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# $2^n$ positive periodic solutions to $n$ species non-autonomous Lotka-Volterra competition systems with harvesting terms

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*Abstract*—By using Mawhin's continuation theorem of coincidence degree theory, we establish the existence of $2^n$ positive periodic solutions for $n$ species non-autonomous Lotka-Volterra competition systems with harvesting terms. An example is given to illustrate the effectiveness of our results.

*Keywords*—Positive periodic solutions; Lotka-Volterra competition system; Coincidence degree; Harvesting term.

## I. INTRODUCTION

THE $n$ species Lotaka-Volterra competition model with harvesting terms is described as follows ([1,2]):

$$\dot{x_i}(t) = x_i(t)\left(a_i - b_i x_i(t) - \sum_{j=1,j\neq i}^{n} c_{ij}x_j(t)\right) - h_i, \quad i = 1,2,\ldots,n,$$

where $x_i(t)(i = 1,2,\ldots,n)$ is the densities functions of the $i$th species; $h_i(i = 1,2,\ldots,n)$ is the $i$th species harvesting terms standing for the harvests. Realistic models require the inclusion of the effect of changing environment. This motivates us to consider the following nonautonomous model

$$\dot{x_i}(t) = x_i(t)\left(a_i(t) - b_i(t)x_i(t) - \sum_{j=1,j\neq i}^{n} c_{ij}(t)x_j(t)\right) - h_i(t), \; i = 1,2,\ldots,n, \tag{1}$$

In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc ), which leads us to assume that $a_i(t), b_i(t), c_{ij}(t)$ and $h_i(t)(i,j = 1,2,\ldots,n)$ are all positive continuous $\omega$-periodic functions.

Since a very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model, also, on the existence of positive periodic solutions to system (1), few results are found in literatures. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [3], to establish the existence of $2^n$ positive periodic solutions for system (1). For the work concerning the multiple existence of periodic solutions of periodic population models which was done using coincidence degree theory, we refer to [4-6].

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## II. EXISTENCE OF $2^n$ POSITIVE PERIODIC SOLUTIONS

In this section, by using Mawhms continuation theorem, we shall show the existence of positive periodic solutions of (1). To do so, we need to make some preparations.

Let $X$ and $Z$ be real normed vector spaces. Let $L : \mathrm{Dom}\,L \subset X \to Z$ be a linear mapping and $N : X \times [0,1] \to Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if dim Ker $L$ = codim Im $L < \infty$ and Im $L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero, then there exists continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}\,P = \mathrm{Ker}\,L$ and $\mathrm{Ker}\,Q = \mathrm{Im}\;L = \mathrm{Im}\,(I-Q)$, and $X = \mathrm{Ker}\,L \bigoplus \mathrm{Ker}\;P, Z = \mathrm{Im}\;L \bigoplus \mathrm{Im}\,Q$. It follows that $L|_{\mathrm{Dom}\;L\cap\mathrm{Ker}\;P} : (I-P)X \to \mathrm{Im}\;L$ is invertible and its inverse is denoted by $K_P$. If $\Omega$ is a bounded open subset of $X$, the mapping $N$ is called $L$-compact on $\bar{\Omega}\times[0,1]$, if $QN(\bar{\Omega}\times[0,1])$ is bounded and $K_P(I-Q)N : \bar{\Omega}\times[0,1] \to X$ is compact. Because Im $Q$ is isomorphic to Ker $L$, there exists an isomorphism $J : \mathrm{Im}\,Q \to \mathrm{Ker}\;L$.

The Mawhin's continuous theorem [3, p.40] is given as follows:

**Lemma 1.** [3] *Let $L$ be a Fredholm mapping of index zero and let $N$ be $L$-compact on $\bar{\Omega}\times[0,1]$. Assume*

$(a)$ *for each $\lambda \in (0,1)$, every solution $x$ of $Lx = \lambda N(x,\lambda)$ is such that $x \notin \partial\Omega \cap \mathrm{Dom}\;L$;*
$(b)$ $QN(x,0)x \neq 0$ *for each $x \in \partial\Omega \cap \mathrm{Ker}\;L$;*
$(c)$ $\deg(JQN(x,0), \Omega\cap\mathrm{Ker}\,L, 0) \neq 0$.

*Then $Lx = Nx$ has at least one solution in $\overline{\Omega}\cap\mathrm{Dom}\,L$.*

For the sake of convenience, we denote by $f^l = \min_{t\in[0,\omega]} f(t), f^M = \max_{t\in[0,\omega]} f(t), \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t$, respectively, here $f(t)$ is a continuous $\omega$-periodic function. Throughout this paper, we need the following assumption.

