World Academy of Science, Engineering and Technology
International Journal of Physical and Mathematical Sciences
Vol:6, No:8, 2012

# Linear-Operator Formalism in the Analysis of Omega Planar Layered Waveguides

António L. Topa

***Abstract*—**A complete spectral representation for the electromagnetic field of planar multilayered waveguides inhomogeneously filled with omega media is presented. The problem of guided electromagnetic propagation is reduced to an eigenvalue equation related to a $2\times2$ matrix differential operator. Using the concept of adjoint waveguide, general bi-orthogonality relations for the hybrid modes (either from the discrete or from the continuous spectrum) are derived. For the special case of homogeneous layers the linear operator formalism is reduced to a simple $2\times2$ coupling matrix eigenvalue problem. Finally, as an example of application, the surface and the radiation modes of a grounded omega slab waveguide are analyzed.

***Keywords*—**Metamaterials, linear operators, omega media, layered waveguide, orthogonality relations



## I. Introduction

ELECTROMAGNETIC characteristics of new complex materials, namely or omega media, have granted considerable attention in the literature [1]-[3]. Although omega media are nonchiral, their properties are governed by constitutive relations similar to chiral media. While chiral artificial material consists of small wire helixes inserted into the host medium, the omega medium contains omega-shaped microstructures in which both the loop and the stamps lie in the same plane. Although the electric field in both the wire and the  elements induces not only electric but also magnetic polarization and vice versa, differences in the mutual placement of the polarization vectors are observed: in the wire element of the chiral medium, these interacting fields are parallel; in the omega microstructure they are perpendicular to each other. This distinctive feature of omega media implies that the orientation of the doping elements in the host isotropic medium cannot be random but must be parallel to a unique preferred direction. In fact, with a random distribution of conducting microstructures, the overall electro-magnetic coupling would result in a null average.

In this paper a linear-operator formalism for the analysis of inhomogeneous omega planar waveguides is presented. For these waveguides, using the theory of linear operators and through a suitable definition of a two-vector transverse mode function, the problem of guided electromagnetic wave propagation is reduced to an eigenvalue equation related to a $2\times2$ matrix differential operator. This mathematical framework is similar to the one developed by the authors for inhomogeneous chiral planar waveguides [4], [5].

A. L. Topa is Instituto de Telecomunicações and Instituto Superior Técnico, Technical University of Lisbon, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal (phone: 351-21 841 8479; fax: 351-21 841 8472; e-mail: antonio.topa@lx.it.pt).

Using the concept of adjoint waveguide, bi-orthogonality relations are derived for the hybrid modes. In order to have a complete field representation in open omega waveguides, these relations are of utmost importance when choosing an appropriate set of mutually orthogonal radiation modes. As an example of application, a general analysis of the surface and radiation modes of a grounded omega slab is also presented. Finally, one should note that this linear-operator formalism is applicable to multilayered planar waveguides with inhomogeneous layers. However no specific example of application is worked out for this general case. For the case of homogeneous layers the general formalism reduces to an algebraic $2\times2$ coupling matrix eigenvalue problem.

## II. Linear-Operator Formalism

The aim of this section is to reduce the problem of guided electromagnetic wave propagation in (open or closed) inhomogeneous omega waveguides to a linear-operator formalism. Based on the transverse electromagnetic field equations an eigenvalue problem is obtained. For each eigenvalue the corresponding eigenfunction represents a transverse mode function of the waveguide. Hence, the orthogonality properties of these eigenfunctions can be used to represent the electromagnetic field as a superposition of mode functions, as long as completeness is guaranteed.

In this section, the general layered grounded open waveguide depicted in Fig. 1 will be considered. It is uniform in the  direction and is inhomogeneously filled with spatially nondispersive lossless omega medium.

