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Some results of sign patterns allowing simultaneous unitary diagonalizability

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Abstract—Allowing diagonalizability of sign pattern is still an open problem. In this paper, we make a carefully discussion about allowing unitary diagonalizability of two sign pattern. Some sufficient and necessary conditions of allowing unitary diagonalizability are also obtained.

Keywords—Sign pattern; Unitary diagonalizability ; Eigenvalue; Allowing diagonalizability

I. Introduction and preliminaries

THE origins of sign pattern matrix are the need to solve certain problems in economics and other areas based only on the signs of the entries of the matrices. Now this matrix branch has been widely developed. The eigen-problem is an important research field in both the tradition and sign pattern matrix, and this often establish relationships with the diagonalizability of matrix. In this paper, we mainly consider sign patterns that allow simultaneously unitary diagonalizability. The question of characterizing sign patterns that allow diagonalizability is an open problem(see [1]). Here we introduce some definitions and notations.

A sign pattern (matrix) is a matrix whose entries are in the set $\{+,-,0\}$. The set of all $n \times n$ sign patterns is denoted by Q_n. For $A = (a_{ij}) \in Q_n$, associated with A is a class of real matrices, called the qualitative class of A, defined by $Q(A) = \{B = (b_{ij}) \in M_n(R) \,|\, \mathrm{sign} b_{ij} = a_{ij}\ for$ all i and $j\}$ and $S(B) = A$, for any $B \in Q(A)$.

A generalized sign pattern (matrix) is a matrix whose entries are in the set $\{+,-,0,\#\}$, where $\#$ indicates an ambiguous sum (the result of adding $+$ with $-$). In this paper, we mainly study sign pattern. Although the matrices we study are sign patterns, the product of sign patterns may be generalized. In this paper, for generalized sign pattern, we say, two matrix is equal to, if the corresponding entries whose are in the set $\{+,-,0,\#\}$ are uniform in the two matrix.

Let P be a property referring to a real matrix. For a sign pattern A, if there exists a real matrix $B \in Q(A)$ such that B has property P, then we say A allows P. The signed digraph of an $n \times n$ sign pattern $A = (a_{ij})$, denoted by $D(A)$, is the digraph with vertex set $\{1, 2, \cdots, n\}$, where (i, j) is an arc if only and if $a_{ij} \neq 0$. Let $A = (a_{ij})$ be an $n \times n$ sign pattern. A nonzero product of the form

$$P = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_{k+1}},$$

in which the index set $\{i_1, \cdots i_{k+1}\}$ consists of distinct indices is called a path of length k (or k-path). i_1 and i_{k+1} are called initial vertex and terminal vertex of P. A nonzero product of the form

$$\gamma = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1},$$

in which the index set $\{i_1, \cdots i_k\}$ consists of distinct indices is called a simple cycle of length k (or simple k-cycle). Each $i_m (m = 1, \cdots, k)$ is called a vertex of γ. A composite k-cycle is a product of simple cycles whose total length is k and whose index sets are mutually disjoint.

Let $A \in Q_n$. We define $MR(A)$, the maximal rank of A by

$$MR(A) = \max\{\mathrm{rank} B \,|\, B \in Q(A)\}\ .$$

Similarly, the minimal rank of A, $mr(A)$, is

$$mr(A) = \min\{\mathrm{rank} B \,|\, B \in Q(A)\}\ .$$

A sign pattern A is called normal, if $AA^T = A^T A$.

II. Allowing simultaneously unitary diagonalizability of sign patterns

In this section, we consider two sign patterns allowing simultaneous unitary diagonalizability.

Definition 2.1. *Let $A \in Q_n$. If there exists a real matrix $B \in Q(A)$ such that B has property $BB^T = B^T B$, then we say A allows unitary diagonalizability.*

Lemma 2.1. *$A, B \in Q_n$ are sign patterns allowing simultaneous unitary diagonalizability if and only if there exist $A_0 \in Q(A), B_0 \in Q(B)$ such that A_0 and B_0 are simultaneous diagonalizable and $A_0 B_0 = B_0 A_0$.*

Proof. $A, B \in Q_n$ are sign patterns allowing simultaneous diagonalizability if and only if there exist $A_0 \in Q(A), B_0 \in Q(B)$ such that A_0 and B_0 are simultaneous diagonalizable. This holds if and only if $A_0 B_0 = B_0 A_0$.

