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On weakly prime and weakly quasi-prime fuzzy left ideals in ordered semigroups

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Abstract—In this paper, we first introduce the concepts of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup S. Furthermore, we give some characterizations of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup S by the ordered fuzzy points and fuzzy subsets of S.

Keywords—Ordered semigroup; ordered fuzzy point; weakly prime fuzzy left ideal; weakly quasi-prime fuzzy left ideal.

I. INTRODUCTION

There has been a rapid growth worldwide in the interest of fuzzy set theory and its applications from the past several years. Evidence of this can be found in the increasing number of high-quality research articles on fuzzy sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences held every year. It seems that the fuzzy set theory deals with the applications of fuzzy technology in information processing. The information processing is already important and it will certainly increase in importance in the future. The fundamental concept of a fuzzy set, introduced by L. A. Zadeh, provides a natural frame-work for generalizing several basic notions of algebra. Following the terminology given by Zadeh, if S is an ordered semigroup, fuzzy sets in ordered semigroups S were first considered by Kehayopulu and Tsingelis in [1], then they defined " fuzzy" analogous for several notations, which have proven useful in the theory of ordered semigroups. Moreover, they proved that each ordered groupoid can be embedded into an ordered groupoid having the greatest element (poe-groupoid) in terms of fuzzy sets [2]. A theory of fuzzy sets on ordered semigroups has been recently developed (see [3-8,12,13]). The concept of ordered fuzzy points of an ordered semigroup S was introduced by Xie and Tang [3], and prime fuzzy ideals of an ordered semigroup S were studied in [4]. Authors also introduced the concepts of weakly prime fuzzy ideals, completely prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of an ordered semigroup S, and established the relations among five types ideals. Furthermore, Xie and Tang characterize weakly prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of S by their level ideals [3].

As we know, fuzzy ideals (left, right ideals) with special properties of ordered semigroups always play an important role in the study of ordered semigroups structure. The ordered fuzzy points of an ordered semigroup S are key tools to describe the algebraic subsystems of S. Motivated by the study of prime fuzzy ideals in rings, semigroups and ordered semigroups, and also motivated by Kehayopulu and Tsingelis's works in ordered semigroups in terms of fuzzy subsets, in this paper we attempt to introduce and give a detailed investigation of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup S. We characterize weakly prime and weakly quasi-prime fuzzy left ideals of S by ordered fuzzy points of S. Furthermore, we prove that in commutative ordered semigroups the weakly prime, quasi-prime and weakly quasi-prime fuzzy left ideals of S are the same. Finally, we describe the largest fuzzy ideal $i(f)$ of S contained in a fuzzy left ideal f of S and the largest fuzzy subsemigroup $I(f)$ of S such that the fuzzy left ideal f is a fuzzy ideal of $I(f)$. The paper illustrates that one can pass from the theory of semigroups or ordered semigroups to the theory of "fuzzy" ordered semigroups. As an application of the results of this paper, the corresponding results of semigroup (without order) are also obtained.

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II. PRELIMINARIES AND SOME NOTATIONS

Throughout this paper unless stated otherwise S stands for an ordered semigroup, that is, a semigroup S with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$ (for example, see [9]). For convenience we use the notation $S^1 := S \cup \{1\}$, where $1 \cdot a = a \cdot 1 := 1$ for all $a \in S$ and $1 \cdot 1 = 1$. A nonempty subset I of S is called a *left* (resp. *right*) *ideal* of S if

(1) $SI \subseteq I($ resp. $IS \subseteq I)$, and

(2) If $a \in I, b \leq a$ with $b \in S$, then $b \in I$.

I is called an *ideal* of S if I is both a left ideal and a right ideal of S [9]. Let L be a left ideal of S. L is called *weakly prime* if for any two ideals A, B of S such that $AB \subseteq L$, it implies that $A \subseteq L$ or $B \subseteq L$; L is called *quasi-prime* if for any two left ideals L_1 and L_2 of S, $L_1L_2 \subseteq L$, then $L_1 \subseteq L$ or $L_2 \subseteq L$ and L is called *weakly quasi-prime* if for any two left ideals L_1 and L_2 of S such that $L \subseteq L_1, L_2$ and $L_1L_2 \subseteq L$, we have $L_1 = L$ or $L_2 = L$ [10].

For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$. We denote by $L(a)$ (resp. (a)) the left ideal (resp. ideal) of S generated by $a \in S$. Then $L(a) = (a \cup Sa] = (S^1a]$ and $(a) = (a \cup Sa \cup aS \cup SaS] = (S^1aS^1]$ [9].