For bianisotropic omega media the constitutive relations may be written as

$$\mathbf{D} = \varepsilon_0 \left( \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E} + Z_0 \overline{\boldsymbol{\xi}} \cdot \mathbf{H} \right) \tag{1a}$$

$$\mathbf{B} = \mu_0 \left( Y_0 \overline{\boldsymbol{\zeta}} \cdot \mathbf{E} + \overline{\boldsymbol{\mu}} \cdot \mathbf{H} \right) \tag{1b}$$

with $Z_0 = Y_0^{-1} = k_0 / (\omega\varepsilon_0) = (\omega\mu_0)/k_0$, where $\overline{\boldsymbol{\varepsilon}}$ and $\overline{\boldsymbol{\mu}}$ are the relative dielectric permittivity and relative magnetic permeability dimensionless tensors, and $\overline{\boldsymbol{\xi}}$ and $\overline{\boldsymbol{\zeta}}$ are the magnetoelectric coupling dimensionless tensors. As the medium is considered spatially nondispersive these relations are local. The structure depicted in Fig. 1 is uniform and infinite in the $y$ direction (hence $\partial/\partial y \equiv 0$) and can be inhomogeneously filled with omega media. More precisely $\overline{\boldsymbol{\varepsilon}}(\omega,x)$, $\overline{\boldsymbol{\mu}}(\omega,x)$, $\overline{\boldsymbol{\xi}}(\omega,x)$ and $\overline{\boldsymbol{\zeta}}(\omega,x)$ are piece-wise-continuous functions of $x$ (*i.e.*, the general case of inhomogeneous layers is included).

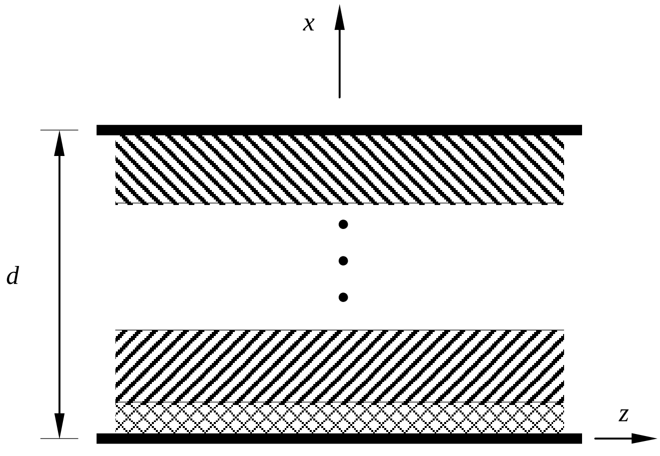


Fig. 1 Multilayered omega waveguide closed by electric and/or magnetic walls placed at $x=0$ and $x=d$

Introducing normalized distances marked with primes (*e.g.*, $x'=k_0x$, $y'=k_0y$, $z'=k_0z$) and a normalized magnetic field $\mathscr{H}$ such that

$$\mathscr{H}=Z_0\mathbf{H} \tag{2}$$

then, from Maxwell's curl equations for source free regions together with (1a) and (1b), one may write

$$-j\nabla'\times\mathscr{H}=\bar{\boldsymbol{\varepsilon}}\cdot\mathbf{E}+\bar{\boldsymbol{\xi}}\cdot\mathscr{H} \tag{3a}$$

$$j\nabla'\times\mathbf{E}=\bar{\boldsymbol{\zeta}}\cdot\mathbf{E}+\bar{\boldsymbol{\mu}}\cdot\mathscr{H} \tag{3b}$$

where time-harmonic field variation of the form $\exp(j\omega t)$ was assumed and $\nabla'=\nabla/k_0$. Considering forward plane wave propagation of the form $\exp(-j\beta z')$, where $\beta$ is the normalized longitudinal wavenumber given by

$$\beta=k/k_0 \tag{4}$$

one has

$$\nabla'=\partial_{x'}\hat{\mathbf{x}}-j\beta\hat{\mathbf{z}} \tag{5}$$

where $\partial_{x'}$ stands for $\partial/\partial x'$.

In this paper, only the case of $\Omega$-shaped perfectly conducting microstructures oriented as in Fig. 2 in a isotropic host material, will be considered. The normal to the planes of the loops points in the $x$ direction while the stamps are aligned along the $z$ direction, and the loops are oriented in the positive $y$ direction. Therefore tensors $\bar{\boldsymbol{\varepsilon}}$, $\bar{\boldsymbol{\mu}}$, $\bar{\boldsymbol{\xi}}$ and $\bar{\boldsymbol{\zeta}}$ have the following dyadic representation

$$\bar{\boldsymbol{\varepsilon}}=\varepsilon_{xx}\hat{\mathbf{x}}\hat{\mathbf{x}}+\varepsilon_{yy}\hat{\mathbf{y}}\hat{\mathbf{y}}+\varepsilon_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}} \tag{6a}$$

$$\bar{\boldsymbol{\mu}}=\mu_{xx}\hat{\mathbf{x}}\hat{\mathbf{x}}+\mu_{yy}\hat{\mathbf{y}}\hat{\mathbf{y}}+\mu_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}} \tag{6b}$$

$$\bar{\boldsymbol{\xi}}=j\Omega\hat{\mathbf{z}}\hat{\mathbf{x}} \tag{6c}$$

$$\bar{\boldsymbol{\zeta}}=-j\Omega\hat{\mathbf{x}}\hat{\mathbf{z}} \tag{6d}$$

where $\Omega$ is the dimensionless omega parameter which is positive. If the loops were oriented in the negative $y$ direction, one should have $\Omega<0$.