$(H)$ $a_i^l > 2\sqrt{b_i^M h_i^M}, i = 1, 2, \ldots, n.$

For simplicity, we also introduce the following positive numbers

$$l_i^{\pm} = \frac{a_i^M \pm \sqrt{(a_i^M)^2 - 4b_i^l h_i^l}}{2b_i^l},\ K_i = \frac{a_i^M b_i^M}{a_i^l b_i^l},\ i = 1, 2, \ldots, n.$$

**Lemma 2.** *For the following equation*

$$a_i(t) - b_i(t)e^{u_*(t)} - h_i(t)e^{-u_*(t)} = 0,$$

*where $t \in R, i = 1, 2, \ldots, n$. If assumption $(H)$ holds, then for all $t \in R$, we have the following inequality*

$$\ln l_i^- < u_*^- < \ln\Big(\frac{l_i^+ + l_i^-}{2K_i}\Big) < u_*^+ < \ln l_i^+,$$

*where $u_*^{\pm} = \ln \frac{a_i(t) \pm \sqrt{(a_i(t))^2 - 4b_i(t)h_i(t)}}{2b_i(t)}, i = 1, 2, \ldots, n.$*

*Proof:* By the assumption $(H)$ and the expression of $u_*^{\pm}$ and $l_i^{\pm}$, $\ln l_i^- < u_*^-$ and $u_*^+ < \ln l_i^+$ obviously hold. Now let us prove $u_*^- < \ln\big(\frac{l_i^+ + l_i^-}{2K_i}\big) < u_*^+$. In fact,

$$\begin{aligned}
&\frac{u_*^-}{(l_i^+ + l_i^-)/2K_i}\\
&= \frac{a_i^M b_i^M}{a_i^l b_i^l} \times \frac{a_i(t) - \sqrt{(a_i(t))^2 - 4b_i(t)h_i(t)}}{2b_i(t)} \times \frac{2b_i^l}{a_i^M}\\
&= \frac{b_i^M}{a_i^l} \times \frac{a_i(t) - \sqrt{(a_i(t))^2 - 4b_i(t)h_i(t)}}{b_i(t)}\\
&= \frac{b_i^M}{a_i^l} \times \frac{4h_i(t)}{a_i(t) + \sqrt{(a_i(t))^2 - 4b_i(t)h_i(t)}}\\
&< \frac{b_i^M}{a_i^l} \times \frac{4h_i^M}{a_i^l} = \frac{4b_i^M h_i^M}{(a_i^l)^2} < 1
\end{aligned}$$

and

$$\begin{aligned}
&\frac{u_*^+}{(l_i^+ + l_i^-)/2K_i}\\
&= \frac{a_i^M b_i^M}{a_i^l b_i^l} \times \frac{a_i(t) + \sqrt{(a_i(t))^2 - 4b_i(t)h_i(t)}}{2b_i(t)} \times \frac{2b_i^l}{a_i^M}\\
&= \frac{b_i^M}{a_i^l} \times \frac{a_i(t) + \sqrt{(a_i(t))^2 - 4b_i(t)h_i(t)}}{b_i(t)}\\
&> \frac{b_i^M}{a_i^l} \times \frac{a_i^l}{b_i^M} = 1,\ i = 1, 2, \ldots, n,
\end{aligned}$$

which imply that $u_*^- < \ln\big(\frac{l_i^+ + l_i^-}{2K_i}\big) < u_*^+, i = 1, 2, \ldots, n.$ The proof of Lemma 2 is complete. ■

**Theorem 1.** *Assume that $(H)$ hold. Then system (1) has at least $2^n$ positive $\omega$-periodic solutions.*

*Proof:* By making the substitution

$$x_i(t) = \exp\{u_i(t)\},\ i = 1, 2, \ldots, n, \tag{2}$$

then system (1) can be reformulated as

$$\begin{aligned}\dot{u_i}(t) &= a_i(t) - b_i(t)e^{u_i(t)} - \sum_{j=1, j\neq i}^{n} c_{ij}(t)e^{u_j(t)}\\ &\quad -h_i(t)e^{-u_i(t)},\ i = 1, 2, \ldots, n.\end{aligned} \tag{3}$$