Lemma 2.1 ([9]): Let S be an ordered semigroup. Then the following statements hold:

(1) $A \subseteq (A] \ \ \forall A \subseteq S$.
(2) *If* $A \subseteq B \subseteq S$*, then* $(A] \subseteq (B]$.
(3) $(A](B] \subseteq (AB] \ \ \forall A, B \in S$.
(4) $((A]] = (A] \ \ \forall A \subseteq S$.
(5) *For every left (resp. right) ideal* T *of* S*, we have* $(T] = T$.
(6) *If* A, B *are left ideals of* S, *then* $(AB]$, $A \cap B$, $A \cup B$ *are left ideals of* S.
(7) $(SaS]$, $(Sa]$ *are an ideal and a left deal of* S, $\forall a \in S$, *respectively.*

A function f from S to the real closed interval $[0, 1]$ is a *fuzzy subset* of S. The ordered semigroup S itself is a fuzzy subset of S such that $S(x) \equiv 1$ for all $x \in S$. Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $f \cap g$, $f \cup g$ are fuzzy subsets of S are defined by

$$(f \cap g)(x) = \min(f(x), g(x)) = f(x) \wedge g(x),$$
$$(f \cup g)(x) = \max(f(x), g(x)) = f(x) \vee g(x)$$

for all $x \in S$, respectively. The set of all fuzzy subsets of S is denoted by $F(S)$. One can easily show that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice with the maximum element S and the minimum element 0, which is a mapping from S into $[0, 1]$ defined by

$$0: \ S \to [0, 1], \ x \mapsto 0(x) := 0, \ \forall x \in S.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $x \in S$, we define $A_x := \{(y, z) \in S \times S | \ x \leq yz\}$. The product $f \circ g$ of f and g is defined by

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} \min\{f(y), g(z)\}, & \text{if } A_x \neq \emptyset \ , \\ \qquad 0 \qquad , & \text{if } A_x = \emptyset. \end{cases}$$

for all $x \in S$. It is well known (cf. [2, Theorem]), that this operation "$\circ$" is associative and $(F(S), \circ, \subseteq)$ is a *poe*-semigroup.

Let A be a nonempty subset of S. We denote by f_A the characteristic mapping of A, that is the mapping of S into $[0, 1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Lemma 2.2 ([1, 3]): Let S *be an ordered semigroup and* $A, B \subseteq S$. *Then* $f_A \circ f_B = f_{(AB]}$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *ideal* of S if (1) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) for all $x, y \in S$ and (2) $x \leq y$ implies $f(x) \geq f(y)$. Equivalent definition: (1) $S \circ f \subseteq f$ (resp. $f \circ S \subseteq f$) and (2) $x \leq y$ implies $f(x) \geq f(y)$ [1, 5]. A *fuzzy ideal* of S is a fuzzy subset of S which is both a fuzzy left and a fuzzy right ideal of S. Let f be a fuzzy left ideal of S. Then f is called is called *quasi-prime* if for any two fuzzy left ideals f_1, f_2 of S such that $f_1 \circ f_2 \subseteq f$, it implies that $f_1 \subseteq f$ or $f_2 \subseteq f$ [11].

Lemma 2.3 ([1]): Let S *be an ordered semigroup and* $\emptyset \neq A \subseteq S$. *Then* A *is a left (resp. right) ideal of* S *if and only if the characteristic mapping* f_A *of* A *is a fuzzy left (resp. right) ideal of* S.

Lemma 2.4 ([3]): Let f *be a fuzzy subset of an ordered semigroup* S. *Then* f *is a strongly convex fuzzy subset of* S *if and only if* $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.

Definition 2.5 ([3]): Let S be an ordered semigroup, $a \in S$ and $\lambda \in [0, 1]$. An *ordered fuzzy point* a_λ of S is defined by the rule that

$$a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

Clearly, an ordered fuzzy point a_λ of S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in sequel.

Lemma 2.6 ([3]): Let a_λ *be an ordered fuzzy point of* S. *Then*

(1) *The fuzzy left ideal generated by* a_λ, *denoted by* $L(a_\lambda)$, *is*

$$(\forall x \in S) \ \ L(a_\lambda)(x) = \begin{cases} \lambda, & \text{if } x \in L(a), \\ 0, & \text{if } x \notin L(a), \end{cases}$$

where $L(a)$ *is a left ideal of* S *generated by* a.

(2) *The fuzzy ideal generated by* a_λ, *denoted by* (a_λ), *is*

$$(\forall x \in S) \ \ (a_\lambda)(x) = \begin{cases} \lambda, & \text{if } x \in (a), \\ 0, & \text{if } x \notin (a), \end{cases}$$

where (a) *is an ideal of* S *generated by* a.

Lemma 2.7 ([3]): Let a_λ, b_μ $(\lambda \neq 0, \mu \neq 0)$ *be ordered fuzzy points of* S, *and* f, g *fuzzy subsets of* S. *Then the following statements are true:*

(1) $(\forall x \in S)(S \circ a_\lambda \circ S)(x) = \begin{cases} \lambda, & \text{if } x \in (SaS], \\ 0, & \text{if } x \notin (SaS], \end{cases}$ *and* $S \circ a_\lambda \circ S$ *is a fuzzy ideal of* S.