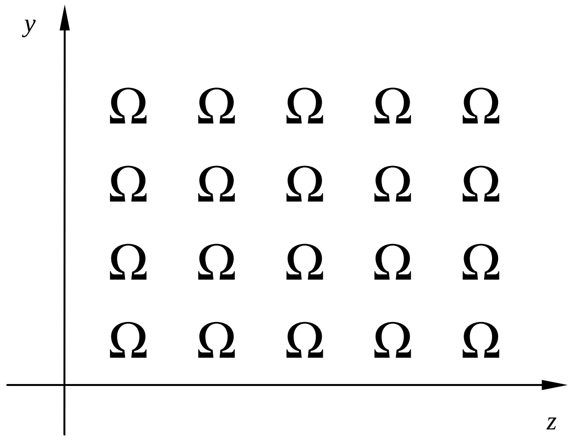


Fig. 2 Spatial orientation of planar, $\Omega$-shaped, conducting microstructures in the hosting isotropic material

From (6) one has $\bar{\boldsymbol{\xi}}=-\bar{\boldsymbol{\zeta}}^T$, which shows that the $\Omega$-medium is reciprocal. Moreover, when the medium is lossless the constitutive parameters are all real.

After substituting (6) into Maxwell's equations (3) and eliminating the components directed along $\boldsymbol{x}$, one obtains the following set of coupled partial differential equations

$$\partial_{x'}\mathbf{f}_t=-j\bar{\mathbf{C}}\cdot\mathbf{f}_t \tag{7}$$

where $\mathbf{f}_t$ is a column vector with the electric and magnetic field components tangential to the $yz$ plane

$$\mathbf{f}_t=\begin{bmatrix}E_y & \mathscr{H}_z & \mathscr{H}_y & E_z\end{bmatrix}^{\mathrm{T}}, \tag{8}$$

(T stands for transpose), whereas $\bar{\mathbf{C}}$ is a $4\times4$ coupling matrix given by

$$\bar{\mathbf{C}}=\begin{bmatrix}0 & \mu_{zz} & 0 & 0\\ \varepsilon_{yy}-\dfrac{\beta^2}{\mu_{xx}} & 0 & 0 & j\beta\dfrac{\Omega}{\mu_{xx}}\\ j\beta\dfrac{\Omega}{\mu_{xx}} & 0 & 0 & -\varepsilon_{zz}+\dfrac{\Omega^2}{\mu_{xx}}\\ 0 & 0 & -\mu_{yy}+\dfrac{\beta^2}{\varepsilon_{xx}} & 0\end{bmatrix} \tag{9}$$

The transverse field components may be algebraically expressed in terms of $\mathbf{f}_t$ as follows

$$\mathbf{f}_n=\bar{\mathbf{G}}\cdot\mathbf{f}_t \tag{10}$$

where

$$\mathbf{f}_n = \begin{bmatrix} E_x & \mathscr{H}_x \end{bmatrix}^{\mathrm{T}}. \tag{11}$$

In (10) $\bar{\mathbf{G}}$ is a $2 \times 4$ matrix given by

$$\bar{\mathbf{G}} = \begin{bmatrix} 0 & 0 & \dfrac{\beta}{\varepsilon_{xx}} & 0 \\ -\dfrac{\beta}{\mu_{xx}} & 0 & 0 & j\dfrac{\Omega}{\mu_{xx}} \end{bmatrix}. \tag{12}$$

One should note that, according to (8)-(12), only hybrid modes can propagate in the planar structure. Moreover one has

$$\mathrm{tr}\bar{\mathbf{C}} = 0 \tag{13a}$$

$$\mathrm{tr}\left(\mathrm{adj}\bar{\mathbf{C}}\right) = 0, \tag{13b}$$

and hence the eigenvalues of $\bar{\mathbf{C}}$ are anti-symmetric therefore allowing and (7) to be recast as a $2 \times 2$ matrix eigenvalue problem.

