

# An iterative updating method for damped gyroscopic systems

Yongxin Yuan

**Abstract**—The problem of updating damped gyroscopic systems using measured modal data can be mathematically formulated as following two problems. **Problem I:** Given  $M_a \in \mathbf{R}^{n \times n}$ ,  $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_p\} \in \mathbf{C}^{p \times p}$ ,  $X = [x_1, \dots, x_p] \in \mathbf{C}^{n \times p}$ , where  $p < n$  and both  $\Lambda$  and  $X$  are closed under complex conjugation in the sense that  $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$ ,  $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$  for  $j = 1, \dots, l$ , and  $\lambda_k \in \mathbf{R}$ ,  $x_k \in \mathbf{R}^n$  for  $k = 2l+1, \dots, p$ , find real-valued symmetric matrices  $D, K$  and a real-valued skew-symmetric matrix  $G$  (that is,  $G^T = -G$ ) such that  $M_a X \Lambda^2 + (D + G) X \Lambda + K X = 0$ . **Problem II:** Given real-valued symmetric matrices  $D_a, K_a \in \mathbf{R}^{n \times n}$  and a real-valued skew-symmetric matrix  $G_a$ , find  $(\hat{D}, \hat{G}, \hat{K}) \in \mathbf{S}_E$  such that  $\|\hat{D} - D_a\|^2 + \|\hat{G} - G_a\|^2 + \|\hat{K} - K_a\|^2 = \min_{(D, G, K) \in \mathbf{S}_E} (\|D - D_a\|^2 + \|G - G_a\|^2 + \|K - K_a\|^2)$ , where  $\mathbf{S}_E$  is the solution set of Problem I and  $\|\cdot\|$  is the Frobenius norm. This paper presents an iterative algorithm to solve Problem I and Problem II. By using the proposed iterative method, a solution of Problem I can be obtained within finite iteration steps in the absence of roundoff errors, and the minimum Frobenius norm solution of Problem I can be obtained by choosing a special kind of initial matrices. Moreover, the optimal approximation solution  $(\hat{D}, \hat{G}, \hat{K})$  of Problem II can be obtained by finding the minimum Frobenius norm solution of a changed Problem I. A numerical example shows that the introduced iterative algorithm is quite efficient.

**Keywords**—model updating, iterative algorithm, gyroscopic system, partially prescribed spectral data, optimal approximation.

## I. INTRODUCTION

**T**HROUGHOUT this paper, we shall adopt the following notation.  $\mathbf{C}^{m \times n}$  and  $\mathbf{R}^{m \times n}$  denote the set of all  $m \times n$  complex and real matrices,  $\mathbf{SR}^{n \times n}$  and  $\mathbf{SSR}^{n \times n}$  denote the set of all  $n \times n$  symmetric and skew-symmetric matrices in  $\mathbf{R}^{n \times n}$ .  $A^T$ ,  $\text{tr}(A)$  and  $R(A)$  stand for the transpose, the trace and the column space of the matrix  $A$ , respectively.  $I_n$  represents the identity matrix of order  $n$ . For  $A, B \in \mathbf{R}^{m \times n}$ , an inner product in  $\mathbf{R}^{m \times n}$  is defined by  $(A, B) = \text{tr}(B^T A)$ , then  $\mathbf{R}^{m \times n}$  is a Hilbert space. The matrix norm  $\|\cdot\|$  induced by the inner product is the Frobenius norm. Given two matrices  $A = [a_{ij}] \in \mathbf{R}^{m \times n}$  and  $B \in \mathbf{R}^{p \times q}$ , the Kronecker product of  $A$  and  $B$  is defined by  $A \otimes B = [a_{ij} B] \in \mathbf{R}^{mp \times nq}$ . Also, for an  $m \times n$  matrix  $A = [a_1, a_2, \dots, a_n]$ , where  $a_i, i = 1, \dots, n$ , is the  $i$ -th column vector of  $A$ , the stretching function  $\text{vec}(A)$  is defined as  $\text{vec}(A) = [a_1^T, a_2^T, \dots, a_n^T]^T$ . Let  $A, B$  and  $X$

be some matrices with appropriate dimensions, then we have the following well-known identity [1].

$$\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X).$$

Vibrating structures such as bridges, highways, buildings, automobiles and rotating machinery are often modeled using finite element techniques. These techniques generate structured matrix second-order differential equations

$$M_a \ddot{q}(t) + (D_a + G_a) \dot{q}(t) + K_a q(t) = 0. \quad (1)$$

The vector  $q(t)$  represents the generalized coordinates of the system.  $M_a, K_a, D_a$  and  $G_a$  are, respectively, called the analytical mass, stiffness, damping, and gyroscopic matrices. In many practical applications,  $M_a$  is symmetric and positive definite ( $M_a > 0$ ),  $K_a$  and  $D_a$  are real-valued symmetric, and  $G_a$  is always real-valued skew-symmetric (that is,  $G_a^T = -G_a$ ). If a fundamental solution to (1) is represented by

$$q(t) = x e^{\lambda t},$$

then the scalar  $\lambda$  and the vector  $x$  must solve the quadratic eigenvalue problem (QEP)

$$(\lambda^2 M_a + \lambda(D_a + G_a) + K_a)x = 0. \quad (2)$$

Complex numbers  $\lambda$  and nonzero complex vectors  $x$  for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. It is known that the equation of (2) has  $2n$  finite eigenvalues over the complex field, provided that the leading matrix coefficient  $M_a$  is nonsingular. Note that the signification of the system (1) usually can be interpreted via the eigenvalues and eigenvectors of Eq.(2). Because of this connection, a lot of efforts have been devoted to the QEP in the literature. Many applications, properties and numerical methods for the QEP are surveyed in the thesis by Tisseur and Meerbergen [2].

Accurate models are essential in analyzing systems under various excitations, boundary conditions and parameter changes. Analytical models, obtained by finite element techniques, inevitably deviate from the true model due to uncertainties in geometry, boundary conditions, discretization error, modeling error of joints, variation of material properties, ignorance of nonlinear effect, and other simplifications, etc. Consequently, eigenvalues and eigenvectors that are extracted

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from the test results do not agree with the predicted values from the analytical model. Model updating is the process of using the test results to correct the model so that it agrees, either completely or approximately, with the experimental data.

In the past decades, various techniques for updating mass and stiffness matrices for undamped systems (i.e.,  $D_a = 0, G_a = 0$ ) using measured response data have been discussed by Baruch [3], Baruch and Bar-Itzhack [4], Berman [5], Berman and Nagy [6], Wei [7, 8, 9], Yang et al. [10], Yang and Chen [11], and Yuan [12]. For an account of the earlier methods, see the authoritative book by Friswell and Mottershead [13], an integral introduction of the basic theory of finite element model updating is given. For damped structured systems, the theory and computation have been considered by Friswell et al. [14], Pilkey [15], Kuo et al. [16], Chu et al. [17] and Yuan [18]. It is well known that the damped gyroscopic systems are another important class of nonproportionally damped systems. They correspond to spinning structures where the Coriolis inertia forces are taken into account. Examples of such systems include helicopter rotor blades and spin-stabilized satellites with flexible elastic appendages such as solar panels or antennas. The numerical methods for quadratic eigenvalue problems of gyroscopic systems can see [2, 19-24]. In view of in analytical model (1) for structure dynamics, the effect of damping and Coriolis forces on structural dynamic systems is not well understood because it is purely dynamics property that can not be measured statically. Therefore, the correction of damped gyroscopic systems is very important. However, we observe that the iterative methods for model updating have received little attention in these years. In this paper we will develop an iterative method for the finite element model updating of damped gyroscopic systems which can incorporate the measured model data into the finite element model to produce an adjusted finite element model on the damping, gyroscopic and stiffness matrices that closely match the experimental modal data. The problem of updating damping, gyroscopic and stiffness matrices simultaneously can be mathematically formulated as follows.