Theorem 2.1. *If A and B are two nonnegative sign patterns allowing simultaneous unitary diagonalizability, then $AB = BA$.*

Proof. By Lemma 2.1, $A, B \in Q_n$ are two sign patterns allowing simultaneous diagonalizability if and only if there exist $A_0 \in Q(A), B_0 \in Q(B)$ such that $A_0 B_0 = B_0 A_0$. Because A and B are nonnegative, the proof is similar to that of Lemma 2.1. If $(A_0 B_0)_{ij} = 0 (i, j = 1, \cdots, n)$, then

$$(AB)_{ij} = 0.$$

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Likewise, if $(A_0B_0)_{ij} > 0$, then

$$(AB)_{ij} = +.$$

Vice versa, if $(B_0A_0)_{ij} = 0$, then $(BA)_{ij} = 0$, and if $(B_0A_0)_{ij} > 0$, then $(BA)_{ij} = +$. Therefore, according to $A_0B_0 = B_0A_0$, $AB = BA$ holds.
Similarly, we can easily obtain the following result:
Corollary 2.1. *Let $A, B \in Q_n$ be sign patterns allowing simultaneous diagonalizability. $(AB)_{ij} = \#$ if and only if $(BA)_{ij} = \#,\ i, j = 1, \cdots, n$, then $AB = BA$.*

Lemma 2.2. [2, Corollary 3.7] *Let $A, B \in M_n$ be two nonsingular Hermitian matrices simultaneously unitary diagonalizable. Then, there is a Hermitian matrix $X \in M_n$ such that $B = XAX$ if and only if there is a unitary matrix $V \in M_n$ such that V^*AV and V^*BV are diagonal matrices of the forms*

$$V^*AV = S_A \oplus A_1 \oplus \cdots \oplus A_l,\ \ V^*BV = S_B \oplus B_1 \oplus \cdots \oplus B_l,$$

where $sign(S_A) = sign(S_B)$, and $A_i, B_i \in M_2$ are indefinite matrices such that B_i is a negative multiple of $A_i^{-1}, i = 1, \cdots, l$.

By Lemma 2.2, we can easily obtain the following theorem:
Theorem 2.2. *Let $A, B \in Q_n$ be two symmetric sign patterns allowing simultaneous unitary diagonalizability and $MR(A) = MR(B) = n$. Then, there exist a symmetric matrix $X \in M_n$ and nonsingular $A_0 \in Q(A), B_0 \in Q(B)$ such that $B_0 = XA_0X$ if and only if there is an orthogonal matrix $V \in M_n$ such that V^*A_0V and V^*B_0V are diagonal matrices of the forms*

$$V^*A_0V = S_A \oplus A_1 \oplus \cdots \oplus A_l,\ \ V^*B_0V = S_B \oplus B_1 \oplus \cdots \oplus B_l,$$

where $sign(S_A) = sign(S_B)$, and $A_i, B_i \in M_2$ are indefinite matrices such that B_i is a negative multiple of $A_i^{-1}, i = 1, \cdots, l$.

Theorem 2.3. *Let $A, B \in Q_n$ be two nonnegative symmetric sign patterns allowing simultaneous unitary diagonalizability. If there are nonsingular $A_0 \in Q(A), B_0 \in Q(B)$ and a nonnegative real symmetric matrix X_0 such that $B_0 = X_0A_0X_0$, then there exists a symmetric sign pattern matrix X such that $B = XAX$.*
Proof. This theorem can be proved by using similar methods of Theorem 2.1.
Corollary 2.2. *Let $A, B \in Q_n$ be two symmetric sign patterns allowing simultaneous unitary diagonalizability. If there are $A_0 \in Q(A), B_0 \in Q(B)$ and a nonnegative real symmetric matrix X_0 such that $B_0 = X_0A_0X_0$, and there is not $\#$ in product of $S(X_0)AS(X_0)$, then there exists a symmetric sign pattern matrix $X = S(X_0)$ such that $B = XAX$.*
Proof. If there is not $\#$ in product of $S(X_0)BS(X_0)$, by $B_0 = X_0A_0X_0$, we have

$$sign((X_0A_0X_0)_{ij}) = (XAX)_{ij},\ for\ all\ i, j = 1, \cdots, n.$$

Moreover, $sign((B_0)_{ij}) = (B)_{ij}$, for all $i, j = 1, \cdots, n$. Thus $B = XAX$ holds.

