Robust Camera Calibration using Discrete Optimization

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Abstract—Camera calibration is an indispensable step for augmented reality or image guided applications where quantitative information should be derived from the images. Usually, a camera calibration is obtained by taking images of a special calibration object and extracting the image coordinates of projected calibration marks enabling the calculation of the projection from the 3d world coordinates to the 2d image coordinates. Thus such a procedure exhibits typical steps, including feature point localization in the acquired images, camera model fitting, correction of distortion introduced by the optics and finally an optimization of the model’s parameters. In this paper we propose to extend this list by further step concerning the identification of the optimal subset of images yielding the smallest overall calibration error. For this, we present a Monte Carlo based algorithm along with a deterministic extension that automatically determines the images yielding an optimal calibration. Finally, we present results proving that the calibration can be significantly improved by automated image selection.

Keywords—Camera Calibration, Discrete Optimization, Monte Carlo Method.

I. INTRODUCTION

Generically, calibration is the problem of estimating values for the unknown parameters in a sensor model in order to determine the exact mapping between sensor input and output. A wide range of computer vision applications exist, which require an accurate calibration of the visual system. In these applications, certain quantitative information is extracted from the 2d images and overall performance depends on calibration accuracy [5]. In the context of three-dimensional machine vision, the sensor is represented by the camera including its optics. Hence, calibration is the process to determine the internal camera geometric and optical characteristics and/or the 3d position and orientation of the camera frame relative to a certain world coordinate system [9].

Camera calibration is usually performed by observing a special calibration object, which in most cases is a flat plate with a regular pattern marked on it using colors causing a high contrast between the marks and the background. The pattern is chosen such that the image coordinates of the projected reference points can be measured with high accuracy. Once the relationship between the 2d image coordinates and 3d world coordinates is known, the perspective transformation from the 3d world coordinates is known, the perspective transformation from the 3d world coordinates to a certain world coordinate system [9].

Camera calibration has been studied intensively in the past years, starting in the photogrammetry community [1] and more recently in computer vision [9], [8], [10], [3], [4]. According to Heikkilä and Silvén [3], there are four main problems when designing a whole calibration procedure: control point localization in the images, camera model fitting, image correction for radial and tangential distortion and estimating the errors originated in these stages. Most of the research has been devoted to model fitting and only few works can be found in the literature about the other stages of the process such as feature point localization, cf. [5]. Additionally, the literature neglects the problem of image selection, that is to determine the images that are likely to result in small model fit errors. However, this is an important topic since ill-posed configurations or poor image quality can negatively influence the calibration procedure and thus lead to significant errors. Hence, we suggest to extend Heikkilä and Silvén’s list by another task concerning the identification of the images that yield the best calibration result. In this scope, the work being closest to ours is that of Ouellet et. al. [6] who address the problem of predicting the quality of calibration images by analyzing their feature points. Ouellet et. al. present an acutance-based quality measure [7] for circular calibration marks that can quickly indicate static or motion blur in the images. This in combination with an interactive assistant tool for geometric camera calibration eliminates the needs to carefully examine each of the images and thus facilitate the calibration process [6].

However, the proposal of Ouellet et. al. focusses on user interaction and is not able to predict degenerated configurations that typically result in high errors. Therefore, we suggest to apply a stochastic and deterministic optimization technique in order to automatically determine the optimal subset of the pool of acquired images yielding the best calibration result with respect to the model fit error.

III. CALIBRATION

The characteristics of the imaging system are determined by the well-known calibration technique by Zhang [10] modelling the relationship between the 2d pixel coordinates and 3d world coordinates of the camera and the corresponding point on the object surface.
coordinates by a projection matrix \( P \), which maps points from the projection space \( P^3 \) to the projective plane \( P^2 \):
\[
P = \lambda_w \begin{pmatrix}
\alpha_w & 0 & u_0 & 1 \\
0 & \alpha_w & v_0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
R_w \\
\mathbf{t}_w \\
D_w
\end{pmatrix}
\]

The 3 \times 3 matrix \( A \), whose four entries are called intrinsic parameters models the imaging process whereas the 4 \times 4 displacement matrix \( D_w \) describes the external orientation of the camera (extrinsic parameters).