### *A. Eigenvalue Equation for Inhomogeneous Waveguides*

In order to recast the electromagnetic field equations in terms of a single eigenvalue equation, the following definition of a two-vector transverse mode function (or eigenfunction) is introduced:

$$\mathbf{\Phi} = \begin{bmatrix} E_z & \mathscr{H}_x \end{bmatrix}^{\mathrm{T}}. \tag{14}$$

Hence, from (7)-(12) one obtains the eigenvalue equation

$$\bar{\mathscr{L}} \cdot \mathbf{\Phi} = \beta^2 \bar{\mathscr{W}} \cdot \mathbf{\Phi} \tag{15}$$

where $\bar{\mathscr{L}}$ is a $2 \times 2$ matrix differential operator given by

$$\bar{\mathscr{L}} = \begin{bmatrix} \left(\partial_{x'} \dfrac{1}{\mu_{zz}} \partial_{x'} + \varepsilon_{yy}\right) \dfrac{\Omega}{j} & \left(\partial_{x'} \dfrac{1}{\mu_{zz}} \partial_{x'} + \varepsilon_{yy}\right) \mu_{xx} \\ \partial_{x'}^{2} + \mu_{yy}\varepsilon_{zz} & j\mu_{yy}\Omega \end{bmatrix} \tag{16}$$

and $\bar{\mathscr{W}}$ is the weight operator

$$\bar{\mathscr{W}} = \begin{bmatrix} 0 & 1 \\ \dfrac{\varepsilon_{zz}}{\varepsilon_{xx}} & j\dfrac{\Omega}{\varepsilon_{xx}} \end{bmatrix}. \tag{17}$$

Once the field components $E_z$ and $\mathscr{H}_x$ have been determined through (15), the remaining components can also be determined.

In everything that follows within this section, three classes of waveguides will be considered: (i) *closed* waveguides with electric and/or magnetic walls placed at $x' = 0$ and $x' = d'$; (ii) *open* waveguides extending from $x' = -\infty$ to $x' = +\infty$; (iii) *open grounded* waveguides extending from an electric or magnetic wall placed at $x' = 0$ to $x' = +\infty$. Hence, a finite, infinite or semi-infinite interval $I$ on $x'$ will be introduced as follows: (i) $I = [0, d']$ for closed waveguides,; (ii) $I = ]-\infty, +\infty[$ for open waveguides: (iii) $I = [0, +\infty[$ for open grounded waveguides. In order to define the domain $D$ of $\bar{\mathscr{L}}$, only surface modes will be considered for the two classes (ii) and (iii) of open waveguides. Consequently, $E_z$ and $\mathscr{H}_x$ always have finite energy and hence they belong to the vector space of square integrable functions over $I$. However , only for closed waveguides (*i.e.*, for regular problems corresponding to finite interval $I$), a complete spectral representation is possible within $D$.

### *B. Bi-Orthogonality Relation*

Introducing the following real type inner product

$$\left\langle \mathbf{u}, \mathbf{u}^a \right\rangle = \int_I \left(u_1 u_1{}^a + u_2 u_2{}^a\right) dx' \tag{18}$$

it is possible to determine the adjoint operators $\bar{\mathscr{L}}_a$ and $\bar{\mathscr{W}}_a$ of $\bar{\mathscr{L}}$ and $\bar{\mathscr{W}}$, respectively, with $\mathbf{\Phi}_a$ satisfying the *same* boundary conditions. In fact, making use of (18) with $\mathbf{u}_1 = [u_1, u_2]^T \in D$ and $\mathbf{u}_1{}^a = \left[u_1{}^a, u_2{}^a\right]^T \in D^a$, where $D^a$ denotes the domain of $\bar{\mathscr{L}}_a$, one can easily see that

$$\bar{\mathscr{L}}^a = \bar{\mathscr{L}}^T \tag{19a}$$

$$\bar{\mathscr{W}}^a = \bar{\mathscr{W}}^T \tag{19b}$$

according to (16) and (17).