Lemma 2.3. *Let A and B be two $n \times n$ nonsingular simultaneous diagonalizable normal real matrices. Let the eigenvalues of A be $a_1, \cdots, a_k, \alpha_{k+1} + i\beta_{k+1}, \cdots, a_p + i\beta_p$, and the eigenvalues of B be $b_1, \cdots, b_k, \gamma_{k+1} + i\omega_{k+1}, \cdots, \gamma_p + i\omega_p$. If*

$$\begin{cases} a_ib_i = a_jb_j & i, j = 1, \cdots, k, \\ \alpha_i\omega_j = \beta_i\gamma_j & i, j = k+1, \cdots, n, \end{cases}$$

then there exists a nonsingular symmetric matrix X such that $B = XAX$.

Proof. Let A and B be two nonsingular simultaneous diagonalizable normal real matrices, and there exists real orthogonal matrix Q such that

$$Q^TAQ = \begin{pmatrix} a_1 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & a_k & & & & & \\ & & & \alpha_{k+1} & \beta_{k+1} & & & \\ & & & -\beta_{k+1} & \alpha_{k+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \alpha_p & \beta_p \\ 0 & & & & & & -\beta_p & \alpha_p \end{pmatrix}$$

and

$$Q^TBQ = \begin{pmatrix} b_1 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & b_k & & & & & \\ & & & \gamma_{k+1} & \omega_{k+1} & & & \\ & & & -\omega_{k+1} & \gamma_{k+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \gamma_p & \omega_p \\ 0 & & & & & & -\omega_p & \gamma_p \end{pmatrix}.$$

Suppose that there exist a nonsingular symmetric matrix X such that $Q^TAQ = XQ^TBQX$, then

$$X^{-1} = \begin{pmatrix} a_1 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & a_k & & & & & \\ & & & \alpha_{k+1} & \beta_{k+1} & & & \\ & & & -\beta_{k+1} & \alpha_{k+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \alpha_p & \beta_p \\ 0 & & & & & & -\beta_p & \alpha_p \end{pmatrix}^{-1} X$$

$$\cdot \begin{pmatrix} b_1 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & b_k & & & & & \\ & & & \gamma_{k+1} & \omega_{k+1} & & & \\ & & & -\omega_{k+1} & \gamma_{k+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \gamma_p & \omega_p \\ 0 & & & & & & -\omega_p & \gamma_p \end{pmatrix}.$$

We partition the three matrices into 2×2 blocks with the suitable dimension. Then, their product will have the following three kind of equations.

Case 1: $$\begin{pmatrix} a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & a_k \end{pmatrix}^{-1} X_{11} \begin{pmatrix} b_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & b_k \end{pmatrix} =$$

$$\begin{pmatrix} b_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & b_k \end{pmatrix}^T X_{11} \cdot \left(\begin{pmatrix} a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & a_k \end{pmatrix}^{-1} \right)^T$$

where $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ and X_{11} is a symmetric square matrix.

From above equation, we find that, only need let $a_i b_i = a_j b_j,\ i,j = 1,\cdots,k$, the above equation constantly holds. Thus, there exists solution x_{11}.

Case 2:

$$\begin{pmatrix} a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & a_k \end{pmatrix}^{-1} X_{12} \begin{pmatrix} \gamma_{k+1} & \omega_{k+1} & \cdots & 0 & 0 \\ -\omega_{k+1} & \gamma_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_p & \omega_p \\ 0 & 0 & \cdots & -\omega_p & \gamma_p \end{pmatrix} =$$

$$\begin{pmatrix} b_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & b_k \end{pmatrix}^{T} X_{12}.$$

$$\left(\begin{pmatrix} \alpha_{k+1} & \beta_{k+1} & \cdots & 0 & 0 \\ -\beta_{k+1} & \alpha_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_p & \beta_p \\ 0 & 0 & \cdots & -\beta_p & \alpha_p \end{pmatrix}^{-1} \right)^{T}.$$