The Zhang method requires \( n \geq 2 \) shots each containing the image of the calibration pattern with \( m \) feature points on it. Each feature point represents a mapping from the 3d plane with the projection matrix \( P \) to the 2d image plane and yields an equation in a linear equation system that is solved for the components of \( P \). These camera model’s parameters are then adjusted within a non-linear optimization procedure incorporating a correction of radial and tangential lens distortion. As a merit function for the optimization the mean \( \epsilon \) of all projection errors \( \epsilon_{ij} \) is considered and its minimization is pursued yielding an improvement of the overall model fitting quality:
\[
\epsilon = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_{ij} = \left\| \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} - P \begin{pmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{pmatrix} \right\|_2
\]

The projection error of a single calibration feature \( \epsilon_{ij} \) is defined by the Euclidean distance between its initially extracted image coordinates \( (u_{ij}, v_{ij}) \) and the corresponding 3d world coordinates \( (X_{ij}, Y_{ij}, Z_{ij}) \) being projected to the image plane with the projection matrix \( P \) acquired by the calibration procedure. The mean \( \epsilon \) of all these errors yields an appropriate measure for the quality of the calibration [10].

### IV. SUBSET DETERMINATION

In everyday calibration work, usually a set of \( n \) input images \( I = \{\iota_1, \iota_2, \ldots, \iota_n\} \) is considered for calibration whereas some of the acquired images may originate from ill-posed configurations. Typically, these images are seldomly known beforehand, so that neither considering all the \( n \) images nor a human-made subset selection will in general yield the best calibration result.

In the following we address this problem and apply the mean projection error in order to determine a subset of images that yields a minimal overall projection error with respect to the whole image set. We present a Monte Carlo based method and a deterministic extension in order to identify the optimal subset \( I_{opt} \).

Performing an exhaustive search in order to determine the global optimum is not feasible since the parameter space \( P_I \) is very large as it consists of all \( N_{ex} \) possible combinations of the \( n \) input images with at least two images and hence is
\[
N_{ex} = |P_I| = |\{X : X \subseteq I \land 1 < |X|\}| = 2^l - |\{\emptyset\}| - |I| = 2^n - n = 2^n - (n + 1).
\]

Due to the fact that the calibration procedure requires at least two images (cf. section III), the power set \( 2^I \) of the image set \( I \) needs to be degraded by the empty set and all the singletons.

For an example, let us assume a set of \( n = 20 \) calibration images and an estimated average calibration time of \( t = 0.5s \) per image. Then the expected time \( t \) for the determination of the global maximum by an exhaustive search will take \( t = (2^{20} - 21) \cdot 0.5s = 524277.5s \) that is approximately 146 hours or 6 days, and thus will be impracticable for every day use.

For the remainder, we assume the \( n \) elements of the calibration image set \( I \) being partially ordered by an arbitrary relation. We identify an element at position \( i \) (the \( i \)-th image \( \iota_i \)) with the \( i \)-th unit vector
\[
\{\iota_i\} \in I \implies \mathbf{e}_i = \begin{pmatrix} 0 & \ldots & 1 & \ldots & 0 \end{pmatrix}^T
\]

and model a certain subset by the coordinate vector \( s = (s_1 \ldots s_n)^T, s_j \in [0,1] \), i.e.
\[
s = (0 1 \ldots 0 1)^T = 0 \mathbf{e}_0 + 1 \mathbf{e}_1 + \ldots + 0 \mathbf{e}_k + \ldots + 1 \mathbf{e}_n.
\]

Here, \( s_j = 1 \) denotes the containedness of the \( j \)-th image in the corresponding subset. With this modelling, the subset selection is equivalent to the optimization problem
\[
\min_s \bar{\epsilon}(s), \bar{\epsilon}(s) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\| \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} - P_s \begin{pmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{pmatrix} \right\|_2,
\]

where \( \bar{\epsilon}(s) \) denotes the mean projection error for the whole input image set \( I \) with respect to the projection matrix \( P_s \) that was obtained by the calibration from the image subset represented by \( s \).