At this point it is useful to introduce the concept of adjoint waveguide [6], as the one which has the same geometry and dimensions of the original waveguide, with identical boundaries, and satisfying to the following eigenvalue problem

$$\bar{\mathscr{L}}^a \cdot \mathbf{\Phi}^a = \beta_a{}^2 \bar{\mathscr{W}}_a \mathbf{\Phi}^a \tag{20}$$

where plane wave propagation of the form $\exp(-j\beta_a z)$ was considered. According to the fact that every eigenvalue $\beta^2$ of $\bar{\mathscr{L}}$ is an eigenvalue of $\bar{\mathscr{L}}^a$ [7], one can readily prove that

$$\left(\beta_m{}^2 - \beta_n{}^2\right)\left\langle \bar{\mathbf{W}} \cdot \mathbf{\Phi}_m, \mathbf{\Phi}_n^a \right\rangle = 0 \tag{21}$$

if $\mathbf{\Phi}_m \in D$ and $\mathbf{\Phi}_n^a \in D^a$. Hence, after a suitable normalization, the following bi-orthogonality relation holds

$$\left\langle \bar{\mathscr{W}} \cdot \mathbf{\Phi}_m, \mathbf{\Phi}_n^a \right\rangle = \delta_{mn} \tag{22}$$

where $\delta_{mn}$ is the Kronecker delta.

## III. Homogeneous Layers

For the special case of homogeneous layers, the linear operator formalism herein derived is reduced to a $2\times 2$ coupling matrix eigenvalue problem. In fact, for this case, one obtains from (15)-(17)

$$\partial_{x'}{}^{2}\mathbf{\Phi} = -\bar{\mathbf{R}}\cdot\mathbf{\Phi} \tag{23}$$

where

$$\bar{\mathbf{R}} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{24}$$

with

$$R_{11} = \varepsilon_{zz}\left(\mu_{yy} - \frac{\beta^2}{\varepsilon_{xx}}\right) \tag{25a}$$

$$R_{12} = j\Omega\left(\mu_{yy} - \frac{\beta^2}{\varepsilon_{xx}}\right) \tag{25b}$$

$$R_{21} = j\frac{\Omega\varepsilon_{zz}}{\mu_{xx}}\left(\mu_{yy} - \frac{\beta^2}{\varepsilon_{xx}}\right) - j\Omega\varepsilon_{yy}\frac{\mu_{zz}}{\mu_{xx}} \tag{25c}$$

$$R_{22} = \mu_{zz}\left(\varepsilon_{yy} - \frac{\beta^2}{\mu_{xx}}\right) - \frac{\Omega^2}{\mu_{xx}}\left(\mu_{yy} - \frac{\beta^2}{\varepsilon_{xx}}\right). \tag{25d}$$

Hence, in a similar way as shown in [4], one may write for each homogeneous omega layer:

$$\mathbf{\Phi}(x') = \bar{\mathbf{M}}\cdot\mathbf{\Psi}(x') \tag{26}$$

where

$$\bar{\mathbf{M}} = \begin{bmatrix} 1 & 1 \\ \tau_a & \tau_b \end{bmatrix} \tag{27}$$

is the modal matrix of $\bar{\mathbf{R}}$, such that

$$\partial_{x'}{}^{2}\mathbf{\Psi} = -\bar{\mathbf{\Lambda}}\cdot\mathbf{\Psi} \tag{28}$$

with $\mathbf{\Psi} = [\Psi_a \quad \Psi_b]^{\mathrm{T}}$ and $\bar{\mathbf{\Lambda}} = \mathrm{diag}\left(h_a{}^2, h_b{}^2\right)$. Therefore, one has

$$h_s{}^2 = \frac{(R_{11}+R_{22}) \pm \sqrt{(R_{11}-R_{22})^2 + 4R_{12}R_{21}}}{2} \tag{29}$$

and

$$\tau_s = \frac{h_s{}^2 - R_{11}}{R_{12}} = \frac{R_{21}}{h_s{}^2 - R_{22}} \tag{30}$$

with $s = a,\ b$.

## IV. Grounded Slab Waveguide

As an example of application of the previous formalism, the grounded omega slab waveguide depicted in Fig. 3 will be analyzed.

Since this waveguide is an *open structure* extending from the perfectly conducting plane at $x' = 0$ to $x' = +\infty$, the operator $\bar{\mathscr{L}}$ is defined over a semi-infinite interval, and has a discrete spectrum as well as a continuous spectrum. One should stress that, for the sake of completeness, the radiation modes must be included in the analysis. Nevertheless, the radiation modes do not actually belong to the domain of the operator: indeed they are improper eigenfunctions.