Let

$$X = Z = \big\{u = (u_1, u_2, \ldots, u_n)^T \in C(R, R^n) : u(t+\omega) = u(t), t \in R\big\}$$

with the norm defined by $\|u\| = \sum_{i=1}^n \max_{t\in[0,\omega]} |u_i(t)|, u \in X$, then $X$ and $Z$ are Banach spaces. Let

$$N(u,\lambda) = \begin{bmatrix} a_1(t) - b_1(t)e^{u_1(t)} - \lambda\sum_{j=2}^n c_{1j}(t)e^{u_j(t)} - h_1(t)e^{-u_1(t)}\\ a_2(t) - b_2(t)e^{u_2(t)} - \lambda\sum_{j=1, j\neq 2}^n c_{2j}(t)e^{u_j(t)} - h_2(t)e^{-u_2(t)}\\ \vdots\\ a_n(t) - b_n(t)e^{u_n(t)} - \lambda\sum_{j=1}^{n-1} c_{nj}(t)e^{u_j(t)} - h_n(t)e^{-u_n(t)} \end{bmatrix}_{n\times 1},\quad u \in X.$$

$Lu = \dot{u} = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$. We put $Pu = \frac{1}{\omega}\int_0^\omega u(t)\,\mathrm{d}t,\ u \in X;\ Qz = \frac{1}{\omega}\int_0^\omega z(t)\,\mathrm{d}t,\ z \in Z$. Similar to the proof of Theorem 1 in [7], it is easy to prove that $L$ is a Fredholm mapping of index 0. $N$ is $L$-compact on $V$ with any $V$ open bounded in X.

In order to use Lemma 1, we have to find at least $2^n$ appropriate open bounded subsets in $X$. Considering the operator equation $Lu = \lambda N(u,\lambda), \lambda \in (0,1)$, we have

$$\begin{aligned}\dot{u_i}(t) &= \lambda\bigg(a_i(t) - b_i(t)e^{u_i(t)} - \lambda\sum_{j=1, j\neq i}^{n} c_{ij}(t)e^{u_j(t)}\\ &\quad -h_i(t)e^{-u_j(t)}\bigg), i = 1, 2, \ldots, n.\end{aligned} \tag{4}$$

Assume that $u \in X$ is an $\omega$-periodic solution of system (4) for some $\lambda \in (0,1)$. Then there exist $\xi_i, \eta_i \in [0,\omega]$ such that $u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), i = 1, 2, \ldots, n$. It is clear that $\dot{u_i}(\xi_i) = 0, \dot{u_i}(\eta_i) = 0, i = 1, 2, \ldots, n$. From this and (4), we have

$$\begin{aligned}&a_i(\xi_i) - b_i(\xi_i)e^{u_i(\xi_i)} - \lambda\sum_{j=1, j\neq i}^{n} c_{ij}(\xi_i)e^{u_j(\xi_i)}\\ &-h_i(\xi_i)e^{-u_i(\xi_i)} = 0,\ i = 1, 2, \ldots, n\end{aligned} \tag{5}$$

and

$$\begin{aligned}&a_i(\eta_i) - b_i(\eta_i)e^{u_i(\eta_i)} - \lambda\sum_{j=1, j\neq i}^{n} c_{ij}(\eta_j)e^{u_j(\eta_j)}\\ &-h_i(\eta_i)e^{-u_i(\eta_i)} = 0,\ i = 1, 2, \ldots, n.\end{aligned} \tag{6}$$

According to (5), we have

$$\begin{aligned} & b_i^l e^{u_i(\xi_i)} + h_i^l e^{-u_i(\xi_i)} \\ & < b_i(\xi_i)e^{u_i(\xi_i)} + \sum_{j=1,j\neq i}^{n} c_{ij}(\xi_j)e^{u_j(\xi_j)} + h_i(\xi_i)e^{-u_i(\xi_i)} \\ & = a_i(\xi_i) \le a_i^M,\ i=1,2,\ldots,n, \end{aligned}$$

namely,

$$b_i^l e^{2u_i(\xi_i)} - a_i^M e^{u_i(\xi_i)} + h_i^l < 0,\ i=1,2,\ldots,n,$$

which imply that

$$\ln l_i^- < u_i(\xi_i) < \ln l_i^+,\ i=1,2,\ldots,n. \tag{7}$$

Similarly, by (6), we obtain

$$\ln l_i^- < u_i(\eta_i) < \ln l_i^+,\ i=1,2,\ldots,n. \tag{8}$$

From (7) and (8), we obtain

$$\ln l_i^- < u_i(t) \le \ln\left(\frac{l_i^+ + l_i^-}{2K_i}\right),\ i=1,2,\ldots,n \tag{9}$$

or

$$\ln\left(\frac{l_i^+ + l_i^-}{2K_i}\right) < u_i(t) < \ln l_i^+,\ i=1,2,\ldots,n. \tag{10}$$