(2) $(\forall x \in S)(S \circ a_\lambda)(x) = \begin{cases} \lambda, & \text{if } x \in (Sa], \\ 0, & \text{if } x \notin (Sa], \end{cases}$ *and* $S \circ a_\lambda$ *is a fuzzy left ideal of* S.

(3) $a_\lambda \circ b_\mu = (ab)_{\lambda \wedge \mu}$. *In particular,* $a_\lambda \circ a_\lambda = (a^2)_\lambda$.

(4) $(a_\lambda) = a_\lambda \cup a_\lambda \circ S \cup S \circ a_\lambda \cup S \circ a_\lambda \circ S$, $L(a_r) = a_r \cup S \circ a_r$.

(5) $(a_\lambda)^3 \subseteq S \circ a_\lambda \circ S$.

(6) *If* f, g *are fuzzy ideals of* S, *then* $f \circ g$, $f \cup g$ *are fuzzy ideals of* S.

(7) *If* $f \subseteq g$, *and* $h \in F(S)$, *then* $f \circ h \subseteq g \circ h$, $h \circ f \subseteq h \circ g$.

(8) *If* S *is commutative, then, for every ordered fuzzy point* a_λ *of* S, $S \circ a_\lambda = a_\lambda \circ S$.

The reader is referred to [3, 14, 15] for notation and terminology not defined in this paper.

III. Weakly prime fuzzy left ideals of ordered semigroups

Definition 3.1 : A fuzzy left ideal f of an ordered semigroup S is called *weakly prime* if for any two fuzzy ideals f_1 and f_2, $f_1 \circ f_2 \subseteq f$ implies that $f_1 \subseteq f$ or $f_2 \subseteq f$.

Theorem 3.2: Let S *be an ordered semigroup. Then a fuzzy left ideal* f *of* S *is weakly prime if and only if for any two ordered fuzzy points* $x_r, y_t \in S (rt > 0)$, $x_r \circ S \circ y_t \circ S \subseteq f$ *implies that* $x_r \in f$ *or* $y_t \in f$.

Proof. Let x_r and y_t are ordered fuzzy points of S such that $x_r \circ S \circ y_t \circ S \subseteq f$. Then

$$(S \circ x_r \circ S) \circ (S \circ y_t \circ S) \subseteq S \circ f \subseteq f.$$

Since f is weakly prime and, $S\circ x_r\circ S$ and $S\circ y_t\circ S$ are fuzzy ideals of S, then we have $S\circ x_r\circ S\subseteq f$ or $S\circ y_t\circ S\subseteq f$. Say $S\circ x_r\circ S\subseteq f$, then by Lemma 2.7(5),

$$(x_r)^3\subseteq S\circ x_r\circ S\subseteq f.$$

Thus $x_r\in(x_r)\subseteq f$.

Conversely, let f_1,f_2 be fuzzy ideals of S such that $f_1\circ f_2\subseteq f$. If $f_1\not\subseteq f, f_2\not\subseteq f$, then there exist $x,y\in S$ such that $f_1(x)>f(x)$, $f_2(y)>f(y)$. Let $r=f_1(x), t=f_2(y)$. Then $rt>0$, $x_r\in f_1$, $y_t\in f_2$ and so we have

$$x_r\circ S\circ y_t\circ S\subseteq f_1\circ f_2\subseteq f.$$

By hypothesis, $x_r\in f$ or $y_t\in f$. Say $x_r\in f$, then $f(x)\geq r=f_1(x)$, which is impossible.

Theorem 3.3: Let S be an ordered semigroup. Then a left ideal L of S is weakly prime if and only if f_L is a weakly prime fuzzy left ideal of S.

Proof. Let L be a weakly prime left ideal of S. By Lemma 2.3, f_L is a fuzzy left ideal of S. For any two fuzzy ideals f_1 and f_2 of S, if $f_1\circ f_2\subseteq f_L$, then $f_1\subseteq f_L$, or $f_2\subseteq f_L$. In fact:

If $f_1\not\subseteq f_L$, $f_2\not\subseteq f_L$. Then there exist $x,y\in S$ such that $f_1(x)>f_L(x)$, $f_2(y)>f_L(y)$. Thus $f_1(x)>0, f_2(y)>0$, and $f_L(x)=f_L(y)=0$. It implies that $x,y\notin L$. We now show that there exist $r_1,r_2\in S$ such that $(xr_1yr_2]\not\subseteq L$. Indeed:

If $(xSyS]\subseteq L$, then $(SxS](SyS]\subseteq L$. Since $(SxS]$ and $(SyS]$ are ideals of S and L is a weakly prime left ideal of S, then we have $(SxS]\subseteq L$ or $(SyS]\subseteq L$. Say $(SxS]\subseteq L$, then $(x)^3\subseteq(SxS]\subseteq L$. Thus $x\in(x)\subseteq L$. Impossible. Now let $a\in(xr_1yr_2]$ such that $a\notin L$. Then $f_L(a)=0$, and

$$\begin{aligned}(f_1\circ f_2)(a) &= \bigvee_{(p,q)\in A_a}\min\{f_1(p),f_2(q)\}\\ &\geq \min\{f_1(xr_1),f_2(yr_2)\}\\ &\geq \min\{f_1(x),f_2(y)\}>0,\end{aligned}$$

which contradicts with the fact that $f_1\circ f_2\subseteq f_L$. Therefore f_L is a weakly prime fuzzy left ideal of S.

