World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:2, No:8, 2008

Stochastic Resonance in Nonlinear Signal Detection

Youguo Wang, and Lenan Wu

***Abstract*—**Stochastic resonance (SR) is a phenomenon whereby the signal transmission or signal processing through certain nonlinear systems can be improved by adding noise. This paper discusses SR in nonlinear signal detection by a simple test statistic, which can be computed from multiple noisy data in a binary decision problem based on a maximum a posteriori probability criterion. The performance of detection is assessed by the probability of detection error Per . When the input signal is subthreshold signal, we establish that benefit from noise can be gained for different noises and confirm further that the subthreshold SR exists in nonlinear signal detection. The efficacy of SR is significantly improved and the minimum of Per can dramatically approach to zero as the sample number increases. These results show the robustness of SR in signal detection and extend the applicability of SR in signal processing.

***Keywords*—**Probability of detection error, signal detection, stochastic resonance.

I. Introduction

STOCHASTIC resonance (SR) is a phenomenon whereby the signal transmission or signal processing through certain nonlinear systems can be improved by adding noise. Most occurrences of SR involve a signal, which is subthreshold (weak signal) and is too weak to elicit a strong response from a single nonlinear system. Addition of noise then brings assistance to the subthreshold signal in eliciting a stronger beneficial response from the single nonlinear system [1]-[3], SR has been observed in a diverse range of physical and biological systems, including neurons and neuron models. The study of SR in signal detection has also received some attentions [4]-[11]. This paper discuss SR by a simple test statistic, which can be computed from multiple noisy data in a binary decision problem based on a maximum a posteriori probability (MAP) criterion. The performance of detection is assessed by the probability of detection error Per . Recently, an equivalent test statistic has been used to improve signal detection in non-Gaussian noise [12]. We calculate the detection error Per for four representative noises. We establish that benefit from noise can be gained and confirm further that the subthreshold SR exists in nonlinear signal detection. When the sample number is raised, the efficacy of SR is significantly improved and the minimum of Per can dramatically approach to zero. These results show the robustness of SR in signal detection and extend the applicability of SR in signal processing.

Manuscript received March 7, 2005.

Youguo Wang is with Department of Applied Mathematics and Physics, Nanjing University of Posts and Telecommunications, Nanjing, 210003, and with Department of Radio Engineering, Southeast University, Nanjing 210096, People's Republic of China (e-mail: guowy2489@126.com).

Lenan Wu is with Department of Radio Eng., Southeast University, Nanjing 210096, People's Republic of China (e-mail: wuln@seu.edu.cn).

II. Nonlinear Signal Detection

Let x is a random signal which assumes values s_1 (hypothesis H_1) or s_0 (hypothesis H_0) with the prior probability P_1 and $P_0 = 1 - P_1$, η is threshold white noise, u is a fixed threshold level, y is a binary output: $y = \begin{cases} 1 & if \quad x+\eta > u \\ 0 & if \quad x+\eta < u \end{cases}$. The binary output y is sampled so as to yield the binary data set ($y_{1,} y_2, \cdots, y_N$). We want to use these data ($y_{1,} y_2, \cdots, y_N$) to detect x . Recently, an important and simple test statistic: $\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ has been used to improve signal detection in non-Gaussian noise [12]. Here, we use an equivalent test statistic $y = \sum_{i=1}^{N} y_i$, which assumes integer values between 0 and N ($y \in R = \{0,1,\cdots,N\}$). A given detector will decide hypothesis H_0 whenever the observation y falls in region R_0 ($\subset R$) or decide H_1 when y falls in the complementary region R_1 ($R_0 \cup R_1 = R$). In doing so, the detector achieves the probability of detection error P_{er} given by:

$$Per = P_1 \sum_{n\in R_0} P_r(y=n \mid H_1) + P_0 \sum_{n\in R_1} P_r(y=n \mid H_0) . \quad (1)$$

Where $\Pr(y=n \mid H_1)$ (respectively $\Pr(y=n \mid H_0)$) is the conditional probability for observing y when H_1 (respectively H_0) holds. Since R_0 and R_1 are complementary, so we have

$$\sum_{n\in R_0} \Pr(y=n \mid H_1) = 1 - \sum_{n\in R_1} \Pr(y=n \mid H_1) . \quad (2)$$