#### A. Monte Carlo Method (MCM)

Due to the huge discrete search space we propose a stochastic selection scheme for solving the discrete optimization problem. We use a Monte Carlo Method (MCM) which is inspired by the well-known Random Sampling Consensus (RANSAC) [2] and thus makes use of random choices.

The method’s key idea is to randomly choose combinations of the input images and keep the combination that yields the minimum mean projection error with respect to all the input images \( s_i \in I \).

In more detail, \( r \) unique subsets \( s_l, l = 1..r \), are randomly choosen from the search space \( P_l = 2^I \setminus \{I \cup \{\emptyset\}\} \), that is the power set of input images excluding the singletons and the empty set. In order to completely cover the search space and enlarge the convergence range, the size \( k_l = \lfloor |s_l|/2 \rfloor \) of individual subsets \( s_l \) is also randomly choosen from the closed interval \([0_{\min}, 0_{\max}] \) with \( 2 \leq 0_{\min} \leq 0_{\max} \leq |I| \).

For each of the subsets \( s_l \) the projection matrix \( P_{s_l} \) is determined by calibration. In our experiments, we apply the well-known calibration method by Zhang (see section III). Once the projection for a subset \( s_l \) has been determined the mean projection error \( \bar{\epsilon}(s_l) \) is calculated for all the images in the input image set \( I \) from their correspondences. Finally,
Algorithm 1 Monte-Carlo-Method based image subset determination for an image set \( I \)

**Require:** \( r, n_{\text{min}}, n_{\text{max}}, I \)

**Ensure:** \( 2 \leq n_{\text{min}} \leq n_{\text{max}} \leq |I| \)

\[ D = \emptyset \quad \text{// set of drawn subsets} \]

\[ s_{\text{opt}} = (0 0 \ldots 0)^T \]

\[ \epsilon_{\text{opt}} = \infty \]

while \( r \leq |D| \) do

draw a subset \( s \) with \( s \notin D \) and \( n_{\text{min}} \leq ||s||^2 \leq n_{\text{max}} \)

\[ D = D \cup s \]

\[ P_s = \text{Calibrate}(s) \]

\[ \epsilon_s = \text{MeanProjectionError}(P_s, I) \]

if \( \epsilon_s < \epsilon_{\text{opt}} \) then

\[ \epsilon_{\text{opt}} = \epsilon_s \]

\[ s_{\text{opt}} = s \]

end if

end while

return \( s_{\text{opt}} \)

---

The subset \( s^* \) that yields the minimal mean projection error is chosen as the (sub)optimal combination.

Once all the \( r \) subsets have been evaluated, the current combination is considered to be the optimal solution \( s_{\text{opt}} = s^* \). Thus, the calculation of the (locally) optimal solution is determined with \( N_{\text{MCM}} = r \) evaluations.

### B. Deterministic Extension

In order to improve the optimization procedure, we extend the former approach by a deterministic search strategy that refines a given solution. This strategy starts from an initial subset and deterministically adds or removes new elements in order to identify an improved combination. The selection strategy is a combination of the sequential forward selection and sequential backward selection algorithm, which are briefly explained now.

In an initial step, the sequential forward selection strategy identifies the best element within the set of \( n \) elements with respect to a certain score. In a subsequent step, the best combination of two elements is selected by testing this very element with the remaining elements. Repeating this until all the \( n \) elements have been selected, a (locally) optimal combination has been identified with \( N_{\text{SF}} \) evolutions:

\[ N_{\text{SF}} = \sum_{i=1}^{n} (n - i + 1) = \frac{n(n + 1)}{2}. \]

This, in comparison to (2) reduces the effort to identify an optimum significantly.

The sequential backward selection algorithm behaves similarly, but starts with the whole set of \( n \) elements and omits repeatedly the elements that do not maximize the overall score. The number of evaluations \( N_{\text{SB}} \) is equal to that of the sequential forward algorithm (eq. (4)).