**Problem I.** Let  $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_p\} \in \mathbf{C}^{p \times p}$  and  $X = [x_1, \dots, x_p] \in \mathbf{C}^{n \times p}$  be the measured eigenvalue and eigenvector matrices, where  $p < n$  and both  $\Lambda$  and  $X$  are closed under complex conjugation in the sense that  $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$ ,  $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$  for  $j = 1, \dots, l$ , and  $\lambda_k \in \mathbf{R}$ ,  $x_k \in \mathbf{R}^n$  for  $k = 2l + 1, \dots, p$ , find real-valued symmetric matrices  $D, K$  and a real-valued skew-symmetric matrix  $G$  such that

$$M_a X \Lambda^2 + (D + G) X \Lambda + K X = 0. \quad (3)$$

It is well known that  $D_a, G_a$  and  $K_a$  are good approximations of  $D, G$  and  $K$ . The strategy for obtaining an improved model is to find  $D, G$  and  $K$  that satisfy (3) and deviate as little as possible from  $D_a, G_a$  and  $K_a$ . Thus, we should further solve

the following optimal approximation problem.

**Problem II.** Let  $\mathbf{S}_E$  be the solution set of Problem I. Find  $(\hat{D}, \hat{G}, \hat{K}) \in \mathbf{S}_E$  such that

$$\begin{aligned} & \|\hat{D} - D_a\|^2 + \|\hat{G} - G_a\|^2 + \|\hat{K} - K_a\|^2 = \\ & \min_{(D,G,K) \in \mathbf{S}_E} (\|D - D_a\|^2 + \|G - G_a\|^2 + \|K - K_a\|^2). \end{aligned} \quad (4)$$

The paper is organized as follows. In Section 2, an efficient iterative method is presented to solve Problem I and Problem II. Then several properties of Algorithm 1 are proved. By using the proposed iterative method, a solution of Problem I can be obtained within finite iteration steps in the absence of roundoff errors, and the minimum Frobenius norm solution of Problem I can be obtained by choosing a special kind of initial matrices. In addition, the optimal approximation solution of Problem II is provided by finding the minimum Frobenius norm solution of a new matrix equation. In Section 3, a numerical example is used to test the effectiveness of the proposed algorithm.

## II. THE SOLUTION OF PROBLEM I AND PROBLEM II

Let  $\alpha_i = \text{Re}(\lambda_i)$  (the real part of the complex number  $\lambda_i$ ),  $\beta_i = \text{Im}(\lambda_i)$  (the imaginary part of the complex number  $\lambda_i$ ),  $y_i = \text{Re}(x_i)$ ,  $z_i = \text{Im}(x_i)$  for  $i = 1, 3, \dots, 2l - 1$ , and

$$\tilde{\Lambda} = \text{diag} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{2l-1} & \beta_{2l-1} \\ -\beta_{2l-1} & \alpha_{2l-1} \end{bmatrix}, \lambda_{2l+1}, \dots, \lambda_p \right\} \in \mathbf{R}^{p \times p}, \quad (5)$$

$$\tilde{X} = [y_1, z_1, \dots, y_{2l-1}, z_{2l-1}, x_{2l+1}, \dots, x_p] \in \mathbf{R}^{n \times p}. \quad (6)$$

Then, the equation of (3) can be equivalently written as

$$D \tilde{X} \tilde{\Lambda} + G \tilde{X} \tilde{\Lambda} + K \tilde{X} = F, \quad (7)$$

$$\text{s. t. } G \in \mathbf{SSR}^{n \times n}, D, K \in \mathbf{SR}^{n \times n},$$

where  $F = -M_a \tilde{X} \tilde{\Lambda}^2$ .

Now, we can describe an iterative algorithm for solving Problem I as follows.

### Algorithm 1

S 1. Input matrices  $\tilde{X} \in \mathbf{R}^{n \times p}$ ,  $\tilde{\Lambda} \in \mathbf{R}^{p \times p}$  and  $M_a \in \mathbf{SR}^{n \times n}$ , and choose arbitrary  $n \times n$  symmetric matrices  $D_1, K_1$  and a skew-symmetric matrix  $G_1$ .

S 2. Calculate

$$\begin{aligned} R_1 &= F - D_1 \tilde{X} \tilde{\Lambda} - G_1 \tilde{X} \tilde{\Lambda} - K_1 \tilde{X}; \\ P_1 &= \frac{1}{2}(R_1 \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_1^T); \\ Q_1 &= \frac{1}{2}(R_1 \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_1^T); \\ W_1 &= \frac{1}{2}(R_1 \tilde{X}^T + \tilde{X} R_1^T); \\ s &:= 1. \end{aligned}$$

S 3. If  $R_s = 0$ , then stop and  $(D_s, G_s, K_s)$  is a solution to the equation of (7), that is, a solution of Problem I; elseif  $R_s \neq 0$  but  $P_s = 0, Q_s = 0$  and  $W_s = 0$ , then stop and the equation of (7) has no solution; else  $s := s + 1$ .

S 4. Calculate

$$\begin{aligned} D_s &= D_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2 + \|Q_{s-1}\|^2 + \|W_{s-1}\|^2} P_{s-1}; \\ G_s &= G_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2 + \|Q_{s-1}\|^2 + \|W_{s-1}\|^2} Q_{s-1}; \\ K_s &= K_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2 + \|Q_{s-1}\|^2 + \|W_{s-1}\|^2} W_{s-1}; \\ R_s &= F - D_s \tilde{X} \tilde{\Lambda} - G_s \tilde{X} \tilde{\Lambda} - K_s \tilde{X} \\ &= R_{s-1} - \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2 + \|Q_{s-1}\|^2 + \|W_{s-1}\|^2} \\ &\quad (P_{s-1} \tilde{X} \tilde{\Lambda} + Q_{s-1} \tilde{X} \tilde{\Lambda} + W_{s-1} \tilde{X}); \\ P_s &= \frac{1}{2} (R_s \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} P_{s-1}; \\ Q_s &= \frac{1}{2} (R_s \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} Q_{s-1}; \\ W_s &= \frac{1}{2} (R_s \tilde{X}^T + \tilde{X} R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} W_{s-1}. \end{aligned}$$

S 5. Go to S 3.

From Algorithm 1, we can easily see that  $Q_s, G_s \in \mathbf{SSR}^{n \times n}$  and  $P_s, D_s, W_s, K_s \in \mathbf{SR}^{n \times n}$  for  $s = 1, 2, \dots$ .

**Definition 1** Assume that  $Y, Z \in \mathbf{R}^{m \times n}$ . The matrices  $Y, Z$  are called orthogonal each other if  $\text{tr}(Y^T Z) = 0$ .

About Algorithm 1, we present the following basic properties.

*Lemma 1:* The sequences  $\{R_i\}, \{P_i\}, \{Q_i\}$  and  $\{W_i\}$  generated by Algorithm 1 satisfy

$$\begin{aligned} \text{tr}(R_j^T R_i) &= 0, \quad \text{and} \quad \text{tr}(P_j^T P_i) + \text{tr}(Q_j^T Q_i) \\ &+ \text{tr}(W_j^T W_i) = 0 \quad \text{for} \quad i, j = 1, 2, \dots, s, \quad i \neq j. \end{aligned} \quad (8)$$

**Proof.** Since  $\text{tr}(R_j^T R_i) = \text{tr}(R_i^T R_j)$ ,  $\text{tr}(P_j^T P_i) = \text{tr}(P_i^T P_j)$ ,  $\text{tr}(Q_j^T Q_i) = \text{tr}(Q_i^T Q_j)$  and  $\text{tr}(W_j^T W_i) = \text{tr}(W_i^T W_j)$ , then we only need to show that

$$\begin{aligned} \text{tr}(R_j^T R_i) &= 0, \quad \text{and} \quad \text{tr}(P_j^T P_i) + \text{tr}(Q_j^T Q_i) \\ &+ \text{tr}(W_j^T W_i) = 0 \quad \text{for} \quad 1 \leq i < j \leq s. \end{aligned}$$

We use the mathematical induction to prove this conclusion, and we do it in two steps.

