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Solution of Two Dimensional Quasi-Harmonic Equations with CA Approach

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***Abstract*—**Many computational techniques were applied to solution of heat conduction problem. Those techniques were the finite difference (FD), finite element (FE) and recently meshless methods. FE is commonly used in solution of equation of heat conduction problem based on the summation of stiffness matrix of elements and the solution of the final system of equations. Because of summation process of finite element, convergence rate was decreased. Hence in the present paper Cellular Automata (CA) approach is presented for the solution of heat conduction problem. Each cell considered as a fixed point in a regular grid lead to the solution of a system of equations is substituted by discrete systems of equations with small dimensions. Results show that CA can be used for solution of heat conduction problem.

***Keywords*—**Heat conduction, Cellular automata, convergence rate, discrete system.

I. Introduction

IN general, partial differential equations are much more difficult to solve analytically than are ordinary differential equations. They may sometimes be solved using a Bäcklund transformation, characteristics, Green's function, integral transform, Lax pair, separation of variables, when all else fails (which it frequently does) numerical methods such as finite differences, finite element and meshless methods [1, 2] is used.

Cellular Automata (CA) is a simple mathematical model which provides powerful and interesting tools for describing complex space-time phenomena. Since their introduction by Stanislaw Ulam [3, 4 and 5] and the work of John Von Neumann [6] to provide a formal framework for investigating the behavior of complex and adaptive systems, many other scientists have applied CA method to a wide range of problems. Stephen Wolfram [7, 8, 9, 10 and 11] in the publication of his book "A new kind of science" gave a classification of CA and developed a very good study establishing that CA's evolution may reproduce behaviors of many continuous systems. In recent years, CA has already become a very popular tool for simulating the behavior of complex physical processes [12, 13 and 14]. Gürdel and Tatting [15] used cellular automata for design of truss structures with linear and nonlinear response. Missoum et.al [16] worked on optimization of nonlinear trusses using a displacement based approach. Missoum et.al [17] addressed nonlinear topology design of trusses using cellular automata. Cortés et.al [18] employed cellular automata for the topology optimization of truss structures using Accelerated Simultaneous Analysis and Design (ASAND). Tovar et.al [19] studied optimality conditions of the hybrid cellular automata for structural optimization. Setoodeh et.al [20, 21 and 22] worked on optimal design of variable stiffness fiber reinforced composites using cellular automata. In civil and mechanical engineering, the heat flow component of greatest concern in the conductive heat flow. This is especially of concern in the design of structures that will affect the thermal region of the soil. That includes warm foundations, hockey rinks, refrigerated storage, pipelines, etc.

In the present paper CA is used for the solution of heat conduction problem. In what follows in section 2 CA approach is first explained. Then in sections 3 the construction of CA formulation for solution of heat conduction problem presented. In section 4 some numerical examples are solved with this approach and the results compared with exact analytical solution and finally in section 5 some concluding remarks are addressed.

II. Cellular Automata

A. Introduction

Cellular automaton is a decentralized computing model for carrying out complex computation with local information, also is dynamic system in which space and time are discrete. In this approach each cell is considered as a fixed point in a regular grid. The state of each cell is updated at discrete time steps, based upon conditions in previous time steps. All of the cells are updated every time step, thus the state of the entire grid is updated every time step. In the CA approach, the physical domain is decomposed into a grid of cells. Each cell is governed by rules that depend on the neighboring cells only. All computations are done locally, and the connectivity of cells can be directly mapped into inter-processor connectivity. When the correct update rules are used, CA is hoped to converge to the correct global state (solution) of the problem. Each cell contains some initial states and the same sets of

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rules are applied for each cell. The rules specify how these states are to be changed at the next computer clock interval. The logical rules of a cell specifies the new state of that cell on the next clock period, based on the cells current state, and on that of all the cell's immediate neighbors. The number of neighbors that influence a given cell is what we call the connectivity of the cellular automata. In other words, the number of neighbors that connect (or influence) a given cell is called the CA connectivity. The connectivity can be any positive integer number.

