,\Gamma_F)\\&=\sum_{R\subseteq M\setminus k}\sum_{S\subseteq B_k\setminus\{U(i)\}}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\left(u_T(S\vee Q\vee U(i))\right.\\&\quad\left.-u_T(S\vee Q)\right)\\&=\sum_{M'\setminus k\subseteq R\subseteq M\setminus k}\ \sum_{\mathrm{Supp}B_k^{'}\setminus i\subseteq\mathrm{Supp}S\subseteq\mathrm{Supp}B_k\setminus i}\tfrac{1}{2^{m-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k|-1}}\\&=\tfrac{1}{2^{m'-1}}\tfrac{1}{2^{|\mathrm{Supp}B_k^{'}|-1}}\\&=\tfrac{1}{2^{m'-1}}.\end{aligned}$$

Thus, we have$\beta(U,u_T,\Gamma_F)=f(U,u_T,\Gamma_F)$.

Case (iii): When $m'>1$, and $\mathrm{Supp}B_k^{'}>\{i\}$. Without loss of generality, suppose $|\mathrm{Supp}B_k^{'}|=h$, where $\mathrm{Supp}B_k^{'}=\{i,j_1,j_2,\ldots,j_{h-1}\}$. Let $\Gamma_F^{-j_1}=\{B_1^{'},B_2^{'},\ldots,B_{m+1}^{'}\}$, where $B_{m+1}^{'}=\{U(j_1)\}$, $B_k=B_k^{'}\vee U(j_1)$ and $B_l=B_l^{'}$ for any $l\in\{1,2,\ldots,k-1,k+1,\ldots,m\}$; Let $\Gamma_F^{-\{j_1,j_2\}}=\{B_1^{''},B_2^{''},\ldots,B_{m+1}^{''},B_{m+2}^{''}\}$, where $B_{m+2}^{''}=\{U(j_2)\}$, $B_k^{'}=B_k^{''}\vee U(j_2)$ and$B_l^{'}=B_l^{''}$ for any $l\in\{1,2,\ldots,k-1,k+1,\ldots,m,m+1\}$,.... From recursive

Open Science Index, Mathematical and Computational Sciences Vol:5, No:8, 2011 publications.waset.org/4821.pdf

method and Axioms 5, 6, we get

$$\begin{aligned} & f_i(U, u_T, \Gamma_F) \\ & = \sum_{R \subseteq \{M \cup H'\} \setminus k} \tfrac{1}{2^{m+h-2}} (u_T^B(R \cup k) - u_T^B(R)) \\ & = \sum_{M' \setminus k \subseteq R \subseteq \{M \cup H'\} \setminus k} \tfrac{1}{2^{m+h-2}} \\ & = \tfrac{1}{2^{h-1}} \sum_{M' \setminus k \subseteq R \subseteq M \setminus k} \tfrac{1}{2^{m-1}} \\ & = \tfrac{1}{2^{m'+h-2}}, \end{aligned}$$

where $H' = \{1, 2, ..., h-1\}$. From Eq.(2), we obtain

$$\beta_i(U, u_T, \Gamma_F) = \frac{1}{2^{m'-1}} \frac{1}{2^{h-1}} = \frac{1}{2^{m'+h-2}}.$$

Thus, Eq.(2) is unique on the unanimity games, and the proof is finished.

■

IV. The Banzhaf-Owen value for two special kinds of fuzzy games with a coalition structure

In this section, we shall discuss the Banzhaf- Owen value for two special kinds of fuzzy games with a coalition structure, which are proposed by Owen [10] and Tsurumi et al. [11]. The fuzzy coalition value for fuzzy games with "multilinear extension form" is expressed by (Owen [10])

$$v_O(U) = \sum_{T \subseteq U} \{\Pi_{i \in \mathrm{Supp}T} U(i) \Pi_{i \in \mathrm{Supp}U \setminus \mathrm{Supp}T} (1 - U(i))\} \\ v_0(\mathrm{Supp}T) \qquad \forall U \in L(N). \tag{3}$$

The fuzzy coalition value for fuzzy games with "Choquet integral form" is written as (Tsurumi et al. [11])

$$v_C(U) = \sum\nolimits_{l=1}^{q(U)} v_0([U]_{h_l})(h_l - h_{l-1}) \quad \forall U \in L(N). \tag{4}$$

where $[U]_{h_l} = \{i \in N | U(i) \geq h_l\}$ and $Q(U) = \{U(i) | U(i) > 0, i \in N\}$. $q(U)$ denotes the cardinality of $Q(U)$. The elements in $Q(U)$ are written in the increasing order as $0 = h_0 \leq h_1 \leq \cdots \leq h_{q(U)}$.