We first show that

$$\begin{aligned} \text{tr}(R_{i+1}^T R_i) &= 0, \quad \text{and} \quad \text{tr}(P_{i+1}^T P_i) + \text{tr}(Q_{i+1}^T Q_i) \\ &+ \text{tr}(W_{i+1}^T W_i) = 0 \quad \text{for} \quad i = 1, 2, \dots, s. \end{aligned} \quad (9)$$

For  $i = 1$ , by Algorithm 1 and noting that  $P_1, W_1 \in \mathbf{SR}^{n \times n}$  and  $Q_1 \in \mathbf{SSR}^{n \times n}$ , we have

$$\begin{aligned} &\text{tr}(R_2^T R_1) \\ &= \text{tr}((R_1 - \delta_1 (P_1 \tilde{X} \tilde{\Lambda} + Q_1 \tilde{X} \tilde{\Lambda} + W_1 \tilde{X}))^T R_1) \\ &= \text{tr}(R_1^T R_1) - \delta_1 \text{tr}(\tilde{\Lambda}^T \tilde{X}^T P_1^T R_1 \\ &+ \tilde{\Lambda}^T \tilde{X}^T Q_1^T R_1 + \tilde{X}^T W_1^T R_1) \\ &= \|R_1\|^2 - \frac{1}{2} \delta_1 \text{tr}(\tilde{\Lambda}^T \tilde{X}^T P_1^T R_1 + R_1^T P_1 \tilde{X} \tilde{\Lambda} \\ &+ \tilde{\Lambda}^T \tilde{X}^T Q_1^T R_1 + R_1^T Q_1 \tilde{X} \tilde{\Lambda} + \tilde{X}^T W_1^T R_1 + R_1^T W_1 \tilde{X}) \\ &= \|R_1\|^2 - \frac{1}{2} \delta_1 \text{tr}(P_1^T R_1 \tilde{\Lambda}^T \tilde{X}^T + P_1^T \tilde{X} \tilde{\Lambda} R_1^T \\ &+ Q_1^T R_1 \tilde{\Lambda}^T \tilde{X}^T - Q_1^T \tilde{X} \tilde{\Lambda} R_1^T + W_1^T R_1 \tilde{X}^T + W_1^T \tilde{X} R_1^T) \\ &= \|R_1\|^2 - \delta_1 \text{tr}(P_1^T P_1 + Q_1^T Q_1 + W_1^T W_1) \\ &= 0, \end{aligned}$$

where  $\delta_1 = \frac{\|R_1\|^2}{\|P_1\|^2 + \|Q_1\|^2 + \|W_1\|^2}$ .

Applying the proved result  $\text{tr}(R_2^T R_1) = 0$ , we get

$$\begin{aligned} &\text{tr}(P_2^T P_1) + \text{tr}(Q_2^T Q_1) + \text{tr}(W_2^T W_1) \\ &= \frac{1}{2} \text{tr}((R_2 \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_2^T) P_1) + \frac{\|R_2\|^2}{\|R_1\|^2} \|P_1\|^2 \\ &+ \frac{1}{2} \text{tr}((R_2 \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_2^T)^T Q_1) + \frac{\|R_2\|^2}{\|R_1\|^2} \|Q_1\|^2 \\ &+ \frac{1}{2} \text{tr}((R_2 \tilde{X}^T + \tilde{X} R_2^T) W_1) + \frac{\|R_2\|^2}{\|R_1\|^2} \|W_1\|^2 \\ &= \frac{1}{2} \text{tr}(R_2 (P_1 \tilde{X} \tilde{\Lambda} + Q_1 \tilde{X} \tilde{\Lambda} + W_1 \tilde{X})^T \\ &+ (P_1 \tilde{X} \tilde{\Lambda} + Q_1 \tilde{X} \tilde{\Lambda} + W_1 \tilde{X}) R_2^T) \\ &+ \frac{\|R_2\|^2}{\|R_1\|^2} \|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2} \|Q_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2} \|W_1\|^2 \\ &= \frac{1}{2} \frac{\|P_1\|^2 + \|Q_1\|^2 + \|W_1\|^2}{\|R_1\|^2} \\ &\quad \text{tr}((R_2 (R_1 - R_2))^T + (R_1 - R_2) R_2^T) \\ &+ \frac{\|R_2\|^2}{\|R_1\|^2} \|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2} \|Q_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2} \|W_1\|^2 \\ &= 0. \end{aligned}$$

Suppose that (9) holds for  $i = t - 1$ . For  $i = t$ , we have

$$\begin{aligned} &\text{tr}(R_{t+1}^T R_t) \\ &= \text{tr}((R_t - \delta_t (P_t \tilde{X} \tilde{\Lambda} + Q_t \tilde{X} \tilde{\Lambda} + W_t \tilde{X}))^T R_t) \\ &= \text{tr}(R_t^T R_t) - \delta_t \text{tr}(\tilde{\Lambda}^T \tilde{X}^T P_t^T R_t \\ &+ \tilde{\Lambda}^T \tilde{X}^T Q_t^T R_t + \tilde{X}^T W_t^T R_t) \\ &= \|R_t\|^2 - \frac{1}{2} \delta_t \text{tr}(\tilde{\Lambda}^T \tilde{X}^T P_t^T R_t + R_t^T P_t \tilde{X} \tilde{\Lambda} \\ &+ \tilde{\Lambda}^T \tilde{X}^T Q_t^T R_t + R_t^T Q_t \tilde{X} \tilde{\Lambda} + \tilde{X}^T W_t^T R_t + R_t^T W_t \tilde{X}) \\ &= \|R_t\|^2 - \frac{1}{2} \delta_t \text{tr}(P_t^T R_t \tilde{\Lambda}^T \tilde{X}^T + P_t^T \tilde{X} \tilde{\Lambda} R_t^T \\ &+ Q_t^T R_t \tilde{\Lambda}^T \tilde{X}^T - Q_t^T \tilde{X} \tilde{\Lambda} R_t^T + W_t^T R_t \tilde{X}^T + W_t^T \tilde{X} R_t^T) \\ &= \|R_t\|^2 - \delta_t \text{tr}(P_t^T (P_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2} P_{t-1}) \\ &+ Q_t^T (Q_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2} Q_{t-1}) \\ &+ W_t^T (W_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2} W_{t-1})) \\ &= \|R_t\|^2 - \delta_t \text{tr}(P_t^T P_t + Q_t^T Q_t + W_t^T W_t) \\ &= 0, \end{aligned}$$

where  $\delta_t = \frac{\|R_t\|^2}{\|P_t\|^2 + \|Q_t\|^2 + \|W_t\|^2}$ .

$$\begin{aligned} &\text{tr}(P_{t+1}^T P_t) + \text{tr}(Q_{t+1}^T Q_t) + \text{tr}(W_{t+1}^T W_t) \\ &= \frac{1}{2} \text{tr}((R_{t+1} \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_{t+1}^T) P_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|P_t\|^2 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \text{tr}((R_{t+1} \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_{t+1}^T)^T Q_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|Q_t\|^2 \\
 & + \frac{1}{2} \text{tr}((R_{t+1} \tilde{X}^T + \tilde{X} R_{t+1}^T) W_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|W_t\|^2 \\
 & = \frac{1}{2} \text{tr}(R_{t+1} (P_t \tilde{X} \tilde{\Lambda} + Q_t \tilde{X} \tilde{\Lambda} + W_t \tilde{X})^T \\
 & + (P_t \tilde{X} \tilde{\Lambda} + Q_t \tilde{X} \tilde{\Lambda} + W_t \tilde{X}) R_{t+1}^T) \\
 & + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|P_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|Q_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|W_t\|^2 \\
 & = \frac{1}{2} \frac{\|P_t\|^2 + \|Q_t\|^2 + \|W_t\|^2}{\|R_t\|^2} \\
 & \text{tr}((R_{t+1} (R_t - R_{t+1})^T + (R_t - R_{t+1}) R_{t+1}^T) \\
 & + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|P_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|Q_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2} \|W_t\|^2 \\
 & = 0.
 \end{aligned}$$

Therefore, (9) holds for  $i = t$ . By the principle of induction, we know (9) holds for all  $i$ .