Assuming that the $\Omega$-shaped conducting microstructures have a spatial orientation in the slab as in Fig. 2, all the modes in the waveguide are hybrid.

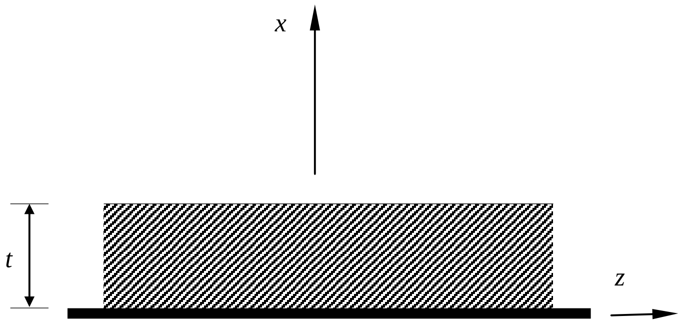


Fig. 3 Grounded omega slab waveguide. The slab of thickness $t$ is an isotropic medium with $\Omega$-shaped conducting microstructures with spatial orientation as in Fig. 2, while the upper medium is the air

According to (26), (27), (29) and (30), for $0 < x' < t'$, where $t'$ is the normalized thickness of the slab, one has,

$$E_z = \Psi_a + \Psi_b \tag{31a}$$

$$\mathscr{H}_x = \tau_a\Psi_a + \tau_b\Psi_b \tag{31b}$$

where

$$\Psi_a = A\left[\sin(h_a x') - Q\cos(h_a x')\right] \tag{32a}$$

$$\Psi_b = A\left[R\sin(h_b x') + Q\cos(h_b x')\right] \tag{32b}$$

which automatically guarantees that $E_z = 0$ for $x' = 0$. Imposing the other boundary condition at $x' = 0$, *i.e.*, $E_y = 0$, one obtains from (A1)

$$Q = 0. \tag{33}$$

In the air region, *i.e.*, for $x' > 0$, one gets

$$E_z = \alpha_1 A\left\{\cos\left[\rho(x'-t')\right] + B_1\sin\left[\rho(x'-t')\right]\right\} \tag{34a}$$

$$\mathscr{H}_x = \alpha_2 A\left\{\cos\left[\rho(x'-t')\right] + B_2\sin\left[\rho(x'-t')\right]\right\} \tag{34b}$$

with

$$\rho^2 = 1 - \beta^2, \tag{35}$$

and where $B_1$ and $B_2$ are arbitrary constants determined according to the type of modes to be considered. For example, when considering the surface hybrid modes one should make $B_1 = -j$ and $B_2 = -j$ while $\rho = -j\alpha$ with $\alpha = \sqrt{\beta^2 - 1}$ real for lossless media. In order to satisfy the radiation condition one must have $\alpha > 0$, *i.e.*, $\beta > 1$.

The coefficients $\alpha_1$ and $\alpha_2$ in (34) are evaluated by imposing the continuity of $E_y$ and $E_z$ at $x' = t'$. By enforcing the remaining boundary conditions, i.e., the continuity of $\mathscr{H}_y$ and $\mathscr{H}_z$ at $x' = t'$, the following linear system is obtained

$$\begin{bmatrix} \eta_a & \eta_b \\ \nu_a & \nu_b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ R \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (36)$$

where

$$\eta_s = \left(\mu_{xx}\tau_s - j\Omega\right)\left[\frac{h_s}{\mu_{zz}}\cos\left(h_s t'\right) - B_2 \rho \sin\left(h_s t'\right)\right] \qquad (37a)$$

$$\nu_s = \left(\mu_{yy} - \frac{\beta^2}{\varepsilon_{xx}}\right) B_1 \sin\left(h_s t'\right) - \rho h_s \cos\left(h_s t'\right) \qquad (37b)$$

with $s = a,\ b$. In order to have a determined system and obtain non trivial solutions, one has to ensure that

$$\eta_a \nu_b - \nu_a \eta_b = 0 . \qquad (38)$$

Furthermore, one also obtains from (36):

$$R = -\frac{\eta_a}{\eta_b} = -\frac{\nu_a}{\nu_b} . \qquad (39)$$

For the surface modes, since $B_1$ and $B_2$ are both defined, (38) becomes the modal equation of the omega waveguide of Fig. 3.