Conversely, let f_L be a weakly prime fuzzy left ideal of S and A,B are ideals of S such that $AB\subseteq L$. Then, by Lemma 2.1, we have $(AB]\subseteq(L]=L$. Thus, By Lemma 2.7, $f_A\circ f_B=f_{(AB]}\subseteq f_L$. By hypothesis and Lemma 2.3, since f_L is weakly prime, then $f_A\subseteq f_L$ or $f_B\subseteq f_L$, that is, $A\subseteq L$ or $B\subseteq L$.

IV. WEAKLY QUASI-PRIME FUZZY LEFT IDEALS OF ORDERED SEMIGROUPS

Definition 4.1: Let S be an ordered semigroup. A fuzzy left ideal f of S is called *weakly quasi-prime* if for any two fuzzy left ideals f_1 and f_2 of S such that $f\subseteq f_1$, $f\subseteq f_2$ and $f_1\circ f_2\subseteq f$, then $f_1=f$ or $f_2=f$.

Theorem 4.2: Let S be an ordered semigroup. Then a left ideal L of S is weakly quasi-prime if and only if f_L is weakly quasi-prime.

Proof. Let L be a weakly quasi-prime left ideal of S, then, by Lemma 2.3, f_L is a fuzzy left ideal of S. Let f_1 and f_2 be any two fuzzy left ideals of S such that $f_L\subseteq f_1$, $f_L\subseteq f_2$ and $f_1\circ f_2\subseteq f_L$. If $f_1\not\subseteq f_L$, $f_2\not\subseteq f_L$, then there exist $x,y\in S$ such that $f_1(x)>f_L(x)$, $f_2(y)>f_L(y)$, so that $x,y\notin L$ and $f_1(x)>0$, $f_2(y)>0$. Furthermore, there exist $r_1,r_2\in S$ such that

$$(\{r_1x\}\cup L](\{r_2y\}\cup L]\not\subseteq L.$$

Indeed: If $(Sx\cup L](Sy\cup L]\subseteq L$, since $(Sx\cup L]$ and $(Sy\cup L]$ are all left ideals of S containing L, by hypothesis, we have $(Sx\cup L]=L$ or $(Sy\cup L]=L$. Say $(Sx\cup L]=L$, we have $Sx\subseteq L$. Thus

$$\begin{aligned}&(L\cup L(x)](L\cup L(x)]\\ \subseteq\ &(L^2\cup LL(x)\cup L(x)L\cup L(x)L(x)]\\ \subseteq\ &(L\cup Sx]=(L]=L.\end{aligned}$$

Hence $x\in(L\cup L(x)]=L$. Impossible.

Since $(\{r_1x\}\cup L](\{r_2y\}\cup L]\not\subseteq L$, we have $(r_1xr_2y]\not\subseteq L$ or $(Lr_2y]\not\subseteq L$.

1) If $(r_1xr_2y]\not\subseteq L$, then there exists $z\in(r_1xr_2y]$ and $z\notin L$, thus we have

$$\begin{aligned}(f_1\circ f_2)(z) &\geq \min\{f_1(r_1x),f_2(r_2y)\}\\ &\geq \min\{f_1(x),f_2(y)\}>0.\end{aligned}$$

But $f_L(z)=0$. Impossible.

2) If $(Lr_2y]\not\subseteq L$, then there exist $l\in L$ and $w\in S$ such that $w\in(lr_2y]$ and $w\notin L$, that is, $f_L(w)=0$. But

$$\begin{aligned}(f_1\circ f_2)(w) &\geq \min\{f_1(l),f_2(r_2y)\}\\ &\geq \min\{f_1(l),f_2(y)\}\\ &= f_2(y)>0,\end{aligned}$$

which is impossible. Therefore we have $f_1\subseteq f_L$ or $f_2\subseteq f_L$.

Conversely, If f_L is weakly quasi-prime left ideal of S, then L is weakly quasi-prime. The proof is a simple modification of the proof of reversed part of Theorem 3.3.

Theorem 4.3: Let S be a commutative ordered semigroup and f a fuzzy left ideal of S. Then the following statements are equivalent:

(1) *f is weakly prime;*

(2) *f is quasi-prime;*

(3) *f is weakly quasi-prime.*

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are obvious.

