

# Order Reduction using Modified Pole Clustering and Pade Approximations

C.B. Vishwakarma

**Abstract**—The authors present a mixed method for reducing the order of the large-scale dynamic systems. In this method, the denominator polynomial of the reduced order model is obtained by using the modified pole clustering technique while the coefficients of the numerator are obtained by Pade approximations. This method is conceptually simple and always generates stable reduced models if the original high-order system is stable. The proposed method is illustrated with the help of the numerical examples taken from the literature.

**Keywords**—Modified pole clustering, Order reduction, Pade approximation, Stability, Transfer function

## I. INTRODUCTION

THE approximation of linear systems plays an important role in many engineering applications, especially in control system design, where the engineer is faced with controlling a physical system for which an analytic model is represented as a high order linear system. In many practical situations a fairly complex and high order system is not only tedious but also not cost effective for on-line implementation. It is therefore desirable that a high system be replaced by a low order system such that it retains the main qualitative properties of the original system.

Several order reduction techniques for linear dynamic systems in the frequency domain are available in the literature [1-4]. Further, some methods have also been suggested by combining the features of two different methods [5-7].

In the clustering technique suggested by Sinha and Pal [8], the poles and zeroes are separately grouped to form clusters and then these clusters are replaced by their cluster centres by using an inverse distance measure (IDM) criterion. In the literature [9], only poles are grouped together to generate cluster centres and then denominator polynomial of the reduced model is synthesized from these cluster centres. In the proposed method, a modified pole clustering technique is suggested, which generates the more effective cluster centres than cluster centres obtained by authors [8, 9]. If a cluster contains  $r$  number of poles, then IDM criterion is repeated  $r$  times with the most dominant pole available in that cluster. The Pade approximation method was originally introduced by Pade [10]. This method is computationally simple and fits initial time moments and matches the steady state values. The disadvantage of this

method is that the reduced model may be unstable even though the original system is stable. To overcome this problem, Shamash [11-12] introduced a method of reduction based on retention of poles of the high order system in the reduced model and Pade approximation about more than one point. In the proposed method, the denominator polynomial of the reduced order model is determined by using modified pole clustering technique while the coefficients of the numerator are obtained by Pade approximations. The proposed method is compared with the other well-known order reduction techniques available in the literature.

## II. STATEMENT OF THE PROBLEM

Let the transfer function of high order original system of the order '  $n$  ' be

$$G_n(s) = \frac{N(s)}{D(s)} = \frac{e_0 + e_1s + e_2s^2 + \dots + e_{n-1}s^{n-1}}{f_0 + f_1s + f_2s^2 + \dots + f_ns^n} \quad (1)$$

where  $e_i; 0 \leq i \leq n-1$  and  $f_i; 0 \leq i \leq n$  are known scalar constants.

Let the transfer function of the reduced model of the order '  $k$  ' be

$$R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{a_0 + a_1s + a_2s^2 + \dots + a_{k-1}s^{k-1}}{b_0 + b_1s + b_2s^2 + \dots + b_ks^k} \quad (2)$$

where  $a_i; 0 \leq i \leq k-1$  and  $b_i; 0 \leq i \leq k$  are unknown scalar constants.

The objective of this paper is to realize the  $k^{\text{th}}$ -order reduced model in the form of (2) from the original system (1) such that it retains the important features of the original high-order system.

## III. REDUCTION METHOD

The reduction procedure for getting the  $k^{\text{th}}$ -order reduced models consist of the following two steps:

*Step-1:* Determination of the denominator polynomial for the  $k^{\text{th}}$ -order reduced model using modified pole clustering:

The following rules are used for clustering the poles of the original system given in frequency domain:

- Separate clusters should be made for real and complex poles.
- Clusters of the poles in the left half  $s$ -plane should not contain any pole of the right half  $s$ -plane and vice-versa.

Dr. C.B. Vishwakarma is a Associate Professor in the Department of Electrical Engineering, Galgotias College of Engineering & Technology, Greater Noida, India. (E-mail: cbv\_gkv@yahoo.co.in.)