For convenience, we denote

$$G_i = \left(\ln l_i^-, \ln\left(\frac{l_i^+ + l_i^-}{2K_i}\right)\right),\ i=1,2,\ldots,n,$$

$$H_i = \left(\ln\left(\frac{l_i^+ + l_i^-}{2K_i}\right), \ln l_i^+\right),\ i=1,2,\ldots,n.$$

Clearly, $l_i^{\pm}, i=1,2,\ldots,n$ are independent of $\lambda$. For each $i=1,2,\ldots,n$, we choose one of the intervals among the two intervals $G_i$ and $H_i$ and denote it as $\Delta_i$, then define the set

$$\begin{aligned} & \{u=(u_1,u_2,\ldots,u_n)^T \in X : \\ & u_i(t) \in \Delta_i, t \in R, i=1,2,\ldots,n\}. \end{aligned}$$

Obviously, the number of the above sets is $2^n$. We denote these sets as $\Omega_k, k=1,2,\ldots,2^n$. $\Omega_k, k=1,2,\ldots,2^n$ are bounded open subsets of $X, \Omega_i \cap \Omega_j = \phi, i \neq j$. Thus $\Omega_k (k=1,2,\ldots,2^n)$ satisfies the requirement $(a)$ in Lemma 1.

Now we show that $(b)$ of Lemma 1 holds, i.e., we prove when $u \in \partial\Omega_k \cap \operatorname{Ker} L = \partial\Omega_k \cap R^n, QN(u,0) \neq (0,0)^T, k=1,2,\ldots,2^n$. If it is not true, then when $u \in \partial\Omega_k \cap \operatorname{Ker} L = \partial\Omega_k \cap R^n, i=1,2,\ldots,2^n$, constant vector $u=(u_1,u_2,\ldots,u_n)^T$ with $u \in \partial\Omega_k, k=1,2,\ldots,2^n$, satisfies

$$\int_0^\omega a_i(t)\,\mathrm{d}t - \int_0^\omega b_i(t)e^{u_i}\,\mathrm{d}t - \int_0^\omega h_i(t)e^{-u_i}\,\mathrm{d}t = 0,$$

where $i=1,2,\ldots,n$. In view of the mean value theorem of calculous, there exist $n$ points $t_i (i=1,2,\ldots,n)$ such that

$$a_i(t_i) - b_i(t_i)e^{u_i} - h_i(t_i)e^{-u_i} = 0,\ i=1,2,\ldots,n. \tag{11}$$

Following the arguments of (7)-(10), we have

$$\ln l_i^- < u_i \le \ln\left(\frac{l_i^+ + l_i^-}{2K_i}\right), i=1,2,\ldots,n$$

or

$$\ln\left(\frac{l_i^+ + l_i^-}{2K_i}\right) < u_i < \ln l_i^+, i=1,2,\ldots,n.$$

Moreover, for (11), we have

$$u_i^{\pm} = \ln\frac{a_i(t_i) \pm \sqrt{(a_i(t_i))^2 - 4b_i(t_i)h_i(t_i)}}{2b_i(t_i)},\ i=1,2,\ldots,n.$$

According to Lemma 2, we obtain for $i=1,2,\ldots,n$,

$$\ln l_i^- < u_i^- < \ln\left(\frac{u_i^- + u_i^+}{2K_i}\right) < u_i^+ < \ln l_i^+.$$

Then $u$ belongs to one of $\Omega_k \cap R^n, k=1,2,\ldots,2^n$. This contradicts the fact that $u \in \partial\Omega_k \cap R^n, k=1,2,\ldots,2^n$. This proves $(b)$ in Lemma 1 holds. Finally, we show that $(c)$ in Lemma 1 holds. Note that the system of algebraic equations:

$$a_i(t_i) - b_i(t_i)e^{x_i} - h_i(t_i)e^{-x_i} = 0,\ i=1,2,\ldots,n$$

has $2^n$ distinct solutions since $(H)$ holds, $(x_1^*, x_2^*, \ldots, x_n^*) = (\ln\hat{x}_1, \ln\hat{x}_2, \ldots, \ln\hat{x}_n)$, where $x_i^{\pm} = \ln\frac{a_i(t_i)\pm\sqrt{(a_i(t_i))^2-4b_i(t_i)h_i(t_i)}}{2b_i(t_i)}, \hat{x}_i = x_i^-$ or $\hat{x}_i = x_i^+, i=1,2,\ldots,n$. It is easy to verify that

$$\ln l_i^- < x_i^- < \ln\left(\frac{l_i^+ + l_i^-}{2K_i}\right) < x_i^+ < \ln l_i^+,\ i=1,2,\ldots,n.$$