(3)$\Longrightarrow$ (1). Let $f_1\circ f_2\subseteq f$ for any two fuzzy ideals f_1 and f_2 of S. Since S is commutative, so we have f is a fuzzy ideal of S and

$$(f_1\cup f)\circ(f_2\cup f)\subseteq f_1\circ f_2\cup f_1\circ f\cup f\circ f_2\cup f^2\subseteq f.$$

By (3), $f_1\cup f=f$ or $f_2\cup f=f$, that is, $f_1\subseteq f$ or $f_2\subseteq f$.

Theorem 4.4: Let f be a fuzzy left ideal of an ordered semigroup S. Then the following statements are equivalent:

(1) *f is a weakly quasi-prime fuzzy left ideal of S.*

(2) *For any two fuzzy left ideals f_1 and f_2 of S, if $(f_1\cup f)\circ(f_2\cup f)\subseteq f$, then $f_1\subseteq f$ or $f_2\subseteq f$.*

(3) *For any two fuzzy left ideals f_1 and f_2 of S, if $f\subseteq f_1$ and $f_1\circ f_2\subseteq f$, then $f_1=f$ or $f_2\subseteq f$.*

(4) *For any two fuzzy left ideals f_1 and f_2 of S, if $(f_1\cup f)\circ f_2\subseteq f$, then $f_1\subseteq f$ or $f_2\subseteq f$.*

(5) *For any two ordered fuzzy points* $a_r, b_t \in S(rt > 0)$, *if* $(a_r \cup f) \circ S \circ (b_t \cup f) \subseteq f$, *then* $a_r \in f$, *or* $b_t \in f$.

Proof. (1)$\Longrightarrow$ (2). Since $f_1 \cup f$ and $f_2 \cup f$ are fuzzy left ideals of S such that $f \subseteq f_1 \cup f$, $f \subseteq f_2 \cup f$ and $(f_1 \cup f) \circ (f_2 \circ f) \subseteq f$, then we have $f = f_1 \cup f$ or $f = f_2 \cup f$, that is, $f_1 \subseteq f$ or $f_2 \subseteq f$.

(2)$\Longrightarrow$ (3). For two fuzzy left ideals f_1 and f_2 of S, if $f \subseteq f_1$ and $f_1 \circ f_2 \subseteq f$, then

$$(f \cup f_1) \circ (f_2 \cup f) \subseteq f_1 \circ (f \cup f_2) \subseteq (f_1 \circ f) \cup (f_1 \circ f_2) \subseteq f.$$

By (2), we have $f_1 \subseteq f$ or $f_2 \subseteq f$, so that $f_1 = f$ or $f_2 \subseteq f$.

(3)$\Longrightarrow$ (4). Since $f \subseteq f \cup f_1$ and $(f \cup f_1) \circ f_2 \subseteq f$, by (3) we have $f_1 \cup f = f$ or $f_2 \subseteq f$. It thus follows that $f_1 \subseteq f$ or $f_2 \subseteq f$.

(4)$\Longrightarrow$ (5). Let $a_r, b_t (rt > 0)$ be two ordered fuzzy points of S such that $(a_r \cup f) \circ S \circ (b_t \cup f) \subseteq f$. Then $a_r \circ S \circ b_t \subseteq f$ and $f \circ S \circ b_t \subseteq f$. Thus

$$\begin{aligned} & (a_r \cup S \circ a_r \cup f) \circ (S \circ b_t) \\ \subseteq\ & (a_r \circ S \circ b_t \cup S \circ a_r \circ S \circ b_t) \cup (f \circ S \circ b_t) \\ \subseteq\ & (f \cup S \circ f) \cup f \subseteq f. \end{aligned}$$

Since $a_r \cup S \circ a_r = L(a_r)$ and $S \circ b_t$ are fuzzy left ideals of S. By (4), we have $L(a_r) \subseteq f$ or $S \circ b_t \subseteq f$. If $L(a_r) \subseteq f$, then $a_r \in L(a_r) \subseteq f$. If $S \circ b_t \subseteq f$, then

$$(L(b_t) \cup f) \circ L(b_t) \subseteq L(b_t)^2 \cup f \circ L(b_t) \subseteq S \circ b_t \subseteq f.$$

By (4), we have $b_t \in L(b_t) \subseteq f$.

(5)$\Longrightarrow$ (1). Let f_1 and f_2 be two fuzzy left ideals of S such that $f \subseteq f_1$, $f \subseteq f_2$ and $f_1 \circ f_2 \subseteq f$. If $f_1 \neq f$, $f_2 \neq f$, then there exist $x, y \in S$ such that $f_1(x) > f(x)$, $f_2(y) > f(y)$. Let $r = f_1(x), t = f_2(y)$. Then $rt > 0$ and

$$(x_r \cup f) \circ S \circ (y_t \cup f) \subseteq f_1 \circ S \circ f_2 \subseteq f_1 \circ f_2 \subseteq f.$$

By (5), we have $x_r \in f$ or $y_t \in f$, which is impossible.