Next, assume that  $\text{tr}(R_{i+d}^T R_i) = 0$ , and  $\text{tr}(P_{i+d}^T P_i) + \text{tr}(Q_{i+d}^T Q_i) + \text{tr}(W_{i+d}^T W_i) = 0$  for  $1 \leq i \leq s$  and  $1 < d < s$ . We will prove  $\text{tr}(R_{i+d+1}^T R_i) = 0$ , and  $\text{tr}(P_{i+d+1}^T P_i) + \text{tr}(Q_{i+d+1}^T Q_i) + \text{tr}(W_{i+d+1}^T W_i) = 0$ .

$$\begin{aligned}
 & \text{tr}(R_{i+d+1}^T R_i) \\
 & = \text{tr}((R_{i+d} - \delta_{i+d} (P_{i+d} \tilde{X} \tilde{\Lambda} \\
 & + Q_{i+d} \tilde{X} \tilde{\Lambda} + W_{i+d} \tilde{X}))^T R_i) \\
 & = -\delta_{i+d} \text{tr}(\tilde{\Lambda}^T \tilde{X}^T P_{i+d}^T R_i \\
 & + \tilde{\Lambda}^T \tilde{X}^T Q_{i+d}^T R_i + \tilde{X}^T W_{i+d}^T R_i) \\
 & = -\frac{1}{2} \delta_{i+d} \text{tr}(\tilde{\Lambda}^T \tilde{X}^T P_{i+d}^T R_i + R_i^T P_{i+d} \tilde{X} \tilde{\Lambda} \\
 & + \tilde{\Lambda}^T \tilde{X}^T Q_{i+d}^T R_i + R_i^T Q_{i+d} \tilde{X} \tilde{\Lambda} \\
 & + \tilde{X}^T W_{i+d}^T R_i + R_i^T W_{i+d} \tilde{X}) \\
 & = -\frac{1}{2} \delta_{i+d} \text{tr}(P_{i+d}^T R_i \tilde{\Lambda}^T \tilde{X}^T + P_{i+d}^T \tilde{X} \tilde{\Lambda} R_i^T \\
 & + Q_{i+d}^T R_i \tilde{\Lambda}^T \tilde{X}^T - Q_{i+d}^T \tilde{X} \tilde{\Lambda} R_i^T \\
 & + W_{i+d}^T R_i \tilde{X}^T + W_{i+d}^T \tilde{X} R_i^T) \\
 & = -\delta_{i+d} \text{tr}(P_{i+d}^T (P_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} P_{i-1}) \\
 & + Q_{i+d}^T (Q_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} Q_{i-1}) \\
 & + W_{i+d}^T (W_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} W_{i-1})) \\
 & = 0,
 \end{aligned}$$

where  $\delta_{i+d} = \frac{\|R_{i+d}\|^2}{\|P_{i+d}\|^2 + \|Q_{i+d}\|^2 + \|W_{i+d}\|^2}$ . From the above results, we have  $\text{tr}(R_{i+d+1}^T R_i) = 0$  and

$$\begin{aligned}
 \text{tr}(R_{i+d+1}^T R_{i+1}) & = 0. \text{ Hence we can get} \\
 & \text{tr}(P_{i+d+1}^T P_i) + \text{tr}(Q_{i+d+1}^T Q_i) + \text{tr}(W_{i+d+1}^T W_i) \\
 & = \frac{1}{2} \text{tr}((R_{i+d+1} \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_{i+d+1}^T) P_i) \\
 & + \frac{1}{2} \text{tr}((R_{i+d+1} \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_{i+d+1}^T)^T Q_i) \\
 & + \frac{1}{2} \text{tr}((R_{i+d+1} \tilde{X}^T + \tilde{X} R_{i+d+1}^T) W_i) \\
 & = \frac{1}{2} \text{tr}(R_{i+d+1} (P_i \tilde{X} \tilde{\Lambda} + Q_i \tilde{X} \tilde{\Lambda} + W_i \tilde{X})^T \\
 & + (P_i \tilde{X} \tilde{\Lambda} + Q_i \tilde{X} \tilde{\Lambda} + W_i \tilde{X}) R_{i+d+1}^T) \\
 & = \frac{1}{2} \xi \text{tr}((R_{i+d+1} (R_i - R_{i+1})^T + (R_i - R_{i+1}) R_{i+d+1}^T) \\
 & = 0,
 \end{aligned}$$

where  $\xi = \frac{\|P_i\|^2 + \|Q_i\|^2 + \|W_i\|^2}{\|R_i\|^2}$ . Thus the conclusion (8) holds by the principle of induction. The proof is completed.

**Lemma 2:** Let Problem I be consistent, and  $(D^*, G^*, K^*)$  be an arbitrary solution of Problem I. Then, for any initial matrix triplet  $(D_1, G_1, K_1)$  with  $G_1 \in \mathbf{SSR}^{n \times n}$  and  $D_1, K_1 \in \mathbf{SR}^{n \times n}$ , we have

$$\begin{aligned}
 & \text{tr}((D^* - D_i)^T P_i) + \text{tr}((G^* - G_i)^T Q_i) \\
 & + \text{tr}((K^* - K_i)^T W_i) = \|R_i\|^2 \text{ for } i = 1, 2, \dots, \quad (10)
 \end{aligned}$$

where the sequences  $\{D_i\}, \{P_i\}, \{G_i\}, \{Q_i\}, \{K_i\}, \{W_i\}$  and  $\{R_i\}$  are generated by Algorithm 1.