The algorithm presented in this paper makes use of a combination of both selection strategies. It requires a so-called start configuration \( s_0 \) and two parameters \( n_{\text{min}} \) and \( n_{\text{max}} \) (with \( 2 \leq n_{\text{min}}, n_{\text{max}} \leq n \)) that allow to constrain the size of the optimal solution. In general, we do not assume any constraint on the subset and therefore \( n_{\text{min}} = 2 \) and \( n_{\text{max}} = n \) holds. Again, the score for the selection process is given by the mean projection error \( \epsilon(s) \) with respect to the whole input image set \( I \) (eq. (3)).

The start configuration \( s_0 = s^* \) is determined by the formerly described, stochastic algorithm (alg. 1). Starting from such a configuration containing the images \( i_2, i_4 \) and \( i_6 \), repeatedly one image is removed and added. The new set \( \{i_2, i_4, i_6, i_5\} \) yields the smallest projection error and thus is considered for further use. In a subsequent step, again one image is removed and added repeatedly, yielding the new optimal selection \( \{i_2, i_5, i_6\} \).

Fig. 1. Deterministic image selection process en detail. a) Starting from a configuration containing the images \( i_2, i_4 \) and \( i_6 \), repeatedly one image is removed and added. The new set \( \{i_2, i_4, i_5, i_6\} \) yields the smallest projection error and thus is considered for further use. b) In a subsequent step, again one image is removed and added repeatedly, yielding the new optimal selection \( \{i_2, i_5, i_6\} \).

The start configuration \( s_0 = s^* \) is determined by the formerly described, stochastic algorithm (alg. 1). Starting from such a configuration \( s_c \) with \( k_c = ||s_c||^2 \) images, \( (n - k_c) \) new configurations with \( (k_c + 1) \) images are created by adding each of the remaining images \( I \setminus I_c \). Likewise, \( k_c \) new configurations of size \( n_{\text{min}} \leq (k_c - 1) \leq n_{\text{max}} \) are set up by repeatedly removing one image. From these \( k_c \) \( (n - k_c) \) new configurations the new optimal configuration \( s_{c+1} \) is determined, that is again the one with the smallest projection error. The selection process terminates as soon as the newly determined configuration equals the current configuration \( s_{c+1} = s_c \) and thus the optimal combination \( s_{\text{opt}} = s_c \) has been identified.

The overall number of evaluations arises from the number \( N_{\text{MCM}} \) of evaluations of the Monte Carlo approach and the amount of evaluations \( N_{\text{Det}} \) for the deterministic extension:

\[ N = N_{\text{MCM}} + N_{\text{Det}} = r + n(n + 1), \tag{5} \]

with \( N_{\text{Det}} = N_{\text{SF}} + N_{\text{SB}} \).

### V. Experiments and Results

For an evaluation of our approach, we calibrated several cameras of different resolution and manufacturers. For any given camera we recorded \( n = 20 \) images of a 14-by-10 checkerboard (with \( m = 117 \) calibration marks) from different
directions whereas some of them originated from ill-posed configurations.

With these images we determined the globally optimal solution for subsets of fixed size. For this, we performed an exhaustive search of the search space and identified the minimum projection errors for the configurations that comprise of two images, three images and so on up to 20 images. Starting with configurations of only two images is due to the fast convergence of the Monte-Carlo method that requires at least two different views of the calibration pattern. In contrast to taking the minimum number of images, considering all images corresponds to the procedure typically pursued in everyday calibration work.

In the following, the Monte-Carlo approach with and without deterministic refinement has been applied to the image set and subsets of varying size were considered. For the optimizations random choices \( r_1 = 10, r_2 = 25, r_3 = 50, r_4 = 100, r_5 = 250, r_6 = 500 \) and \( r_7 = 1000 \) have been considered. In order to get statistically representative results, the experiment for a given setup was repeated 100 times and the resulting projection errors were arithmetically averaged.