Let $G_O(U)$ and $G_C(U)$ denote the set of all fuzzy games in $U \in L(N)$ given by Owen [10] and Tsurumi et al. [11], respectively. For (U, v, Γ_F), when we restrict $v \in G(N)$ in the setting of $v_O \in G_O(U)$ and $v_C \in G_C(U)$, then we get the coalition structure (U, v_O, Γ_F) and (U, v_C, Γ_F), respectively.

Theorem 2: Let $v_O \in G_O(U)$, the function $\beta^O : (U, v_O, \Gamma_F) \to \Re_+^{L(U)}$ is defined by

$$\begin{aligned} & \beta_i^O(U, v_O, \Gamma_F) = \sum_{R \subseteq M \setminus k} \sum_{S \subseteq B_k \setminus \{U(i)\}} \frac{1}{2^{m-1}} \frac{1}{2^{|\mathrm{Supp}B_k| - 1}} \\ & (\sum_{W \in L(R)} \sum_{\substack{i \in T_0 \subseteq \mathrm{Supp}\{S \vee U(i)\}, \{T_0 \cup \mathrm{Supp}(\vee_{p \in W} B_p)\} \\ \in \mathrm{Supp}L(S \vee U(i) \vee_{p \in W} B_p, \Gamma_F^{S \vee U(i) \vee_{p \in W} B_p})}} \\ & (\prod_{j \in \{T_0 \cup \mathrm{Supp}(\vee_{p \in W} B_p)\}} U(j) \prod_{\substack{j \in \mathrm{Supp}\{S \vee Q \vee U(i)\} \setminus \\ \{T_0 \cup \mathrm{Supp}(\vee_{p \in W} B_p)\}}} (1 - U(j)) \\ & v_0(T_0 \ )) - \sum_{W \in L(R)} \sum_{\substack{T_0 \subseteq \mathrm{Supp}S, \{T_0 \cup \mathrm{Supp}(\vee_{p \in W} B_p)\} \\ \in \mathrm{Supp}L(S \vee_{p \in W} B_p, \Gamma_F^{S \vee_{p \in W} B_p})}} \\ & (\prod_{j \in \{T_0 \cup \mathrm{Supp}(\vee_{p \in W} B_p)\}} U(j) \prod_{\substack{j \in \mathrm{Supp}\{S \vee Q\} \setminus \\ \{T_0 \cup \mathrm{Supp}(\vee_{p \in W} B_p)\}}} (1 - U(j))) \\ & v_0(T_0)) \qquad \forall i \in \mathrm{Supp}U, \end{aligned} \tag{5}$$

where $L(R)$ denotes the power set of R, $\Gamma_F^{S \vee U(i) \vee_{p \in W} B_p}$ and $\Gamma_F^{S \vee_{p \in W} B_p}$ are the fuzzy coalition structures in $S \vee U(i) \vee_{p \in W} B_p$ and $S \vee_{p \in W} B_p$ with respect to Γ_F. $\mathrm{Supp}L(S \vee_{p \in W} B_p, \Gamma_F^{S \vee_{p \in W} B_p})$ and $\mathrm{Supp}L(S \vee U(i) \vee_{p \in W} B_p, \Gamma_F^{S \vee U(i) \vee_{p \in W} B_p})$ denote the support of $L(S \vee_{p \in W} B_p, \Gamma_F^{S \vee_{p \in W} B_p})$ and $L(S \vee U(i) \vee_{p \in W} B_p, \Gamma_F^{S \vee U(i) \vee_{p \in W} B_p})$, respectively. $Q = \vee_{p \in R} B_p$. Then β^O is the unique Banzhaf-Owen value in (U, v_O, Γ_F).

Proof: The proof of Theorem 2 is similar to that of Theorem 1.