- Poles on the  $j\omega$ -axis have to be retained in the reduced order model.
- Poles at the origin have to be retained in the reduced order model.

The brief algorithm for realizing the denominator polynomial by using the modified pole clustering is as follows:

Let there be  $r$  real poles in  $i^{th}$  cluster are  $p_1, p_2, \dots, p_r$ , where  $|p_1| < |p_2| < \dots < |p_r|$ , and then modified cluster centre  $p_{ei}$  can be obtained by using the algorithm of modified pole clustering suggested in this paper.

Let  $m$  pair of complex conjugate poles in the  $j^{th}$  cluster be  $[(\alpha_1 \pm j\beta_1), (\alpha_2 \pm j\beta_2), \dots, (\alpha_m \pm j\beta_m)]$  where  $|\alpha_1| < |\alpha_2| < \dots < |\alpha_m|$ .

Now using the same algorithm separately for real and imaginary parts of the complex conjugate poles, the modified cluster centre is obtained, which is written as  $\Phi_{ej} = A_{ej} \pm jB_{ej}$ ,

where  $\Phi_{ej}^* = A_{ej} + jB_{ej}$  and  $\Phi_{ej}^\circ = A_{ej} - jB_{ej}$ .

An interactive computer oriented algorithm [13] has been developed, which automatically finds the modified cluster centre and is give as follows:

**Step-1** Let  $r$  real poles in a cluster be

$$|p_1| < |p_2| < \dots < |p_r|$$

**Step-2** Set  $j = 1$ .

**Step-3** Find pole cluster centre  $c_j = \left[ \sum_{i=1}^r \left( \frac{-1}{|p_i|} \right) \div r \right]^{-1}$ .

**Step-4** Set  $j = j + 1$ .

**Step-5** Now find a modified cluster centre from

$$c_j = \left[ \left( \frac{-1}{|p_1|} + \frac{-1}{|c_{j-1}|} \right) \div 2 \right]^{-1}$$

**Step-6** Is  $r = j?$ , if No, and then go to *step-4* otherwise go to *step-7*.

**Step-7** Take a modified cluster centre of the  $k^{th}$ -cluster as  $p_{ek} = c_j$ .

For synthesizing the  $k^{th}$ -order denominator polynomial, one of the following cases may occur.

**Case-1:** If all modified cluster centres are real, then denominator polynomial of the  $k^{th}$ -order reduced model can be obtained as

$$D_k(s) = (s - p_{e1})(s - p_{e2}) \dots (s - p_{ek}) \quad (3)$$

where  $p_{e1}, p_{e2}, \dots, p_{ek}$  are  $1^{st}, 2^{nd}, \dots, k^{th}$  modified cluster centre respectively.

**Case-2:** If all modified cluster centers are complex conjugate then the  $k^{th}$ -order denominator polynomial is taken as

$$D_k(s) = (s - \Phi_{e1}^*)(s - \Phi_{e1}^\circ) \dots (s - \Phi_{ek/2}^*)(s - \Phi_{ek/2}^\circ) \quad (4)$$

**Case-3:** If some cluster centers are real and some are complex conjugate. For example,  $(k-2)$  cluster centers are real and one pair of cluster centre is complex conjugate, then  $k^{th}$ -order denominator can be obtained as

$$D_k(s) = (s - p_{e1})(s - p_{e2}) \dots (s - p_{e(k-2)})(s - \Phi_{e1}^*)(s - \Phi_{e1}^\circ) \quad (5)$$

hence, the denominator polynomial  $D_k(s)$  is obtained as

$$D_k(s) = b_0 + b_1s + \dots + b_k s^k \quad (6)$$

**Step-2:** Determination of the numerator of the reduced model by Pade approximations:

The original  $n^{th}$ -order system can be expanded in power series about  $s = 0$  as

$$G_n(s) = \frac{\sum_{i=0}^{n-1} e_i s^i}{\sum_{i=0}^n f_i s^i} = c_0 + c_1s + c_2s^2 + \dots \quad (7)$$