Theorem 4.5: Let f be a fuzzy left ideal of an ordered semigroup S and μ a fuzzy subset of S satisfying that:

(1) $f \cap \mu = 0$;

(2) *For any* $a_t, b_r \in \mu$, $((a_t \cup f) \circ S \circ (b_r \cup f)) \cap \mu \neq 0$.

If g is a maximal fuzzy left ideal of S with respect to containing f and $g \cap \mu = 0$, then g is a weakly quasi-prime fuzzy left ideal of S.

Proof. Let f_1 and f_2 be any two fuzzy left ideals of S such that $g \subseteq f_1, f_2$ and $f_1 \circ f_2 \subseteq g$. Then $g = f_1$ or $g = f_2$. Indeed: If $g \subset f_1$ and $g \subset f_2$, then there exist $a_t \in f_1 \backslash g$, $b_r \in f_2 \backslash g$ $(rt > 0)$. Thus

$$g \subset g \cup L(a_t) \subseteq f_1, g \subset g \cup L(b_r) \subseteq f_2.$$

By hypothesis, we have

$$(g \cup L(a_t)) \cap \mu \neq 0, (g \cup L(b_r)) \cap \mu \neq 0.$$

Let

$$c_k \in (g \cup L(a_t)) \cap \mu (k > 0), d_l \in (g \cup L(b_r)) \cap \mu (l > 0).$$

Then

$$\begin{aligned} (c_k \cup f) \circ S \circ (d_l \cup f) &\subseteq (f_1 \cup f) \circ S \circ (f_2 \cup f) \\ &\subseteq (f_1 \circ S \circ f_2) \cup (f_1 \circ S \circ f) \\ &\quad \cup (f \circ S \circ f_2) \cup (f \circ S \circ f) \\ &\subseteq f_1 \circ S \circ f_2 \subseteq f_1 \circ f_2 \subseteq g. \end{aligned}$$

Consequently, $((c_k \cup f) \circ S \circ (d_l \cup f)] \cap \mu = 0$, which contradicts with the fact that

$$((c_k \cup f) \circ S \circ (d_l \cup f)] \cap \mu \neq 0.$$

V. Fuzzy ideals $i(f)$ and $I(f)$ of ordered semigroups

Let f be a fuzzy left ideal of an ordered semigroup S, we define two fuzzy subsets of S, denoted by $i(f)$ and $I(f)$ respectively, as follows:

$$i(f)(x) = \bigvee \{t_\alpha \mid x_{t_\alpha} \in f, x_{t_\alpha} \circ S \subseteq f, t_\alpha \in [0, 1]\}$$

and

$$I(f)(x) = \bigvee \{t_\alpha \mid f \circ x_{t_\alpha} \subseteq f, t_\alpha \in [0, 1]\}$$

for any $x \in S$.

Theorem 5.1: Let f be a left ideal of an ordered semigroup S. Then $i(f)$ is the largest fuzzy ideal of S contained in f.

Proof. For any $x, y \in S$, let

$$\begin{aligned} A(x) &= \{t_\alpha \mid x_{t_\alpha} \in f, x_{t_\alpha} \circ S \subseteq f, t_\alpha \in [0, 1]\}, \\ B(xy) &= \{w_\beta \mid (xy)_{w_\beta} \in f, (xy)_{w_\beta} \circ S \subseteq f, w_\beta \in [0, 1]\}, \\ C(y) &= \{q_\gamma \mid y_{q_\gamma} \in f, y_{q_\gamma} \circ S \subseteq f, q_\gamma \in [0, 1]\}. \end{aligned}$$

Then $i(f)(x) = \bigvee_{t_\alpha \in A(x)} t_\alpha$, $i(f)(xy) = \bigvee_{w_\beta \in B(xy)} w_\beta$, and $i(f)(y) = \bigvee_{q_\gamma \in C(y)} q_\gamma$. If $t_\alpha \in A(x)$, then

$$\begin{aligned} x_{t_\alpha} \circ y_{t_\alpha} &= (xy)_{t_\alpha} \in x_{t_\alpha} \circ S \subseteq f, \\ (xy)_{t_\alpha} \circ S &= x_{t_\alpha} \circ (y_{t_\alpha} \circ S) \subseteq x_{t_\alpha} \circ S \subseteq f. \end{aligned}$$

Thus $t_\alpha \in B(xy)$. It follows that $i(f)(xy) \geq i(f)(x)$. By the same way, we can prove that $i(f)(xy) \geq i(f)(y)$. Also, let $x \leq y$. For any $q_r \in [0, 1]$, it is clear that $x_{q_r} \subseteq y_{q_r}$. If $q_r \in C(y)$, then $y_{q_r} \in f, y_{q_r} \circ S \subseteq f$. So we have $x_{q_r} \in f, x_{q_r} \circ S \subseteq f$. It implies that $q_r \in A(x)$, it thus follows that $i(f)(x) \geq i(f)(y)$. Therefore, $i(f)$ is a fuzzy ideal of S. For any $x \in S$, let $t_\alpha \in A(x)$. Then $x_{t_\alpha} \in f$, that is, $t_\alpha \leq f(x)$. Thus $i(f)(x) \leq f(x)$. Let g be a fuzzy ideal of S such that $g \subseteq f$, and $x_t \in g \subseteq f$. Then $t \in A(x)$, and so we have

$$g(x) = \bigvee_{x_t \in g} t \leq i(f)(x).$$

Therefore $g \subseteq i(f)$.