### *A. Surface Modes*

The surface modes, which constitute the discrete spectrum of the linear operator $\bar{\mathscr{L}}$ and define its domain as the set of eigenfunctions $\mathbf{\Phi} = \left[E_z \quad \mathscr{H}_x\right]^T$, such that $E_z$ and $\mathscr{H}_x$ are square integrable functions over $\left[0,+\infty\right[$, must satisfy to the radiation condition. Therefore, one must have $B_1 = -j$ and $B_2 = -j$ in (34), while $\rho = -j\alpha$ with $\alpha$ real and positive for lossless media. According to (35), all the surface modes are slow modes, *i.e.*, $\beta > 1$, and reach cutoff when $\alpha = 0$, *i.e.*, for $\beta = 1$.

In Fig. 4, the variation of $t/\lambda_c$ - where $\lambda_c$ denotes the cutoff wavelength - with $\Omega$ is presented. These curves are easily calculated by making $\alpha = 0$ in the modal equation (38). For numerical results the following values of the dimensionless constitutive parameters were considered: $\varepsilon_{xx} = 2$, $\varepsilon_{yy} = 3$, $\varepsilon_{zz} = 4$, $\mu_{xx} = 1$, $\mu_{yy} = 2$, and $\mu_{zz} = 3$. Hereafter, one will use the descriptor $H_p$ for each hybrid mode, where the subscript $p$, with $p \geq 0$, indicates the mode order, where all the modes are ordered after increasing cutoff frequencies. The fundamental mode $H_0$ (*i.e.*, the first propagating mode) has no cutoff, or $t/\lambda_c = 0$. For any value of $t'$ where $t' = 2\pi t/\lambda$, one easily obtains from Fig. 4 the number of propagating modes.

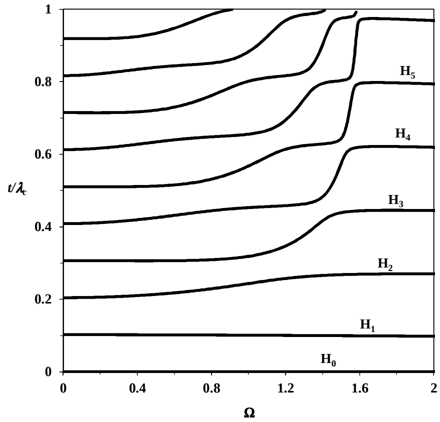


Fig. 4 Variation of $t/\lambda_c$, where $\lambda_c$ denotes the cutoff wavelength, with $\Omega$

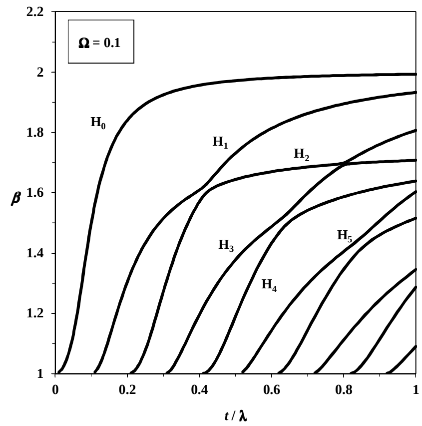


Fig. 5 Variation of $\beta$ with $t/\lambda$ for $\mathbf{\Omega} = 0.1$

In Fig. 5, the variation of $\beta$ - defined in (4) - with $t/\lambda$ is presented, for $\mathbf{\Omega} = 0.1$. In the high frequency regime, when $t/\lambda \to \infty$, there are two asymptotic values for $\beta$ - $\beta_a$ and $\beta_b$ - corresponding to $h_s \to 0$, with $s = a,\ b$. In both cases, when $h_s = 0$, one has $\det(\bar{\boldsymbol{R}}) = 0$ in (23). Nevertheless, for every mode, the dispersion curve converges at last to the highest of these two values. In the present numerical example, one has $\beta_b = \sqrt{\varepsilon_{xx}\mu_{yy}}$ when $h_b = 0$, while $\beta_a = \sqrt{\varepsilon_{yy}\left(\mu_{xx} - \Omega^2/\varepsilon_{zz}\right)}$ when $h_a = 0$.