Therefore, $(x_1^*, x_2^*, \ldots, x_n^*)$ uniquely belongs to the corresponding $\Omega_k$. Since $\operatorname{Ker} L = \operatorname{Im} Q$, we can take $J=I$. A direct computation gives, for $k=1,2,\ldots,2^n$,

$$\begin{aligned} & \deg\{JQN(u,0), \Omega_k \cap \operatorname{Ker}\ L, (0,0)^T\} \\ & = \operatorname{sign}\left[\prod_{i=1}^n \left(-b_i(t_i)x_i^* + \frac{h_i(t_i)}{x_i^*}\right)\right]. \end{aligned}$$

Since $a_i(t_i) - b_i(t_i)x_i^* - \frac{h_i(t_i)}{x_i^*} = 0,\ i=1,2,\ldots,n$, then

$$\begin{aligned} & \deg\{JQN(u,0), \Omega_k \cap \operatorname{Ker} L, (0,0)^T\} \\ & = \operatorname{sign}\left[\prod_{i=1}^n \left(a_i(t_i) - 2b_i(t_i)x_i^*\right)\right] = \pm 1,\ k=1,2,\ldots,2^n. \end{aligned}$$

So far, we have prove that $\Omega_k (k=1,2,\ldots,2^n)$ satisfies all the assumptions in Lemma 1. Hence, system (3) has at least $2^n$ different $\omega$-periodic solutions. Thus by (2) system (1) has at least $2^n$ different positive $\omega$-periodic solutions. This completes the proof of Theorem 1. ■

## III. An example

Now, let us consider the following three species competition system with harvesting terms:

$$\begin{cases} \dot{x}(t) = x(t)\Big(3+\sin t - \dfrac{4+\sin t}{10}x(t) \\ \qquad -c_{12}(t)y(t) - c_{13}(t)z(t)\Big) - \dfrac{9+\cos t}{20}, \\ \dot{y}(t) = y(t)\Big(3+\cos t - \dfrac{5+\cos t}{10}y(t) \\ \qquad -c_{21}(t)x(t) - c_{23}(t)z(t)\Big) - \dfrac{2+\cos t}{5}, \\ \dot{z}(t) = z(t)\Big(3+\sin 2t - \dfrac{8+\sin 2t}{10}z(t) \\ \qquad -c_{31}(t)x(t) - c_{32}(t)y(t)\Big) - \dfrac{8+\cos 2t}{10}. \end{cases} \tag{12}$$

In this case, $a_1(t) = 3+\sin t, b_1(t) = \frac{4+\sin t}{10}, c_{12}(t) = c_{12}(t+2\pi), c_{13}(t) = c_{13}(t+2\pi), h_1(t) = \frac{9+\cos t}{20}, a_2(t) = 3+\cos t, b_2(t) = \frac{5+\cos t}{10}, c_{21}(t) = c_{21}(t+2\pi), c_{23}(t) = c_{23}(t+2\pi), h_2(t) = \frac{2+\cos t}{5}, a_3(t) = 3+\sin 2t, b_3(t) = \frac{8+\sin 2t}{10}, c_{31}(t) = c_{31}(t+2\pi), c_{32}(t) = c_{32}(t+2\pi)$ and $h_3(t) = \frac{8+\cos 2t}{10}$. Since

$$a_1^l = a_2^l = a_3^l = 2,\ 2\sqrt{b_1^M h_1^M} = 1,$$
$$2\sqrt{b_2^M h_2^M} = \frac{6}{5},\ 2\sqrt{b_3^M h_3^M} = \frac{18}{10},$$

then

$$2 = a_1^l > 2\sqrt{b_1^M h_1^M} = 1,\ 2 = a_2^l > 2\sqrt{b_2^M h_2^M} = \frac{6}{5},$$
$$2 = a_3^l > 2\sqrt{b_3^M h_3^M} = \frac{18}{10}.$$

Therefore, all conditions of Theorem 1 are satisfied. By Theorem 1, system (12) has at least eight positive $2\pi$-periodic solutions.

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