The coefficients of the power series expansion can also be calculated as follows:

$$\left. \begin{aligned} c_0 &= \frac{e_0}{f_0} \\ c_i &= \frac{1}{f_0} \left[ e_i - \sum_{j=1}^i f_j c_{i-j} \right], \quad i > 0 \\ e_i &= 0, \quad i > n-1 \end{aligned} \right\} \quad (8)$$

The  $k^{th}$ -order reduced model is taken as

$$R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} a_i s^i}{\sum_{i=0}^k b_i s^i} \quad (9)$$

here,  $D_k(s)$  is known through eqns (3-5)

For  $R_k(s)$  of eqn (9) to be Pade approximants of  $G_n(s)$  of equation (8), we have

$$\left. \begin{aligned} a_0 &= b_0 c_0 \\ a_1 &= b_0 c_1 + b_1 c_0 \\ &\dots\dots\dots \\ &\dots\dots\dots \\ a_{k-1} &= b_0 c_{k-1} + b_1 c_{k-2} + \dots\dots + b_{k-2} c_1 + b_{k-1} c_0 \end{aligned} \right\} \quad (10)$$

the coefficients  $a_j$ ;  $j = 0, 1, 2, \dots, k-1$  can be found by solving the above  $k$  linear equations.

hence, the numerator  $N_k(s)$  is obtained as

$$N_k(s) = a_0 + a_1 s + \dots\dots + a_{k-1} s^{k-1} \quad (11)$$

#### IV. NUMERICAL EXAMPLES

Two numerical examples are taken from the literature to illustrate the algorithm of the proposed method. The first example is solved in details to get second order reduced model while only the results are given in the second example. An integral square error (ISE) [1] in between the transient parts of the original and reduced model is calculated using Matlab to measure the goodness of the reduced order model i.e. lower the ISE, closer the  $R_k(s)$  to  $G_n(s)$ , which is given by

$$ISE = \int_0^{\infty} [y(t) - y_k(t)]^2 dt \quad (11)$$

where,  $y(t)$  and  $y_k(t)$  are the unit step responses of original and reduced system respectively.

*Example-1:* Consider an eight-order system from the literature [5].

$$G_8(s) = \frac{N(s)}{D(s)}, \quad \text{where}$$

$$N(s) = 40320 + 185760s + 222088s^2 + 122664s^3 + 36380s^4 + 5982s^5 + 514s^6 + 18s^7$$

and

$$D(s) = 40320 + 109584s + 118124s^2 + 67284s^3 + 22449s^4 + 4536s^5 + 546s^6 + 36s^7 + s^8$$

The poles are: -1, -2, -3, -4, -5, -6, -7, -8.

Let the 2<sup>nd</sup>-order reduced model is required to be realized, for this purpose only two real clusters are required.

Let the first and second cluster consists the poles (-1, -2, -3, -4) and (-5, -6, -7, -8) respectively. The modified cluster centres are computed as

$$P_{e1} = -1.06371$$

$$P_{e2} = -5.13271$$

Using eqn (3), the denominator polynomial  $D_2(s)$  is obtained as

$$D_2(s) = 5.45971 + 6.19642s + s^2$$

using the eqns (8) and (10), the following coefficients are calculated

$$c_0 = 1 \quad a_0 = 5.45971$$

$$c_1 = 1.8893 \quad a_1 = 16.51145$$

Therefore, finally 2<sup>nd</sup>-order reduced model is obtained as

$$R_2(s) = \frac{5.45971 + 16.51145s}{5.45971 + 6.19642s + s^2}$$

The step responses of the reduced order model and the original system are compared in the Fig.1 and the bode plot comparison is shown in the Fig. 2.

The proposed method is compared with the well-known reduction methods and shown in the Table-1, from which it is concluded that this method is comparable in quality.