Figure 2 depicts the projection errors for the identified optimal solutions whereas Table I shows the computational effort for the chosen sample numbers achieved on a 3Ghz Pentium IV workstation machine. Table II also exhibits the averaged results of the computation time in seconds for both approaches along with the number of evaluations as well as the optimization iterations for the deterministic enhancement for different random samples \( r \). The table depicts the averaged results of 100 experiments.

![Fig. 2. The mean projection error for varying random samples drawn using the Monte-Carlo (dark) and the enhanced Monte-Carlo method (light). With increasing numbers of random samples both methods converge towards the global optimum. However, the enhanced method converges much faster.](image)

**Table I**

<table>
<thead>
<tr>
<th>( r )</th>
<th>( t_{\text{MCM}} )</th>
<th>( t_{\text{Enh. MCM}} )</th>
<th>( N )</th>
<th>( N_{\text{Det}} )</th>
<th>Optimizations</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
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<td>11.32</td>
<td>86.8</td>
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<td>25</td>
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<td>11.68</td>
<td>87.4</td>
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</tr>
<tr>
<td>50</td>
<td>5.33</td>
<td>13.88</td>
<td>110.5</td>
<td>60.5</td>
<td>3.1</td>
</tr>
<tr>
<td>100</td>
<td>10.48</td>
<td>18.70</td>
<td>159.0</td>
<td>59.0</td>
<td>3.0</td>
</tr>
<tr>
<td>250</td>
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<td>81.36</td>
<td>300.9</td>
<td>50.9</td>
<td>2.6</td>
</tr>
<tr>
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</tr>
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<td>272.34</td>
<td>1036.2</td>
<td>36.2</td>
<td>1.8</td>
</tr>
</tbody>
</table>

In addition, Figure 2 shows that the enhanced Monte-Carlo approach is especially useful when only a few samples are drawn \( r \leq 100 \), because of its fast convergence. In contrast, using many random selections \( r > 100 \) only a few deterministic evaluations and optimization iterations are necessary due to the almost optimal results achieved in the stochastic step and vice versa (cf. Table I). No matter if the emphasis is on the stochastic or the deterministic part, the total number of evaluations that is needed to achieve an almost optimal result is orders of magnitude lower than the total number of evaluations \( N_{\text{ex}} = 1048555 \) needed for an exhaustive search. All the exemplary results are with respect to images with \( 1024 \times 768 \) pixel that have been acquired with a Matrox CV-M50 camera.

In addition to the former experiments, we asked different persons with a background in computer vision to calibrate the cameras and compared their calibration results with those that have been obtained with our unconstrained Monte Carlo method with deterministic refinement. Again, we took the mean projection error as a metric for the comparison and repeatedly applied our algorithm. Table II exhibits that the automatic approach yields better results than calibrations from image subsets that have been selected by human-made decision and that the optimal solution does not emerge when considering all the images in the input image set. The number of images in the last two rows of Table II are referred to the minimum mean projection error identified within the 100 experiments and counts 0.178365 and 0.178367 respectively.

Following an intuitive approach - that is choosing all the acquired images - will likely result in large projection errors (cf. Table II, row *All images*). Likewise, considering only the minimum number of images will not yield optimal solutions. For this reason, the method proposed in this paper is not a RANSAC method, since such methods draw samples of minimal size in order to determine the model’s parameters and rate outliers. In contrast, we vary the size of the samples and allow non-minimal sample sizes which in turn enables the algorithm to identify optimal solutions. However, due to its similarity we claim our approach to be inspired by RANSAC.

**VI. DISCUSSION AND CONCLUSION**

In this paper, we addressed the problem to automatically determine the optimal subset of the pool of acquired calibration images yielding the best calibration result. We presented a stochastic selection scheme in order to identify combinations of input images that yield a small projection error. Additionally, we proposed a deterministic algorithm for an improvement of optimal solutions that have been found with the stochastic approach.

Experiments comparing the algorithms’ performance with the global optimum as well as with human-made decisions
exhibit that the calibration can be significantly improved by automated image selection. Furthermore the calibration result is robust with respect to outlier images, i.e. images that were taken from ill-posed views. Hence, selecting good image sets for camera calibration no longer requires long lasting experience or time-consuming trial and error.

### References