Since $\varepsilon_{xx}/\varepsilon_{yy} > \mu_{xx}/\mu_{yy}$, one has always $\beta_b > \beta_a$. Therefore, for all the hybrid modes $1 < \beta < \beta_b$, inasmuch as when $t/\lambda \to \infty$ all the dispersion curves converge to $\sqrt{\varepsilon_{xx}\mu_{yy}}$. When $\Omega \to 0$ all the hybrid modes degenerate into the TE and TM surface modes of the biaxial anisotropic case, with the dispersion curves crossing each other instead of displaying coupling points.

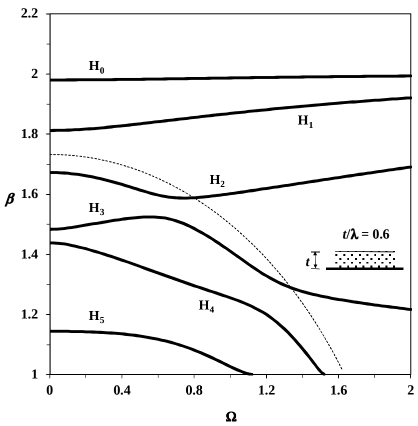


Fig. 6 Variation of $\beta$ with the dimensionless omega parameter $\Omega$

Finally Fig. 6 shows the variation of $\beta$ with the dimensionless omega parameter $\Omega$ for all propagating modes when $t/\lambda = 0.6$. One should stress that mode $H_6$ reaches cutoff while the dashed curve corresponds to $\beta = \beta_a$.

*B. Radiation Modes*

The set of modes described in section A is sufficient to describe any guided field distribution in the slab waveguide provided that there is not any variation along $z$ direction. However, this set is not sufficient to describe the radiation phenomena. For a complete spectral representation the analysis must include an infinite number of radiation modes. The fields of the radiation modes do not decay in the outside of the structure, *i.e.*, they are not bound to the slab, which means that they need not to obey to the radiation condition. Unlike the guided modes each individual radiation modes carries an infinite amount of energy. Therefore the bi-orthogonality relation (22) for these modes must involve the Dirac delta function [8]

$$\left\langle \overline{\mathscr{W}} \cdot \mathbf{\Phi}(x',\rho), \mathbf{\Phi}^a(x',\rho') \right\rangle = \delta(\rho - \rho') \qquad (40)$$

and can be used for the normalization of the radiation modes. According to (34) there are two arbitrary constants $B_1$ and $B_2$ to be chosen, in order to have a complete set of orthogonal radiation modes. This unique degree of freedom shows that only two types of radiation modes need to be considered for a complete spectral representation. One possible choice is the ITE (*Incident Transverse Electric*) and ITM (*Incident Transverse Magnetic*) continuous radiation modes, which can be proved to obey to the bi-orthogonality relation (40).

## References

[1] N. Engheta and M. M. I. Sadaoun, “Novel pseudochiral or $\Omega$-medium and its applications,” *Proc. Conf. PIERS’91*, Cambridge, MA, p. 339, July 1991.
[2] A. Priou (Ed.), *Bianisotropic and Bi-Isotropic Media and Applications*, EMW Publishing, Cambridge, 1994 - Ch. 10: A. Toscano and L. Vegni, “Electromagnetic waves in planar pseudochiral structures,” pp. 181-218.
[3] J. Mazur and D. Pietrzak, “Field displacement phenomenon in a rectangular waveguide containing a thin plate of a $\Omega$-medium,” *IEEE Microwave Guided Lett.*, vol. 6, pp. 34-36, Jan. 1996.
[4] C. R. Paiva and A. M. Barbosa, “A linear-operator formalism for the analysis of inhomogeneous biisotropic planar waveguides,” *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp.672-678, Apr. 1992.
[5] A. L. Topa, C. R. Paiva, and A. M. Barbosa, “New biorthogonality relations for inhomogeneous biisotropic planar waveguides,” *IEEE Trans. Microwave Theory Tech*., vol. 42, pp. 629-634, Apr. 1994.
[6] A. D. Bresler, G. H. Joshi, and N. Marcuvitz, “Orthogonality properties for modes in passive and active uniform waveguides,” *J. Appl. Phys.*, vol. 29, pp. 794-799, May 1958.
[7] B. Friedman, *Principles and Techniques of Applied Mathematics*. New York: Wiley, 1956.
[8] A. L. Topa, C. R. Paiva, and A. M. Barbosa, "NRD directional couplers using omega media," in *Proc. NATO Advanced Research Workshop: Metamaterials for Secure Information and Communication Technologies*, Cairo, Egypt, February 2010.