B. Neighborhood

Two particular 2D neighborhoods, which are often used, are the Moore neighborhood and the Von Neumann neighborhood although there are many other possible variations.

Von Neumann Neighborhood: A diamond -shaped neighborhood that can be used to define a set of cells surrounding a given cell (x_0, y_0) that may affect the evolution of a two-dimensional cellular automaton on a square grid. The von Neumann neighborhood of range r is defined by:

$$N_{(x_0,y_0)}^{Von\ Neumann} = \{(x,y) : |x - x_0| + |y - y_0| \le r\} \quad (1)$$

Von Neumann neighborhoods for ranges r = 0, 1, 2, and 3 are illustrated in Fig.1. The number of cells in the Von Neumann neighborhood of range r is the centered square number $2r(r+1)+1$, the first few of which are 1, 5, 13, 25, 41, 61,… [12].

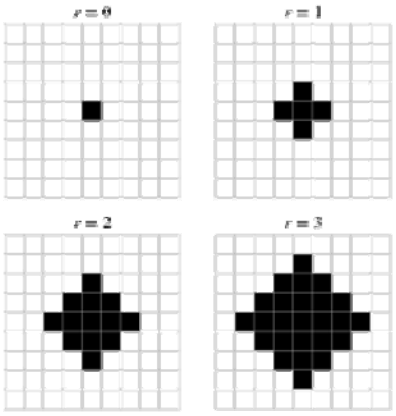


Fig. 1 Von Neumann Neighborhood (r = 0, 1, 2, and 3)

Moore Neighborhood: A square shaped neighborhood that can be used to define a set of cells surrounding a given cell (x_0, y_0) that may affect the evolution of a two-dimensional cellular automaton on a square grid. The Moore neighborhood of range r is defined by,

$$N_{(x_0,y_0)}^{Moore} = \{(x,y) : |x - x_0| \le r, |y - y_0| \le r\} \quad (2)$$

Moore neighborhoods for ranges r = 0, 1, 2, and 3 are illustrated in Fig.2. The number of cells in the Moore neighborhood of range r is the odd squares $(2r+1)^2$, the first few of which are 1, 9, 25, 49, 81,... .

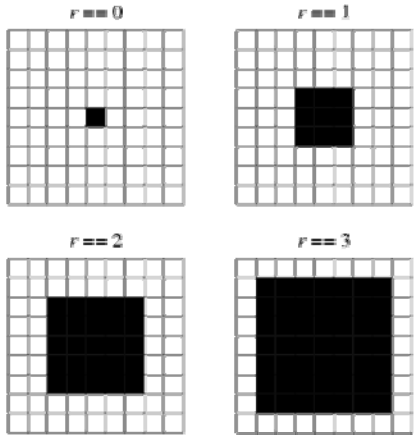


Fig. 2 Moore Neighborhood (r = 0, 1, 2, and 3)

C. CA grid

A CA space consists of a grid of cells; each can be in one of N finite states at any time. The grid can be of any size and shape. It can have any number of dimensions although one and two-dimensional cellular automata are the most common. The cells can also be of various shapes as long as they can be tessellated. Fig.3 shows a two dimensional ground truss and it's sample cell.

D. Update rules

This can be anything from simple on or off (1 or 0) states, to a large set of structured variables such as arrays. Each of the cells can be in one of several states. One cell determines its next state based on the state of itself and its neighborhood using a rule set. The number of possible states depends on the automaton. The cells can change from state to state. The cellular automaton's rules determine how the states change. It works like this: When the time comes for the cells to change state, each cell looks around and collects information on its neighbor's states. Based on its own state, its neighbor's states, and the rules of the CA, the cell decides what its new state should be. Usually all the cells change state at the same time. Standard implementation of CA requires that two copies of the array of cell sates are kept, one to represent iteration t, and the other to represent iteration t+1. The values of the cell states at t+1 are calculated from the cells at t, which are commonly referred to as parallelism in CA. At the end of each iteration, the labels of the arrays are swapped, and the process is repeated. This is here referred to as Jacobi iteration, where all of the new values are calculated from the old values. The Jacobi iteration while convergent can be inefficient. In an alternative method, referred to here as Gauss-Seidel iteration method, the new values are calculated using old and updated values wherever available. This method may lead to faster convergence of the CA method. This means that only one copy of the array is kept.