By $a_i b_i = a_j b_j$, we have

$$X_{12} \begin{pmatrix} \gamma_{k+1} & \omega_{k+1} & \cdots & 0 & 0 \\ -\omega_{k+1} & \gamma_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_p & \omega_p \\ 0 & 0 & \cdots & -\omega_p & \gamma_p \end{pmatrix}$$

$$\cdot \begin{pmatrix} \alpha_{k+1} & \beta_{k+1} & \cdots & 0 & 0 \\ -\beta_{k+1} & \alpha_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_p & \beta_p \\ 0 & 0 & \cdots & -\beta_p & \alpha_p \end{pmatrix}^{T} = a_i b_i X_{12}.$$

Because $\alpha_l \pm i\beta_l$ and $\gamma_l \pm i\omega_l$ are imaginary characteristic root of A and B, X_{12} has a unique solution, $l = k+1,\cdots,p$, and $X_{21} = X_{12}^T$.

Case 3:

$$\begin{pmatrix} \alpha_{k+1} & \beta_{k+1} & \cdots & 0 & 0 \\ -\beta_{k+1} & \alpha_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_p & \beta_p \\ 0 & 0 & \cdots & -\beta_p & \alpha_p \end{pmatrix}^{-1} X_{22}$$

$$\cdot \begin{pmatrix} \gamma_{k+1} & \omega_{k+1} & \cdots & 0 & 0 \\ -\omega_{k+1} & \gamma_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_p & \omega_p \\ 0 & 0 & \cdots & -\omega_p & \gamma_p \end{pmatrix} =$$

$$\begin{pmatrix} \gamma_{k+1} & \omega_{k+1} & \cdots & 0 & 0 \\ -\omega_{k+1} & \gamma_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \gamma_p & \omega_p \\ 0 & 0 & \cdots & -\omega_p & \gamma_p \end{pmatrix}^{T} X_{22}.$$

$$\left(\begin{pmatrix} \alpha_{k+1} & \beta_{k+1} & \cdots & 0 & 0 \\ -\beta_{k+1} & \alpha_{k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_p & \beta_p \\ 0 & 0 & \cdots & -\beta_p & \alpha_p \end{pmatrix}^{-1} \right)^{T}$$

Unfold this equation, then $\alpha_{k+s-1}\omega_{k+s-1} = \gamma_{k+s-1}\beta_{k+s-1}$ $(1 \le s \le n-k+1)$ can make that the above equation has solution X_{22}.

According to above analysis, the proof is completed.

Theorem 2.4. *Let $A, B \in Q_n$ be two nonnegative sign patterns allowing simultaneous unitary diagonalizability. And there exist $A_0 \in Q(A), B_0 \in Q(B)$ such that A_0 and B_0 are two $n \times n$ nonsingular simultaneously diagonalizable normal real matrices. Let the eigenvalues of A_0 be $a_1,\cdots,a_k,\alpha_{k+1}+i\beta_{k+1},\cdots,\alpha_p+i\beta_p$, and the eigenvalues of B_0 be $b_1,\cdots,b_k,\gamma_{k+1}+i\omega_{k+1},\cdots,\gamma_p+i\omega_p$, and*

$$\begin{cases} a_i b_i = a_j b_j & i,j = 1,\cdots,k, \\ \alpha_i \omega_j = \beta_i \gamma_j & i,j = k+1,\cdots,n, \end{cases}$$

then there exists a symmetric sign pattern X such that $B = XAX$ if and only if there does not exist # entries in XAX.

Proof. By Lemma 2.3, we know that there exists X_0 such that $B_0 = X_0 A_0 X_0$. Let $X = X_0$. Because A and B are nonnegative sign patterns, Similar to Corollary 4.2, we can also obtain that $B = XAX$ holds if and only if there does not exist # entries in XAX.

Corollary 2.3. *Let $A, B \in Q_n$ be sign patterns allowing simultaneous unitary diagonalizability. If there are $A_0 \in Q(A), B_0 \in Q(B)$ and a real symmetric matrix X_0 such that $B_0 = X_0 A_0 X_0$, and there is not # in product of $S(X_0)AS(X_0)$, then there exists a symmetric sign pattern matrix $X = S(X_0)$ such that $B = XAX$.*

III. Conclusion

In this paper, we make a discussion about allowing unitary diagonalizability of sign pattern. Some sufficient and necessary conditions of allowing unitary diagonalizability are also obtained. Moreover, the relation of two sign patterns allowing simultaneous unitary diagonalizability is researched.

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