**Proof.** We prove the conclusion by induction. For  $i = 1$ , we have

$$\begin{aligned}
 & \text{tr}((D^* - D_1)^T P_1) + \text{tr}((G^* - G_1)^T Q_1) \\
 & + \text{tr}((K^* - K_1)^T W_1) \\
 & = \frac{1}{2} \text{tr}((D^* - D_1)^T (R_1 \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_1^T)) \\
 & + \frac{1}{2} \text{tr}((G^* - G_1)^T (R_1 \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_1^T)) \\
 & + \frac{1}{2} \text{tr}((K^* - K_1)^T (R_1 \tilde{X}^T + \tilde{X} R_1^T)) \\
 & = \frac{1}{2} \text{tr}(F R_1^T - D_1 \tilde{X} \tilde{\Lambda} R_1^T - G_1 \tilde{X} \tilde{\Lambda} R_1^T - K_1 \tilde{X} R_1^T) \\
 & + \frac{1}{2} \text{tr}(F^T R_1 - \tilde{\Lambda}^T \tilde{X}^T D_1^T R_1 \\
 & - \tilde{\Lambda}^T \tilde{X}^T G_1^T R_1 - \tilde{X}^T K_1^T R_1) \\
 & = \frac{1}{2} \text{tr}(R_1 R_1^T) + \frac{1}{2} \text{tr}(R_1^T R_1) \\
 & = \|R_1\|^2.
 \end{aligned}$$

Now assume the conclusion (10) holds for  $1 \leq i \leq t-1$ . Then we can get

$$\begin{aligned}
 & \text{tr}((D^* - D_t)^T P_t) + \text{tr}((G^* - G_t)^T Q_t) \\
 & + \text{tr}((K^* - K_t)^T W_t) \\
 & = \text{tr}((D^* - D_{t-1})
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2 + \|Q_{t-1}\|^2 + \|W_{t-1}\|^2} P_{t-1}^T P_t) \\
 & + \text{tr}((G^* - G_{t-1} \\
 & - \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2 + \|Q_{t-1}\|^2 + \|W_{t-1}\|^2} Q_{t-1}^T Q_t) \\
 & + \text{tr}((K^* - K_{t-1} \\
 & - \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2 + \|W_{t-1}\|^2 + \|W_{t-1}\|^2} W_{t-1}^T W_t) \\
 & = \text{tr}((D^* - D_{t-1})^T P_t) + \text{tr}((G^* - G_{t-1})^T Q_t) \\
 & + \text{tr}((K^* - K_{t-1})^T W_t) \\
 & = \frac{1}{2} \text{tr}((D^* - D_{t-1})^T (R_t \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_t^T)) \\
 & + \frac{\|R_t\|^2}{\|R_{t-1}\|^2} \text{tr}((D^* - D_{t-1})^T P_{t-1}) \\
 & + \frac{1}{2} \text{tr}((G^* - G_{t-1})^T (R_t \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_t^T)) \\
 & + \frac{\|R_t\|^2}{\|R_{t-1}\|^2} \text{tr}((G^* - G_{t-1})^T Q_{t-1}) \\
 & + \frac{1}{2} \text{tr}((K^* - K_{t-1})^T (R_t \tilde{X}^T + \tilde{X} R_t^T)) \\
 & + \frac{\|R_t\|^2}{\|R_{t-1}\|^2} \text{tr}((K^* - K_{t-1})^T W_{t-1}) \\
 & = \frac{1}{2} \text{tr}((D^* - D_{t-1})^T (R_t \tilde{\Lambda}^T \tilde{X}^T + \tilde{X} \tilde{\Lambda} R_t^T)) \\
 & + \frac{1}{2} \text{tr}((G^* - G_{t-1})^T (R_t \tilde{\Lambda}^T \tilde{X}^T - \tilde{X} \tilde{\Lambda} R_t^T)) \\
 & + \frac{1}{2} \text{tr}((K^* - K_{t-1})^T (R_t \tilde{X}^T + \tilde{X} R_t^T)) + \|R_t\|^2 \\
 & = \frac{1}{2} \text{tr}(F^T R_t - \tilde{\Lambda}^T \tilde{X}^T D_{t-1}^T R_t \\
 & - \tilde{\Lambda}^T \tilde{X}^T G_{t-1}^T R_t - \tilde{X}^T K_{t-1}^T R_t) \\
 & + \frac{1}{2} \text{tr}(F R_t^T - D_{t-1} \tilde{X} \tilde{\Lambda} R_t^T \\
 & - G_{t-1} \tilde{X} \tilde{\Lambda} R_t^T - K_{t-1} \tilde{X} R_t^T) + \|R_t\|^2 \\
 & = \frac{1}{2} \text{tr}(R_{t-1}^T R_t) + \frac{1}{2} \text{tr}(R_{t-1} R_t^T) + \|R_t\|^2 \\
 & = \|R_t\|^2.
 \end{aligned}$$

Thus we complete the proof of Lemma 2 by the principle of induction.

From Lemma 2, we can easily see that if there exists a positive number  $l$  such that  $P_l = 0$ ,  $Q_l = 0$  and  $W_l = 0$  but  $R_l \neq 0$ , then the equation of (7) has no solution. Hence, the solvability of Eq.(7) can be determined automatically by Algorithm 1.

**Theorem 1:** Assume that Problem I is consistent. Then for any arbitrary initial matrix triplet  $(D_1, G_1, K_1)$  with  $G_1 \in \mathbf{SSR}^{n \times n}$  and  $D_1, K_1 \in \mathbf{SR}^{n \times n}$ , a solution of Problem I can be obtained with finite iteration steps in the absence of roundoff errors.

**Proof.** Assume that  $R_l \neq 0$ ,  $l = 1, 2, \dots, np$ . From Lemma 2, we know  $\|P_l\|^2 + \|Q_l\|^2 + \|W_l\|^2 \neq 0$ . Then we can calculate  $R_{np+1}$  and  $(D_{np+1}, G_{np+1}, K_{np+1})$  by Algorithm 1. From Lemma 1, we have

$$\text{tr}(R_{np+1}^T R_t) = 0, \quad t = 1, 2, \dots, np,$$

and

$$\text{tr}(R_j^T R_i) = 0, \quad i, j = 1, 2, \dots, np, \quad i \neq j.$$

Therefore,  $\{R_1, R_2, \dots, R_{np}\}$  forms an orthogonal basis of the real-valued matrix space  $\mathbf{R}^{n \times p}$ , which implies that  $R_{np+1} = 0$ , that is,  $(D_{np+1}, G_{np+1}, K_{np+1})$  is a solution of Problem I.

**Lemma 3:** The equation of (7) has a solution  $(D, G, K)$  with  $G \in \mathbf{SSR}^{n \times n}$  and  $D, K \in \mathbf{SR}^{n \times n}$  if and only if the matrix equations

$$\begin{aligned}
 D \tilde{X} \tilde{\Lambda} + G \tilde{X} \tilde{\Lambda} + K \tilde{X} &= F, \\
 \tilde{\Lambda}^T \tilde{X}^T D - \tilde{\Lambda}^T \tilde{X}^T G + \tilde{X}^T K &= F^T,
 \end{aligned} \quad (11)$$

are consistent.

**Proof.** If the equation of (7) has a solution  $(D^*, G^*, K^*)$  with  $G^* \in \mathbf{SSR}^{n \times n}$  and  $D^*, K^* \in \mathbf{SR}^{n \times n}$ , then  $D^* \tilde{X} \tilde{\Lambda} + G^* \tilde{X} \tilde{\Lambda} + K^* \tilde{X} = F$ , and  $(D^* \tilde{X} \tilde{\Lambda} + G^* \tilde{X} \tilde{\Lambda} + K^* \tilde{X})^T = \tilde{\Lambda}^T \tilde{X}^T D^* - \tilde{\Lambda}^T \tilde{X}^T G^* + \tilde{X}^T K^* = F^T$ . That is to say,  $(D^*, G^*, K^*)$  is a solution of (11).

Conversely, if the matrix equations of (11) has a solution, say,  $D = U$ ,  $G = V$ ,  $K = Z$ . Let  $D^* = \frac{1}{2}(U + U^T)$ ,  $G^* = \frac{1}{2}(V - V^T)$ ,  $K^* = \frac{1}{2}(Z + Z^T)$ , then  $G^*$  is a skew-symmetric matrix and  $D^*, K^*$  are symmetric matrices, and

$$\begin{aligned}
 D^* \tilde{X} \tilde{\Lambda} + G^* \tilde{X} \tilde{\Lambda} + K^* \tilde{X} \\
 &= \frac{1}{2}(U \tilde{X} \tilde{\Lambda} + V \tilde{X} \tilde{\Lambda} + Z \tilde{X}) + \frac{1}{2}(U^T \tilde{X} \tilde{\Lambda} - V^T \tilde{X} \tilde{\Lambda} + Z^T \tilde{X}) \\
 &= \frac{1}{2} F + \frac{1}{2} (F^T)^T = F.
 \end{aligned}$$

Hence,  $(D^*, G^*, K^*)$  is a solution of (7).