Lemma 5.2 ([11]): Lemma 5.2Let S be an ordered semigroup. Then a fuzzy left ideal f of S is quasi-prime if and only if for any two ordered fuzzy points $x_r, y_t \in S(rt > 0)$, $x_r \circ S \circ y_t \subseteq f$ implies that $x_r \in f$ or $y_t \in f$.

Theorem 5.3: Let S be an ordered semigroup with an identity e, and f a weakly prime fuzzy left ideal of S. If $i(f) \neq 0$, then $i(f)$ is a quasi-prime fuzzy ideal of S.

Proof. Let a_t and b_r $(tr > 0)$ be any two ordered fuzzy points of S such that $a_t \circ S \circ b_r \subseteq i(f)$. Then

$$(S \circ a_t \circ S) \circ (S \circ b_r \circ S) \subseteq i(f) \subseteq f.$$

Since f is weakly prime, we have $S \circ a_t \circ S \subseteq f$ or $S \circ b_r \circ S \subseteq f$. Say $S \circ a_t \circ S \subseteq f$. By Theorem 5.1 , we have $S \circ a_t \circ S \subseteq i(f)$. Since S has an identity e, so that

$$a_t = (eae)_t = e_t \circ a_t \circ e_t \subseteq S \circ a_t \circ S \subseteq i(f).$$

By Lemma 5.2, the proof is completed.

Theorem 5.4: Let f be a fuzzy left ideal of an ordered semigroup S. Then $I(f)$ is the largest strongly convex fuzzy subsemigroup of S such that f is a fuzzy ideal of $I(f)$.

Proof. For any $x, y \in S$, let $f(x) = t$. Then $x_t \in f$, and let

$$\begin{aligned} D(x) &= \{t_\alpha \mid f \circ x_{t_\alpha} \subseteq f, t_\alpha \in [0,1]\}, \\ E(y) &= \{r_\beta \mid f \circ y_{r_\beta} \subseteq f, r_\beta \in [0,1]\}. \end{aligned}$$

(1) Since $f \circ x_t \subseteq f \circ f \subseteq f$, thus $t \in D(x)$, it follows that

$$f(x) = t \leq \bigvee_{t_\alpha \in D(x)} t_\alpha = I(f)(x),$$

that is, $I(f)$ is a fuzzy subset of S containing f.

(2) $I(f)$ is a strongly convex fuzzy subsemigroup of S. In fact:

$$\begin{aligned} I(f)(x) \wedge I(f)(y) &= (\bigvee_{t_\alpha \in D(x)} t_\alpha) \wedge (\bigvee_{r_\beta \in E(y)} r_\beta) \\ &= \bigvee_{t_\alpha \in D(x), r_\beta \in E(y)} (t_\alpha \wedge r_\beta). \end{aligned}$$

Since

$$f \circ (x_{t_\alpha} \circ y_{r_\beta}) = (f \circ x_{t_\alpha}) \circ y_{r_\beta} = f \circ (xy)_{t_\alpha \wedge r_\beta} \subseteq f \circ y_{r_\beta} \subseteq f,$$

so we have $t_\alpha \wedge r_\beta \leq I(f)(xy)$. Thus

$$I(f)(x) \wedge I(f)(y) \leq I(f)(xy).$$

Furthermore, let $x \leq y$. For any $r_\beta \in [0,1]$, it is clear that $x_{r_\beta} \subseteq y_{r_\beta}$. If $r_\beta \in E(y)$, then $f \circ x_{r_\beta} \subseteq f \circ y_{r_\beta} \subseteq f$. Which implies that $r_\beta \in D(x)$, it thus follows that $I(f)(x) \geq I(f)(y)$. Therefore, $I(f)$ is a strongly convex fuzzy subsemigroup of S.

(3) f is a fuzzy ideal of $I(f)$. In fact:

Since f is a left ideal of S, then we have $I(f) \circ f \subseteq S \circ f \subseteq f$. Furthermore, for any $x \in S$, if x can be expressible as $x \leq yz$, let

$$F(z) = \{q_\gamma \mid f \circ z_{q_\gamma} \subseteq f, q_\gamma \in [0,1]\}.$$

Then

$$\begin{aligned} (f \circ I(f))(x) &= \bigvee_{(y,z) \in A_x} (f(y) \wedge I(f)(z)) \\ &= \bigvee_{(y,z) \in A_x} (f(y) \wedge \bigvee_{q_\gamma \in F(z)} q_\gamma) \\ &= \bigvee_{(y,z) \in A_x} \bigvee_{q_\gamma \in F(z)} (f(y) \wedge q_\gamma). \end{aligned}$$