III. CA FORMULATION FOR SOLUTION OF THE TWO DIMENSIONAL QUASI-HARMONIC EQUATIONS

A. Two dimensional quasi-harmonic equations

The steady state behavior of many physical phenomena can be described in two-dimensions by the following quasi-harmonic equation:

$$\frac{\partial}{\partial x}(K_x \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial x}(K_y \frac{\partial \phi}{\partial y}) + Q = 0 \quad (3)$$

in which ϕ is the unknown function and K_x , K_y and Q are material parameters which can be functions of x and y. it is noted that, since K_x and K_y can be generally different, we permit orthotropic material behavior. A number of field problems governed by the equation and the physical interpretations of K_x , K_y , ϕ and Q are listed in Table. 1.

TABLE I
PHYSICAL SITUATIONS GOVERNED BY THE QUASI-HARMONIC EQUATION

Physical problem	Unknown ϕ	K_x , K_y	Q
Heat conduction	Temperature	Conductivity	Internal heat generated
Gas Diffusion	Concentration	Diffusivity	$Q=0$
Seepage	Pressure head	Permeability	$Q=0$
Compressible flow	Velocity potential	Density	$Q=0$
Magneto statics	Magnetic potential	Reluctivity	Current density
Torsion	Stress function	(Shear modulus)	Twist
Torsion	Warping function	Shear modulus	$Q=0$
Reynolds film lubrication	Pressure	(Film thickness)/viscosity	Lubricant supply

Two main types of boundary condition are of interest:

1. The value of the unknown to be specified at nodal points on the boundary (Dirichlet boundary condition)

$$\phi = \phi_P \quad (4)$$

2. That a boundary "loading" exists of the form

$$K_x \frac{\partial \phi}{\partial x} L_x + K_y \frac{\partial \phi}{\partial y} L_y + q + \alpha(\phi - \phi_a) = 0 \quad (5)$$

in which q, α and ϕ_a are constants and L_x, L_y are the direction cosines between the outward normal , n , and the x and y axes respectively. This is termed the Cauchy boundary condition. If both boundary conditions (A) and (B) exist in a problem, mathematically the boundary conditions are said to be mixed. The physical significance of this second boundary condition is best illustrated by considering the isotropic case $K_x = K_y = K$ and referring to the particular situation of heat conduction in a two-dimensional medium. In this case ϕ is the temperature at any point and K is the thermal conductivity of the material. The, boundary condition (B) reduces to

$$K \frac{\partial \phi}{\partial n} + q + \alpha(\phi - \phi_a) = 0 \quad (6)$$

where $\partial\phi / \partial n$ is the temperature gradient in a direction normal to the surface at the point under consideration.

B. Finite element discretisation with Galerkin method

If we adopt a finite element discretisation, then the unknown function ϕ may be approximated as

$$\phi \approx \hat{\phi} = \sum_{m=1}^{M} \phi_m N_m \quad (7)$$

Suppose,

$$k_x = k_y = k \quad (8)$$

Therefore,

$$\frac{\partial}{\partial x}(k \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial x}(k \frac{\partial \phi}{\partial y}) + Q = 0 \quad (9)$$

Boundary Conditions:

$$\phi - \bar{\phi} = 0 \qquad \Gamma_\phi$$

$$k \frac{\partial \phi}{\partial n} = -\bar{q} \qquad \Gamma_q$$

Approximation the unknown function with the weak form lead to,

$$\int_\Omega (\frac{\partial N_l}{\partial x} k \frac{\partial \hat{\phi}}{\partial x} + \frac{\partial N_l}{\partial y} k \frac{\partial \hat{\phi}}{\partial y}) dxdy = \int_\Omega N_l Q dxdy - \int_{\Gamma_q^e} N_l^e \bar{q} \, d\Gamma \quad (10)$$