The following lemma comes from [25].

**Lemma 4:** Suppose that the consistent system of linear equations  $Ax = b$  has a solution  $x \in R(A^T)$ , then  $x$  is the unique minimum Frobenius norm solution of the system of linear equations.

Using the Kronecker product and the stretching function, we know that the equations of (11) are equivalent to

$$\begin{aligned}
 \left[ \begin{array}{ccc} \tilde{\Lambda}^T \tilde{X}^T \otimes I_n & \tilde{\Lambda}^T \tilde{X}^T \otimes I_n & \tilde{X}^T \otimes I_n \\ I_n \otimes \tilde{\Lambda}^T \tilde{X}^T & I_n \otimes (-\tilde{\Lambda}^T \tilde{X}^T) & I_n \otimes \tilde{X}^T \end{array} \right] \begin{bmatrix} \text{vec}(D) \\ \text{vec}(G) \\ \text{vec}(K) \end{bmatrix} \\
 = \begin{bmatrix} \text{vec}(F) \\ \text{vec}(F^T) \end{bmatrix}.
 \end{aligned}$$

Assume that  $H \in \mathbf{R}^{n \times n}$  is an arbitrary matrix, then we have

$$\begin{aligned} & \begin{bmatrix} \text{vec}(H\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}H^T) \\ \text{vec}(H\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}H^T) \\ \text{vec}(H\tilde{X}^T + \tilde{X}H^T) \end{bmatrix} \\ &= \begin{bmatrix} \tilde{X}\tilde{\Lambda} \otimes I_n & I_n \otimes \tilde{X}\tilde{\Lambda} \\ \tilde{X}\tilde{\Lambda} \otimes I_n & I_n \otimes (-\tilde{X}\tilde{\Lambda}) \\ \tilde{X} \otimes I_n & I_n \otimes \tilde{X} \end{bmatrix} \begin{bmatrix} \text{vec}(H) \\ \text{vec}(H^T) \end{bmatrix} \\ &= \begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T \otimes I_n & \tilde{\Lambda}^T\tilde{X}^T \otimes I_n & \tilde{X}^T \otimes I_n \\ I_n \otimes \tilde{\Lambda}^T\tilde{X}^T & I_n \otimes (-\tilde{\Lambda}^T\tilde{X}^T) & I_n \otimes \tilde{X}^T \end{bmatrix}^T \\ & \begin{bmatrix} \text{vec}(H) \\ \text{vec}(H^T) \end{bmatrix} \in \\ & R \left( \begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T \otimes I_n & \tilde{\Lambda}^T\tilde{X}^T \otimes I_n & \tilde{X}^T \otimes I_n \\ I_n \otimes \tilde{\Lambda}^T\tilde{X}^T & I_n \otimes (-\tilde{\Lambda}^T\tilde{X}^T) & I_n \otimes \tilde{X}^T \end{bmatrix}^T \right). \end{aligned}$$

It is obvious that if we choose

$$\begin{aligned} D_1 &= H\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}H^T, \quad G_1 = H\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}H^T, \\ K_1 &= H\tilde{X}^T + \tilde{X}H^T, \end{aligned} \quad (12)$$

then all  $D_s, G_s$  and  $K_s$  generated by Algorithm 1 satisfy

$$\begin{aligned} & \begin{bmatrix} \text{vec}(D_s) \\ \text{vec}(G_s) \\ \text{vec}(K_s) \end{bmatrix} \in \\ & R \left( \begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T \otimes I_n & \tilde{\Lambda}^T\tilde{X}^T \otimes I_n & \tilde{X}^T \otimes I_n \\ I_n \otimes \tilde{\Lambda}^T\tilde{X}^T & I_n \otimes (-\tilde{\Lambda}^T\tilde{X}^T) & I_n \otimes \tilde{X}^T \end{bmatrix}^T \right). \end{aligned}$$

It follows from Lemma 4 that if we choose a initial matrix triplet by (12), where  $H$  is an arbitrary matrix, then a solution  $(D^*, G^*, K^*)$  obtained by Algorithm 1 is the minimum Frobenius norm solution of Problem I. In summary of above discussion, we have proved the following result.

**Theorem 2:** Suppose that Problem I is consistent. If we choose the initial matrices by (12), where  $H$  is an arbitrary matrix, or especially,  $D_1 = 0, G_1 = 0$  and  $K_1 = 0$ , then we can obtain the unique minimum Frobenius norm solution of Problem I within finite iterative steps.

Now we show that the solution of Problem II can be derived by finding the minimum norm solution of a new matrix equation. Assume that Problem I is consistent. Obviously the solution set  $\mathbf{S}_E$  of Problem I is nonempty, then for a given matrix triplet  $(D_a, G_a, K_a)$ , we have

$$\begin{aligned} & D\tilde{X}\tilde{\Lambda} + G\tilde{X}\tilde{\Lambda} + K\tilde{X} = -M_a\tilde{X}\tilde{\Lambda}^2 \\ \Leftrightarrow & (D - D_a)\tilde{X}\tilde{\Lambda} + (G - G_a)\tilde{X}\tilde{\Lambda} + (K - K_a)\tilde{X} \\ & = -M_a\tilde{X}\tilde{\Lambda}^2 - D_a\tilde{X}\tilde{\Lambda} - G_a\tilde{X}\tilde{\Lambda} - K_a\tilde{X}. \end{aligned}$$

Let

$$\begin{aligned} \tilde{D} &= D - D_a, \quad \tilde{G} = G - G_a, \quad \tilde{K} = K - K_a, \\ \tilde{F} &= -M_a\tilde{X}\tilde{\Lambda}^2 - D_a\tilde{X}\tilde{\Lambda} - G_a\tilde{X}\tilde{\Lambda} - K_a\tilde{X}, \end{aligned}$$

then the matrix approximation Problem II is equivalent to finding the minimum Frobenius norm solution of the matrix equation

$$\tilde{D}\tilde{X}\tilde{\Lambda} + \tilde{G}\tilde{X}\tilde{\Lambda} + \tilde{K}\tilde{X} = \tilde{F}, \quad (13)$$

$$\text{s. t. } \tilde{G} \in \mathbf{SSR}^{n \times n}, \quad \tilde{D}, \tilde{K} \in \mathbf{SR}^{n \times n}.$$

Applying Algorithm 1, and taking the initial matrices by (12), where  $H$  is an arbitrary matrix, or especially,  $\tilde{D}_1 = 0, \tilde{G}_1 = 0$  and  $\tilde{K}_1 = 0$ , we can obtain the minimum Frobenius norm solution  $(\tilde{D}^*, \tilde{G}^*, \tilde{K}^*)$  of (13). Once  $(\tilde{D}^*, \tilde{G}^*, \tilde{K}^*)$  is obtained, the solution of the matrix optimal approximation Problem II can be computed. In this case, can be expressed as

$$\hat{D} = D_a + \tilde{D}^*, \quad \hat{G} = G_a + \tilde{G}^*, \quad \hat{K} = K_a + \tilde{K}^*. \quad (14)$$

### III. A NUMERICAL EXAMPLE

In this section, we will give a numerical example to illustrate our results. All the tests are performed using MATLAB 6.5. Because of the influence of the error of calculation, the iteration will not stop within finite steps. Hence, we regard  $(D_s, G_s, K_s)$  as a solution of the considered problem if the corresponding residue satisfies  $\|R_s\| \leq 1.0e - 010$ .