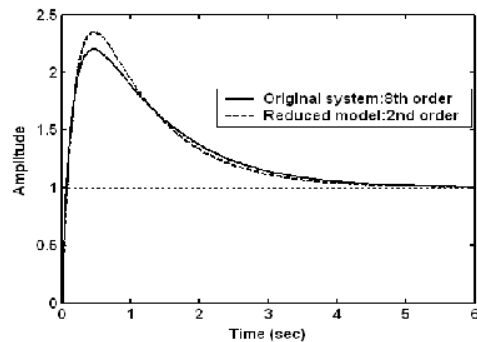


Fig.1 Comparison of the step responses for example 1

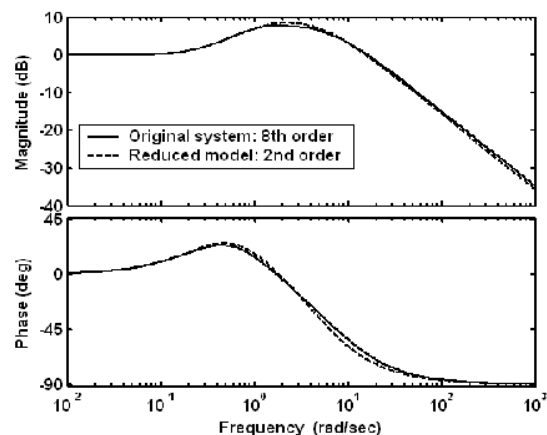


Fig. 2 Comparison of the bode plots for example 1

TABLE I  
COMPARISON OF THE REDUCTION METHODS

Reduction methods	Reduced model	ISE
Proposed method	$R_2(s) = \frac{5.45971 + 16.51145s}{5.45971 + 6.19642s + s^2}$	$1.406 \times 10^{-2}$
G. Parmar et al. [5]	$R_2(s) = \frac{8 + 24.11429s}{8 + 9s + s^2}$	$4.809 \times 10^{-2}$
Mittal et al. [14]	$R_2(s) = \frac{1.9906 + 7.0908s}{2 + 3s + s^2}$	$2.689 \times 10^{-1}$
Mukherjee et al. [15]	$R_2(s) = \frac{4.4357 + 11.3909s}{4.4357 + 4.2122s + s^2}$	$5.6897 \times 10^{-2}$
Prasad and Pal [6]	$R_2(s) = \frac{500 + 17.98561s}{500 + 13.24571s + s^2}$	1.4584

Example-2: Consider a 7<sup>th</sup> –order system taken from the literature [16].

$$G_7(s) = \frac{37500s + 31248.75}{s^7 + 83.64s^6 + 4097s^5 + 70342s^4 + 853703s^3 + 2814271s^2 + 3310875s + 281250}$$

The poles are : -0.0919, -2.0244 ± j0.9646 ,

-7.6744 ± j13.4462 and -32.0753 ± j38.8493 .

The minimum 3<sup>rd</sup> -order reduced model is possible in this case.

The following two cluster centres are computed as

$$p_{e1} = -0.0919$$

$$\Phi_{e1} = -2.3524 \pm j1.1464$$

Finally, the 3<sup>rd</sup> –order reduced model is obtained as

$$R_3(s) = \frac{0.06992 + 0.82487s - 0.16997s^2}{0.62933 + 7.28037s + 4.7967s^2 + s^3}$$

The step response comparison with the original system  $G_7(s)$  is shown in the Fig.3 and the qualitative properties comparison is shown in the Table-2, from which it may be concluded that the proposed method is better in quality and provides good approximation in the transient as well as steady-state gain.

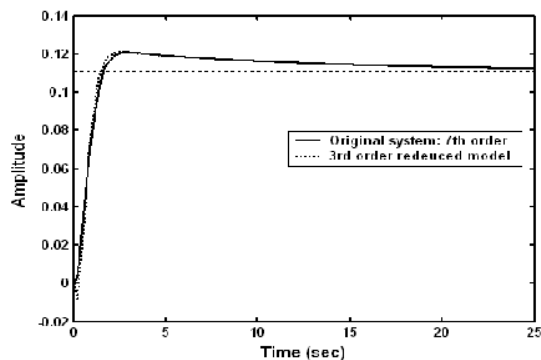


Fig.3 Comparison of the step responses for example 2

TABLE II  
QUALITATIVE COMPARISON WITH THE ORIGINAL SYSTEM

System	Rise Time $t_r$ (sec.)	Peak Overshoot $M_p$ (%)	Settling Time $t_s$ (sec.)
7 <sup>th</sup> Order [16]	1.05	8.61	18.8
3 <sup>rd</sup> Order	0.847	8.90	17.9