Since for any $q_\gamma \in F(z)$, $f \circ z_{q_\gamma} \subseteq f$, we have

$$f(x) \geq f(y) \wedge z_{q_\gamma}(z) \geq f(y) \wedge q_\gamma.$$

Thus we have

$$\begin{aligned} f(x) &\geq \bigvee_{(y,z) \in A_x} \bigvee_{q_\gamma \in F(z)} (f(y) \wedge q_\gamma) \\ &= (f \circ I(f))(x). \end{aligned}$$

If $A_x = \emptyset$, then $f(x) \geq 0 = (f \circ I(f))(x)$. Therefore, $f \circ I(f) \subseteq f$.

(4) Let g be a strongly convex fuzzy subsemigroup of S such that f is a fuzzy ideal of g. For any $x \in S$, if $g(x) = t$, then $x_t \in g$, and $f \circ x_t \subseteq f \circ g \subseteq f$. Thus $t \in D(x)$, which implies that $I(f)(x) \geq t = g(x)$. Therefore $I(f)$ is the largest fuzzy subsemigroup of S such that f is a fuzzy ideal of $I(f)$.

Theorem 5.5: Let S be an ordered semigroup with an identity e and f a fuzzy left ideal of S but not a fuzzy ideal of S. Then the following statements are equivalent:

(1) f is a weakly quasi-prime fuzzy left ideal of S.

(2) If f_1 is a fuzzy left ideal of S such that $f \circ f_1 \subseteq f$, then $f_1 \subseteq f$.

(3) For any ordered fuzzy point $a_t \in S$, if $f \circ (S \circ a_t) \subseteq f$, then $a_t \in f$.

(4) f is the largest fuzzy left ideal of S contained in $I(f)$.

Proof. (1)$\Longrightarrow$ (2). Let f_1 be a fuzzy left ideal of S such that $f \circ f_1 \subseteq f$. Then $(f \circ S) \circ f_1 \subseteq f \circ f_1 \subseteq f$, where $f \circ S$ is a fuzzy ideal of S. Since S has an identity e, for any $x \in S$,

$$(f \circ S)(x) = \bigvee_{(y,z) \in A_x} (f(y) \wedge S(z)) \geq f(x) \wedge S(e) = f(x).$$

Then $f \subseteq f \circ S$. Since f is not a fuzzy ideal of S, then $f \neq f \circ S$. By Theorem 4.4(3), it follows that $f_1 \subseteq f$.

(2)$\Longrightarrow$ (3). For any ordered fuzzy point $a_t \in S$, if $f \circ (S \circ a_t) \subseteq f$, since $S \circ a_t$ is a fuzzy left ideal of S, by (2), $S \circ a_t \subseteq f$. Since S has an identity e, so

$$(S \circ a_t)(a) = \bigvee_{(y,z) \in A_x} (S(y) \wedge a_t(z)) \geq S(e) \wedge a_t(a) = t.$$

Thus, $a_t \in S \circ a_t \subseteq f$.

(3)$\Longrightarrow$ (4). By Theorem 5.4, $f \subseteq I(f)$. Let g be a fuzzy left ideal of S such that $g \subseteq I(f)$. Then $g \subseteq f$. Indeed:

For any $a_t \in g$, since f is a fuzzy ideal of $I(f)$, we have

$$f \circ (S \circ a_t \subseteq f \circ (S \circ g) \subseteq f \circ g \subseteq f \circ I(f) \subseteq f,$$

by (3), we have $a_t \in f$. Thus $g = \bigvee_{a_t \in g} a_t \subseteq f$.

(4)$\Longrightarrow$ (1). Let f_1 and f_2 be any two fuzzy left ideals of S such that $f \subseteq f_1, f \subseteq f_2$ and $f_1 \circ f_2 \subseteq f$. Since $f \circ f_2 \subseteq f_1 \circ f_2 \subseteq f$, then for any $x_t \in f_2$, $f \circ x_t \subseteq f$ holds. Thus $I(f)(x) \geq t = x_t(x)$, which implies that $x_t \in I(f)$. Therefore $f_2 = \bigvee_{x_t \in f_2} x_t \subseteq I(f)$. Since f is the largest fuzzy left ideal of S contained in $I(f)$, then $f_2 \subseteq f$. Thus $f = f_2$.

VI. Conclusion

As we know, fuzzy ideals (left, right ideals) of an ordered semigroup with special properties always play an important role in the study of ordered semigroups structure. The ordered fuzzy points of an ordered semigroup are key tools to describe the algebraic subsystems of ordered semigroups. In this paper we have introduced the concepts of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup. Furthermore, we have given some characterizations of weakly prime and weakly quasi-prime fuzzy left ideals of an ordered semigroup by the ordered fuzzy points and fuzzy subsets. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of ordered semigroups. Hopefully, some new results in these topics can be obtained in the forthcoming paper.

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