**Example 1.** Consider a 7-DOF system modelled analytically with mass, gyroscopic, and stiffness matrices given by

$$M_a = 0.03 \times \begin{bmatrix} 52 & 22 & 18 & -13 & 0 & 0 & 0 \\ 22 & 12 & 13 & -9 & 0 & 0 & 0 \\ 18 & 13 & 104 & 0 & 18 & -13 & 0 \\ -13 & -9 & 0 & 24 & 13 & -9 & 0 \\ 0 & 0 & 18 & 13 & 104 & 0 & 18 \\ 0 & 0 & -13 & -9 & 0 & 24 & 13 \\ 0 & 0 & 0 & 0 & 18 & 13 & 104 \end{bmatrix},$$

$$D_a = \begin{bmatrix} 25.8258 & 15.9831 & -6.8154 & & & & \\ 15.9831 & 59.1804 & -12.6864 & & & & \\ -6.8154 & -12.6864 & 59.6436 & & & & \\ 13.3326 & 13.8510 & 0 & & & & \\ 0.1395 & 0.1053 & -6.2304 & & & & \\ -0.1053 & -0.0792 & 12.9024 & & & & \\ 0 & 0 & 0.1395 & & & & \\ 13.3326 & 0.1395 & -0.1053 & & & & 0 \\ 13.8510 & 0.1053 & -0.0792 & & & & 0 \\ 0 & -6.2304 & 12.9024 & & & & 0.1395 \\ 117.1368 & -12.9024 & 14.0112 & & & & 0.1053 \\ -12.9024 & 59.6436 & 0 & & & & -6.2304 \\ 14.0112 & 0 & 117.1368 & & & & -12.9024 \\ 0.1053 & -6.2304 & -12.902 & & & & 59.6436 \end{bmatrix},$$

$$G_a = \begin{bmatrix} 0 & 15.9831 & -6.8154 & \\ -15.9831 & 0 & -12.6864 & \\ 6.8154 & 12.6864 & 0 & \\ -13.3326 & -13.8510 & 0 & \\ -0.1395 & -0.1053 & 6.2304 & \\ 0.1053 & 0.0792 & -12.9024 & \\ 0 & 0 & -0.1395 & \\ 13.3326 & 0.1395 & -0.1053 & 0 \\ 13.8510 & 0.1053 & -0.0792 & 0 \\ 0 & -6.2304 & 12.9024 & 0.1395 \\ 0 & -12.9024 & 14.0112 & 0.1053 \\ 12.9024 & 0 & 0 & -6.2304 \\ -14.0112 & 0 & 0 & -12.9024 \\ -0.1053 & 6.2304 & 12.9024 & 0 \end{bmatrix},$$

$$D_{14} = \begin{bmatrix} -1.0350 & -7.3983 & 16.0777 & \\ -7.3983 & 13.4950 & 28.2103 & \\ 16.0777 & 28.2103 & -47.6215 & \\ 10.4515 & -23.8379 & -7.9163 & \\ -32.2720 & -10.7902 & 35.6826 & \\ -10.5444 & 25.4311 & 0.3282 & \\ 23.0742 & 4.9442 & -11.3227 & \\ 10.4515 & -32.2720 & -10.5444 & 23.0742 \\ -23.8379 & -10.7902 & 25.4311 & 4.9442 \\ -7.9163 & 35.6826 & 0.3282 & -11.3227 \\ -21.6123 & 31.6409 & 3.7644 & -21.6979 \\ 31.6409 & 57.1259 & 26.1558 & -85.4500 \\ 3.7644 & 26.1558 & 25.3634 & -29.2406 \\ -21.6979 & -85.4500 & -29.2406 & 99.6122 \end{bmatrix},$$

and

$$K_a = 600 \times \begin{bmatrix} 2 & 3 & -2 & 3 & 0 & 0 & 0 \\ 3 & 6 & -3 & 3 & 0 & 0 & 0 \\ -2 & -3 & 4 & 0 & -2 & 3 & 0 \\ 3 & 3 & 0 & 12 & -3 & 3 & 0 \\ 0 & 0 & -2 & -3 & 4 & 0 & -2 \\ 0 & 0 & 3 & 3 & 0 & 12 & -3 \\ 0 & 0 & 0 & 0 & -2 & -3 & 4 \end{bmatrix}.$$

$$G_{14} = \begin{bmatrix} -0.0000 & -6.7377 & -8.0588 & \\ 6.7377 & 0.0000 & -30.9780 & \\ 8.0588 & 30.9780 & -0.0000 & \\ -1.2158 & -34.0188 & 18.5950 & \\ -38.6231 & -19.7042 & 61.9164 & \\ -7.8206 & 21.6911 & 3.8357 & \\ 36.4310 & 15.7616 & -70.7559 & \\ 1.2158 & 38.6231 & 7.8206 & -36.4310 \\ 34.0188 & 19.7042 & -21.6911 & -15.7616 \\ -18.5950 & -61.9164 & -3.8357 & 70.7559 \\ -0.0000 & -67.2390 & -29.2425 & 61.9985 \\ 67.2390 & 0.0000 & -15.1970 & -49.8023 \\ 29.2425 & 15.1970 & 0.0000 & -27.4853 \\ -61.9985 & 49.8023 & 27.4853 & 0.0000 \end{bmatrix},$$

The measured eigenvalue and eigenvector matrices  $\Lambda$  and  $X$  are given by

$$\Lambda = \text{diag}\{-71.087 + 55.495i, -71.087 - 55.495i, -19.507 + 39.177i, -19.507 - 39.177i\}$$

and

$$X = \begin{bmatrix} 0.1714 + 0.3902i & 0.1714 - 0.3902i \\ -0.3786 - 0.3146i & -0.3786 + 0.3146i \\ -0.0063 - 0.1350i & -0.0063 + 0.1350i \\ 0.0855 + 0.5370i & 0.0855 - 0.5370i \\ -0.0657 - 0.0451i & -0.0657 + 0.0451i \\ -0.1868 - 0.4171i & -0.1868 + 0.4171i \\ 0.0738 + 0.0431i & 0.0738 - 0.0431i \\ 0.4628 + 0.2187i & 0.4628 - 0.2187i \\ -0.5110 + 0.0243i & -0.5110 - 0.0243i \\ -0.3022 - 0.2606i & -0.3022 + 0.2606i \\ 0.1293 - 0.0539i & 0.1293 + 0.0539i \\ 0.4070 + 0.1619i & 0.4070 - 0.1619i \\ -0.0065 + 0.0671i & -0.0065 - 0.0671i \\ -0.3158 - 0.0366i & -0.3158 + 0.0366i \end{bmatrix},$$

$$K_{14} = \begin{bmatrix} 0.9201 & -0.2805 & -2.7237 & -0.1034 \\ -0.2805 & -0.0944 & 1.5355 & -0.0387 \\ -2.7237 & 1.5355 & 3.9447 & 0.1669 \\ -0.1034 & -0.0387 & 0.1669 & 1.2746 \\ 4.9203 & -3.3070 & -5.9154 & -0.9910 \\ 0.8169 & -0.4975 & -1.0632 & -1.0895 \\ -3.7965 & 2.6824 & 4.2561 & 0.6674 \\ 4.9203 & 0.8169 & -3.7965 & \\ -3.3070 & -0.4975 & 2.6824 & \\ -5.9154 & -1.0632 & 4.2561 & \\ -0.9910 & -1.0895 & 0.6674 & \\ 8.3247 & 1.9996 & -5.7547 & \\ 1.9996 & 1.0734 & -1.3660 & \\ -5.7547 & -1.3660 & 3.9314 & \end{bmatrix},$$

**Test 1.** Choosing initial iterative matrices  $D_1 = 0$ ,  $G_1 = 0$  and  $K_1 = 0$ . By Algorithm 1, after 13 iteration steps, we get the minimum Frobenius norm solution of Problem I as follows.

with corresponding residual

$$\|R_{14}\| = \|F - D_{14}\tilde{X}\tilde{\Lambda} - G_{14}\tilde{X}\tilde{\Lambda} - K_{14}\tilde{X}\| = 2.5677e - 012.$$

**Test 2.** Choosing initial iterative matrices  $\tilde{D}_1 = 0, \tilde{G}_1 = 0$  and  $\tilde{K}_1 = 0$ . By Algorithm 1, we get the minimum Frobenius norm solution  $(\tilde{D}^*, \tilde{G}^*, \tilde{K}^*)$  of Eq.(13) as follows.