## V. CONCLUSIONS

The authors presented a mixed method for reducing the order of the large-scale single-input-single-output systems. In this method, the denominator polynomial is determined by using the modified pole clustering while the coefficients of the numerator are obtained by Pade approximations. This method has been tested on the two numerical examples chosen from the literature and time and frequency responses of the original and reduced system are compared graphically and shown in the Fig.1, 2 and 3 and error (ISE) and qualitative properties comparison are shown in the Table-1 and Table-2 respectively. From these comparisons, it has been concluded that the proposed method is simple, computer oriented and comparable in quality. This method guarantees stability of the reduced model if the original high-order system is stable and exactly matches the steady-state value of the original system. The proposed method is equally applicable to linear multivariable systems and is reported elsewhere.

## REFERENCES

- [1] V. Singh, D. Chandra and H. Kar, "Improved Routh Pade approximants: A Computer aided approach", IEEE Trans. Autom. Control, 49(2), pp.292-296, 2004.
- [2] S.Mukherjee and R.N. Mishra, "Reduced order modeling of linear multivariable systems using an error minimization technique", Journal of Franklin Inst., 325 (2), pp.235-245, 1988.
- [3] Sastry G.V.K.R Raja Rao G. and Mallikarjuna Rao P., "Large scale interval system modeling using Routh approximants", Electronic Letters, 36(8), pp.768-769, 2000.
- [4] R. Prasad, "Pade type model order reduction for multivariable systems using Routh approximation", Computers and Electrical Engineering, 26, pp.445-459, 2000.
- [5] G. Parmar, S. Mukherjee and R. Prasad, "System reduction using factor division algorithm and eigen spectrum analysis", Applied Mathematical Modelling, Elsevier, 31, pp.2542-2552, 2007.
- [6] R. Prasad and J. Pal, "Stable reduction of linear systems by continued fractions", Journal of Institution of Engineers IE(I) Journal-EL, 72, pp.113-116, 1991.
- [7] R. Prasad, S.P. Sharma and A.K. Mittal, "Improved Pade approximation for multivariable systems using stability equation method", Journal of Institution of Engineers IE (I) Journal-EL, 84, pp.161-165, 2003.
- [8] A.K. Sinha, J. Pal, Simulation based reduced order modeling using a clustering technique, Computer and Electrical Engg., 16(3), 1990, 159-169.
- [9] J. Pal, A.K. Sinha and N.K. Sinha, Reduced-order modelling using pole clustering and time-moments matching, Journal of The Institution of Engineers (India), Pt El, 76, 1995, 1-6.
- [10] Pade H, "Sur La representation approaches dune fonction par des fraction vationnellers", 9, pp.1-32, 1892.
- [11] Shamash Y, "Stable reduced order models using Pade type approximants", IEEE Trans. Autom. Control, Vol.AC-19, No.5, pp.615-616, October 1974.
- [12] Shamash Y, "Linear system reduction using Pade approximation to allow retention of dominant modes", Int. J. Control, Vol.21, No. 2, pp.257-272, 1975.

- [13] C.B. Vishwakarma & R. Prasad, "MIMO system reduction using modified pole clustering and Genetic algorithm", *Modelling and Simulation in Engineering*, Hindawi Pub. Corp., USA, Volume 2009, pp. 1-6, 2009.
- [14] A.K Mittal, R. Prasad, S.P Sharma, "Reduction of linear dynamic systems using an error minimization technique", *Journal of Institution of Engineers IE(I) Journal-EL*, Vol.84, pp.201-206, March 2004.
- [15] S. Mukherjee, Satakshi, R.C Mittal, "Model order reduction using response matching", *Journal of Franklin Inst.*, Vol.342, pp.503-519, 2005.
- [16] Bistritz Y. and Langholz G., "Model reduction by Chebyshev polynomial techniques", *IEEE Trans. Automatic Control*, Vol-AC-25, No.5, pp. 741-747.