With inserting the approximation,

$$K\phi = F \quad (11)$$

Where,

$$K_{lm}^e = \int_{\Omega^e} (\frac{\partial N_l^e}{\partial x} k \frac{\partial N_m^e}{\partial x} + \frac{\partial N_l}{\partial y} k \frac{\partial N_m^e}{\partial y}) dxdy \quad (12)$$

$$f_l^e = \int_{\Omega^e} N_l^e Q dxdy - \int_{\Gamma_q^e} N_l^e \bar{q} \, d\Gamma \quad (13)$$

Triangular elements: The nonzero contribution to $K\phi$ from the element e, with nodes i, j, k

$$k^e \phi^e = \int_0^{x_e} \int_0^{y_e} k \begin{bmatrix} \left[(\frac{\partial N_i^e}{\partial x})^2 + (\frac{\partial N_i^e}{\partial y})^2\right] & \left[\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}\right] & \left[\frac{\partial N_i^e}{\partial x}\frac{\partial N_k^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_k^e}{\partial y}\right] \\ \left[\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}\right] & \left[(\frac{\partial N_j^e}{\partial x})^2 + (\frac{\partial N_j^e}{\partial y})^2\right] & \left[\frac{\partial N_j^e}{\partial x}\frac{\partial N_k^e}{\partial x} + \frac{\partial N_j^e}{\partial y}\frac{\partial N_k^e}{\partial y}\right] \\ \left[\frac{\partial N_i^e}{\partial x}\frac{\partial N_k^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_k^e}{\partial y}\right] & \left[\frac{\partial N_j^e}{\partial x}\frac{\partial N_k^e}{\partial x} + \frac{\partial N_j^e}{\partial y}\frac{\partial N_k^e}{\partial y}\right] & \left[(\frac{\partial N_k^e}{\partial x})^2 + (\frac{\partial N_k^e}{\partial y})^2\right] \end{bmatrix} dxdy \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_k \end{bmatrix} \quad (14)$$

Shape function derivatives,

$$\beta_i^e = \frac{\partial N_i^e}{\partial x} \qquad \gamma_i^e = \frac{\partial N_i^e}{\partial y} \quad (15)$$

Assume k, Q and $\bar{q}$ are constant within an element
Then noting that,

$$\Delta^e = \int_{\Omega^e} dxdy \quad (16)$$

It follows that

$$k^e\phi^e = k^e\Delta^e \begin{bmatrix} [(\beta_i^e)^2+(\gamma_i^e)^2] & [\beta_i^e\beta_j^e+\gamma_i^e\gamma_j^e] & [\beta_i^e\beta_k^e+\gamma_i^e\gamma_k^e] \\ [\beta_i^e\beta_j^e+\gamma_i^e\gamma_j^e] & [(\beta_j^e)^2+(\gamma_j^e)^2] & [\beta_j^e\beta_k^e+\gamma_j^e\gamma_k^e] \\ [\beta_i^e\beta_k^e+\gamma_i^e\gamma_k^e] & [\beta_j^e\beta_k^e+\gamma_j^e\gamma_k^e] & [(\beta_k^e)^2+(\gamma_k^e)^2] \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_j \\ \phi_k \end{bmatrix} \quad (17)$$

The element contribution to the load vector

$$f_l^e = \int_{\Omega^e} N_l^e Q dx dy - \int_{\Gamma_q^e} N_l^e \bar{q} d\Gamma \quad (18)$$

f is calculated using

$$\int_{\Omega^e} N_l^e Q dx dy = \alpha_i^e + \beta_i^e \overline{X}^e + \gamma_i^e \overline{Y}^e \quad (19)$$

Where,

$$\overline{X}^e = \frac{x_i + x_j + x_k}{3} \qquad \overline{Y}^e = \frac{y_i + y_j + y_k}{3} \quad (20)$$

the result,

$$f_i^e = \frac{1}{3}Q^e\Delta^e - \frac{1}{2}\bar{q}^e\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

$$f_j^e = \frac{1}{3}Q^e\Delta^e - \frac{1}{2}\bar{q}^e\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \quad (21)$$

$$f_k^e = \frac{1}{3}Q^e\Delta^e$$

i nd j area assumed to lie on Γ_q.