Therefore, by (14), the optimal approximation solution of Problem II is

$$\tilde{D}^* = \tilde{D}_{14} = \begin{bmatrix} -0.8613 & -1.5193 & 2.7099 & \\ -1.5193 & -1.6506 & 3.2150 & \\ 2.7099 & 3.2150 & -5.9458 & \\ 0.2863 & -3.2481 & -0.9176 & \\ -3.2564 & -1.0203 & 3.7313 & \\ -0.3986 & 2.0852 & -0.0354 & \\ 2.0024 & 0.3289 & -0.7599 & \end{bmatrix},$$

$$\begin{bmatrix} 0.2863 & -3.2564 & -0.3986 & 2.0024 \\ -3.2481 & -1.0203 & 2.0852 & 0.3289 \\ -0.9176 & 3.7313 & -0.0354 & -0.7599 \\ -5.9295 & 2.9333 & 1.9747 & -1.3325 \\ 2.9333 & 3.9074 & 1.2042 & -6.6729 \\ 1.9747 & 1.2042 & 0.2575 & -1.6034 \\ -1.3325 & -6.6729 & -1.6034 & 7.7265 \end{bmatrix},$$

$$\tilde{G}^* = \tilde{G}_{14} = \begin{bmatrix} -0.0000 & -1.2427 & -0.9730 & \\ 1.2427 & 0.0000 & -2.5346 & \\ 0.9730 & 2.5346 & 0.0000 & \\ 0.4217 & -2.9375 & 1.3438 & \\ -3.6149 & -1.7040 & 5.7363 & \\ -0.5062 & 1.6916 & 0.4763 & \\ 3.2792 & 1.4012 & -6.5728 & \end{bmatrix},$$

$$\begin{bmatrix} -0.4217 & 3.6149 & 0.5062 & -3.2792 \\ 2.9375 & 1.7040 & -1.6916 & -1.4012 \\ -1.3438 & -5.7363 & -0.4763 & 6.5728 \\ 0.0000 & -6.1061 & -2.6883 & 5.9064 \\ 6.1061 & 0.0000 & -0.9247 & -4.4698 \\ 2.6883 & 0.9247 & -0.0000 & -1.9688 \\ -5.9064 & 4.4698 & 1.9688 & 0.0000 \end{bmatrix},$$

$$\tilde{K}^* = \tilde{K}_{14} = \begin{bmatrix} 0.1477 & -0.0097 & -0.3159 & \\ -0.0097 & -0.0503 & 0.1516 & \\ -0.3159 & 0.1516 & 0.4305 & \\ -0.0129 & 0.0445 & 0.0358 & \\ 0.4606 & -0.2844 & -0.5687 & \\ 0.0448 & -0.0463 & -0.0820 & \\ -0.3314 & 0.2180 & 0.3875 & \end{bmatrix},$$

$$\begin{bmatrix} -0.0129 & 0.4606 & 0.0448 & -0.3314 \\ 0.0445 & -0.2844 & -0.0463 & 0.2180 \\ 0.0358 & -0.5687 & -0.0820 & 0.3875 \\ 0.1819 & -0.1254 & -0.1320 & 0.0809 \\ -0.1254 & 0.7523 & 0.1703 & -0.5036 \\ -0.1320 & 0.1703 & 0.1043 & -0.1162 \\ 0.0809 & -0.5036 & -0.1162 & 0.3362 \end{bmatrix},$$

with corresponding residual

$$\|R_{14}\| = \|\tilde{F} - \tilde{D}_{14}\tilde{X}\tilde{\Lambda} - \tilde{G}_{14}\tilde{X}\tilde{\Lambda} - \tilde{K}_{14}\tilde{X}\| = 2.2388e-013.$$

$$\hat{D} = \begin{bmatrix} 24.9645 & 14.4638 & -4.1055 & \\ 14.4638 & 57.5298 & -9.4714 & \\ -4.1055 & -9.4714 & 53.6978 & \\ 13.6189 & 10.6029 & -0.9176 & \\ -3.1169 & -0.9150 & -2.4991 & \\ -0.5039 & 2.0060 & 12.8670 & \\ 2.0024 & 0.3289 & -0.6204 & \end{bmatrix},$$

$$\begin{bmatrix} 13.6189 & -3.1169 & -0.5039 & 2.0024 \\ 10.6029 & -0.9150 & 2.0060 & 0.3289 \\ -0.9176 & -2.4991 & 12.8670 & -0.6204 \\ 111.2073 & -9.9691 & 15.9859 & -1.2272 \\ -9.9691 & 63.5510 & 1.2042 & -12.9033 \\ 15.9859 & 1.2042 & 117.3943 & -14.5058 \\ -1.2272 & -12.9033 & -14.5058 & 67.3701 \end{bmatrix},$$

$$\hat{G} = \begin{bmatrix} -0.0000 & 14.7404 & -7.7884 & \\ -14.7404 & 0.0000 & -15.2210 & \\ 7.7884 & 15.2210 & 0.0000 & \\ -12.9109 & -16.7885 & 1.3438 & \\ -3.7544 & -1.8093 & 11.9667 & \\ -0.4009 & 1.7708 & -12.4261 & \\ 3.2792 & 1.4012 & -6.7123 & \end{bmatrix},$$

$$\begin{bmatrix} 12.9109 & 3.7544 & 0.4009 & -3.2792 \\ 16.7885 & 1.8093 & -1.7708 & -1.4012 \\ -1.3438 & -11.9667 & 12.4261 & 6.7123 \\ 0.0000 & -19.0085 & 11.3229 & 6.0117 \\ 19.0085 & 0.0000 & -0.9247 & -10.7002 \\ -11.3229 & 0.9247 & -0.0000 & -14.8712 \\ -6.0117 & 10.7002 & 14.8712 & 0.0000 \end{bmatrix},$$

$$\hat{K} = 1000 \times \begin{bmatrix} 1.2001 & 1.8000 & -1.2003 & \\ 1.8000 & 3.5999 & -1.7998 & \\ -1.2003 & -1.7998 & 2.4004 & \\ 1.8000 & 1.8000 & 0.0000 & \\ 0.0005 & -0.0003 & -1.2006 & \\ 0.0000 & -0.0000 & 1.7999 & \\ -0.0003 & 0.0002 & 0.0004 & \end{bmatrix},$$

$$\begin{bmatrix} 1.8000 & 0.0005 & 0.0000 & -0.0003 \\ 1.8000 & -0.0003 & -0.0000 & 0.0002 \\ 0.0000 & -1.2006 & 1.7999 & 0.0004 \\ 7.2002 & -1.8001 & 1.7999 & 0.0001 \\ -1.8001 & 2.4008 & 0.0002 & -1.2005 \\ 1.7999 & 0.0002 & 7.2001 & -1.8001 \\ 0.0001 & -1.2005 & -1.8001 & 2.4003 \end{bmatrix}.$$

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