■

Example 3: (cf. Example 2). Let the player set $N = \{1, 2, 3, 4\}$, and $U = \{U(i)\}_{i \in N}$ be a fuzzy coalition in as defined in Example 3.1, i.e., $U(i) > 0$ for any $i \in N$, and $\Gamma_F = \{B_1, B_2\}$ is a fuzzy coalition structure in U as defined in Example 3.1, i.e.,$B_1 = \{U(1), U(2)\}$ and $B_2 = \{U(3), U(4)\}$, then

$$\mathrm{Supp}L(U, \Gamma_F) = \{\emptyset, \{i\}(i \in N), \{1,2\}, \{3,4\}, \{1,2,3\}, \\ \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, N\}.$$

If $S = \{U(1), U(2), U(3)\} \subseteq U$, then we get

$$L(S, \Gamma_F^S) = \{\emptyset, \{U(i)\}(i \in \{1,2,3\}), \{U(1), U(2)\}, S\}$$

and

$$\mathrm{Supp}L(S, \Gamma_F^S) = \{\emptyset, \{i\}(i \in \{1,2,3\}), \{1,2\}, \{1,2,3\}\}.$$

Theorem 3: Let $v_C \in G_C(U)$, the function$\beta^C : (U, v_C, \Gamma_F) \to \Re_+^{L(U)}$ is defined by

$$\beta_i^C(U, v_C, \Gamma_F) = \sum_{l=1}^{q(U)} \beta_i([U]_{h_l}, v_0, \Gamma_F^{[U]_{h_l}})(h_l - h_{l-1}) \\ \forall i \in \mathrm{Supp}U \tag{6}$$

where $\beta_i([U]_{h_l}, v_0, \Gamma_F^{[U]_{h_l}})$ denotes the Banzhaf-Owen value in $([U]_{h_l}, \Gamma_F^{[U]_{h_l}})$ shown as Eq.(1). Then β^C is the unique Banzhaf-Owen value in (U, v_C, Γ_F).

Proof: From Theorem 1, it is not difficult to get the result.

■

Example 4: Let the player set $N = \{1, 2, 3, 4, 5\}$, $v \in G(N)$ and $U = \{U(i)\}_{i \in N}$ be a fuzzy coalition in $L(N)$ such that$U(i) > 0$ for any $i \in N$, where $U(1) = 0.6, U(2) = 0.3, U(3) = 0.5, U(4) = 0.7$ and $U(5) = 0.8$. $\Gamma_F = \{B_1, B_2, B_3\}$is a fuzzy coalition structure in U, where$B_1 = \{U(1), U(2)\}, B_2 = \{U(3), U(4)\}$ and $B_3 = \{U(5)\}$. The crisp coalition values are given by Table 1.

When we restrict $v \in G(N)$ in setting of $v_O \in G_O(U)$, from Eq.(5), we get the player Banzhaf-Owen values with respect to v_O are

$$\beta_1^O(U, v_O, \Gamma_F) = 0.84, \beta_2^O(U, v_O, \Gamma_F) = 0.83,$$

$$\beta_3^O(U, v_O, \Gamma_F) = 0.78, \beta_4^O(U, v_O, \Gamma_F) = 1.63,$$

TABLE I
(VALUES FOR CRISP COALITIONS)

S_0	$v_0(S_0)$	S_0	$v_0(S_0)$
{1}	1	{1,2,4}	8
{2}	2	{1,3,4}	5
{3}	1	{1,2,5}	5
{4}	2	{1,4,5}	6
{5}	1	{1,3,5}	8
{1,2}	4	{2,3,4}	7
{1,3}	3	{3,4,5}	6
{1,4}	5	{1,2,3,4}	12
{2,5}	5	{2,3,4,5}	12
{3,4}	3	{1,2,3,5}	10
{3,5}	3	{1,2,4,5}	12
{4,5}	4	{1,2,3,4,5}	25
{1,2,3}	6		

$$\beta_5^O(U, v_O, \Gamma_F) = 1.51.$$

When we restrict $v \in G(N)$ in domain of $v_C \in G_C(U)$, from Eq.(6), we get the player Banzhaf-Owen values with respect to v_C are

$$\beta_1^C(U, v_C, \Gamma_F) = 1.56, \beta_2^C(U, v_C, \Gamma_F) = 1.69,$$

$$\beta_3^C(U, v_C, \Gamma_F) = 1.5, \beta_4^C(U, v_C, \Gamma_F) = 2.6,$$

$$\beta_5^C(U, v_C, \Gamma_F) = 2.2.$$

V. CONCLUSION

We have researched the Banzhaf-Owen value for fuzzy games with coalition structures, which extends the using scope of the Banzhaf-Owen value. Like other payoff indices, there are other axiomatic systems to show the existence and uniqueness of the Banzhaf-Owen value, such as Owen [2], Alonso-Meijide et al. [3]. Moreover, we briefly study two special kinds of fuzzy games with coalition structures.

At present, the research about payoff indices for fuzzy games with coalition structures is not perfect. We shall continue to study in this area. Furthermore, we shall study other kinds of game theory, such as network formation games and playing online games.

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