C. Heat flow through a conductor

The general partial differential equation for conductive heat flow in a soil is given by,

$$\frac{\partial}{\partial x}(\lambda_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda_y \frac{\partial T}{\partial y}) = c\rho\frac{\partial T}{\partial t} \quad (22)$$

Where λ_x and λ_y are the thermal conductivity of the soil in x and y directions

T is temperature

c is the soil mass specific heat

ρ is the density of the soil

The term $c\rho$ is referred to as the volumetric specific heat capacity of the soil.

One method for modifying Eq. (22) uses an apparent specific heat term. The apparent specific heat includes the volumetric specific heat capacity and a term that accounts for the heat released or absorbed by phase change. So, equation 2 can be as follows:

$$\frac{\partial}{\partial x}(\lambda_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda_y \frac{\partial T}{\partial y}) = (c\rho + L_f \frac{\partial \theta_u}{\partial T})\frac{\partial T}{\partial t} \quad (23)$$

L_f : Latent heat of fusion of water, $334\,MJ/m^3$.

θ: Volumetric water content at the initiation of freezing

$\partial\theta_u / \partial T$: Change in unfrozen water content of the soil with temperature

The tern $L_f\theta\partial\theta_u/\partial T$ represents the amount of heat released or absorbed as the temperature of the soil change by ∂T.

Eq. (4) reduce to Eq. (3) for cases where freezing or thawing are not occurring. The term $\partial\theta_u/\partial T$ has been referred to the m_2^i by Newman [5] in analogy to the m_2^w term used in seepage analysis in unsaturated soils. The flowchart of the CA approach for solution of elliptic PDE's is shown in Fig.3.

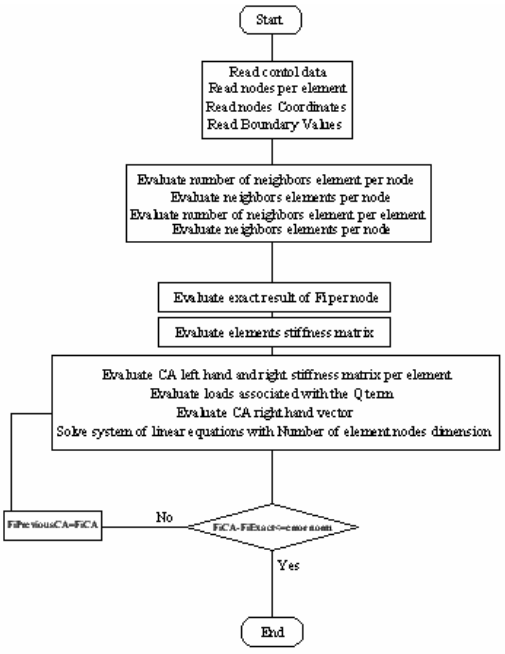


Fig. 3 Flowchart of the CA approach for solution of elliptic PDE's

IV. Results and Discussion

In this section a two dimensional numerical examples are solved using cellular automata to illustrate the efficiency and accuracy of the proposed method. The method's accuracy has been assessed by comparing the CA with analytical solution.

Consider the following equation,

$$2\frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial^2\phi}{\partial y^2} + 2.4 = 0\,, \qquad 0 \le x, y \le 1 \quad (24)$$

The boundary condition are :

$$\phi(x,y)\big|_{x=1} = 0 \qquad \frac{\partial\phi}{\partial x}\big|_{x=0} = 0$$
$$\phi(x,y)\big|_{y=1} = 0 \qquad \frac{\partial\phi}{\partial y}\big|_{y=0} = 0 \quad (25)$$

The boundary with trianglutation is shown in Fig. 4. Number of neighborhood nodes per nodes and number of neighborhood elements per element is calculated, then based on Galerkin descritization

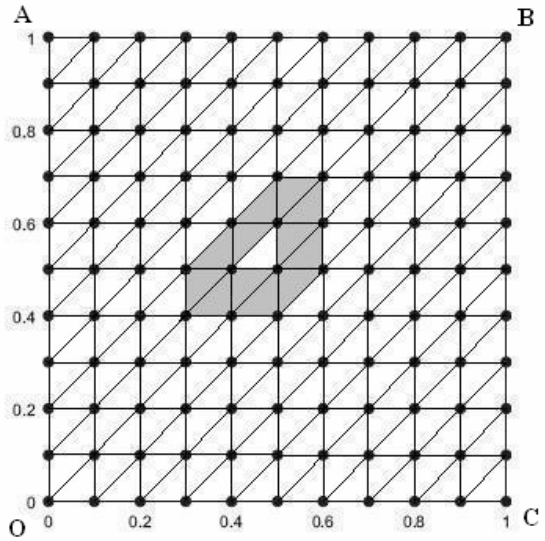
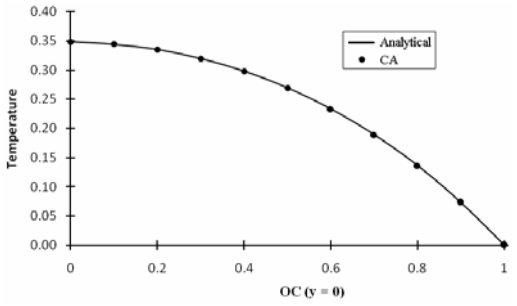
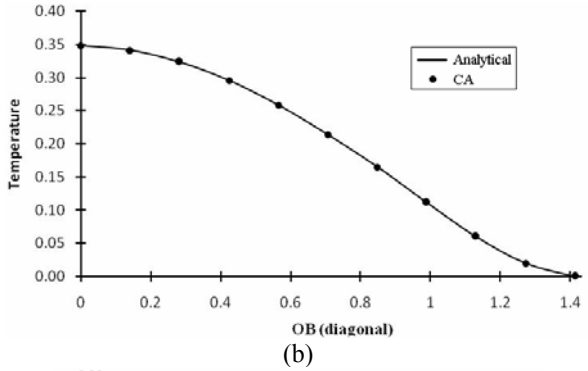


Fig. 4 Regular triangular elements ($\Delta x = \Delta y = 0.1$)

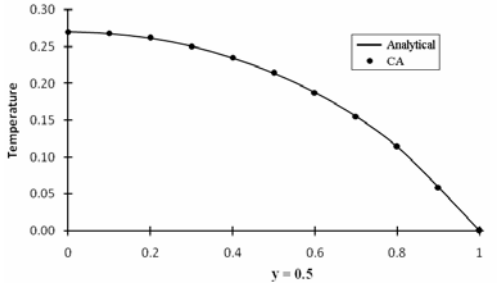
Figure 5 (a) , (b) and (c) shows temperature distribution in a square domain due to internal heat generation in bottom surface, diagonal line, and in y=0.5 respectively.



(a)



(b)



(c)

Fig. 5 Comparing CA and analytical solution (ϕ)
(a)OC (y = 0) (b) OB (diagonal) (c) y = 0.5
(Temperature distribution in a square domain due to internal heat generation)

V. Conclusions

Cellular Automata (CA) approach is presented in this paper for the solution of heat conduction problem. The method is based on the cell neighborhood. Results show that CA can be used for solution of heat conduction problem. The proposed method is shown to lead to symmetric matrices with small dimensions. The efficiency and accuracy of the method is tested against some two dimensional heat flow problem and the results are presented and compared with the exact solutions. In addition CA is faster than FEM for the cases considered while producing the same results as FEM. A topic for future work is to use rectangular element and compare results with the present paper results. More research is warranted to show the efficiency of the method for solution of other type of PDE's.

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