Substituted in (1) yields

$$P_{er} = P_1 + \sum_{n\in R_1} [P_0 \Pr(y=n \mid H_0) - P_1 \Pr(y=n \mid H_1)] . \quad (3)$$

To minimize P_{er} , we let all and only those y , which make $P_0 \Pr(y=n \mid H_0) - P_1 \Pr(y=n \mid H_1)$ be negative, compose R_1 . This gives the optimal detector, also known as the maximum a posterior probability (MAP) detector, which implements the test $L(y=n) \underset{H_0}{\overset{H_1}{\succ \atop \prec}} \frac{P_0}{P_1}$, by use of the likelihood ratio $L(y=n)=\frac{\Pr(y=n\mid H_1)}{\Pr(y=n\mid H_0)}$. The minimal Per reached by the MAP detector of (3) is expressed as:

$$P_{er} = \sum_{n\in R}\min[P_0\Pr(y=n\mid H_0), P_1\Pr(y=n\mid H_1)]$$

$$=\frac{1}{2}-\frac{1}{2}\sum_{n\in R}|P_0\Pr(y=n\mid H_0)-P_1\Pr(y=n\mid H_1)|. \quad (4)$$

The second equality is by $\min(a,b)=\frac{1}{2}(a+b)-\frac{1}{2}|a-b|$.

III. SR and Noise Improve Signal Detection

Given the white noise η is with probability density function $f_\eta(x)$ and cumulative distribution function $F_\eta(x)$. Note that y_i is a Bernoulli random variable, taking on values zero and one with Bernoulli probabilities, and y is binomially distributed. The Bernoulli probabilities depend on the hypothesis H_0 or H_1. We have the conditional probabilities $\Pr(y_i=0\mid H_0)$ $=\Pr(\eta+s_0<u)$ $=F_\eta(u-s_0)$ $=q_0$ and $\Pr(y_i=1\mid H_0)$ $=1-q_0$, $\Pr(y_i=0\mid H_1)$ $=\Pr(\eta+s_1<u)$ $=F_\eta(u-s_1)$ $=q_1$ and $\Pr(y_i=1\mid H_1)$ $=1-q_1$. The conditional probability $\Pr(y=n\mid H_0)$ and $\Pr(y=n\mid H_1)$ then follow, according to the binomial distribution, as

$$\Pr(y=n\mid H_0)=\binom{N}{n}(1-q_0)^n q_0^{N-n}, \text{ and} \quad (5)$$

$$\Pr(y=n\mid H_1)=\binom{N}{n}(1-q_1)^n q_1^{N-n}. \quad (6)$$

Where $\binom{N}{n}$ is the binomial coefficient. The probability of error Per of (4) follows directly from (5) and (6) for a specific noise with density function $f_\eta(x)$ and cumulative distribution $F_\eta(x)$. Here, we discuss four representative noises whose density function is with zero symmetry axes: Uniform noise (finite support PDF), Gaussian noise (thin-tailed PDF), Laplace noise (heavy-tailed PDF), and Cauchy noise (impulsive PDF without mean and variance).

A. Uniform Noise

The Uniform PDF with zero mean and variance σ^2 has the form $f(x)=\begin{cases}\frac{1}{\sqrt{12}\sigma} & -\frac{\sqrt{12}\sigma}{2}<x<\frac{\sqrt{12}\sigma}{2}\\ 0 & otherwise\end{cases}$. Then

$$q_0=F_\eta(u-s_0)=\int_{-\infty}^{u-s_0}f(x)dx$$

$$=\begin{cases}0 & u-s_0<-\sqrt{3}\sigma\\ \frac{u-s_0+\sqrt{3}\sigma}{\sqrt{12}\sigma} & -\sqrt{3}\sigma<u-s_0<\sqrt{3}\sigma\\ 1 & \sqrt{3}\sigma<u-s_0\end{cases}, \quad (7)$$

$$q_1=F_\eta(u-s_1)=\int_{-\infty}^{u-s_1}f(x)dx$$

$$=\begin{cases}0 & u-s_1<-\sqrt{3}\sigma\\ \frac{u-s_1+\sqrt{3}\sigma}{\sqrt{12}\sigma} & -\sqrt{3}\sigma<u-s_1<\sqrt{3}\sigma\\ 1 & \sqrt{3}\sigma<u-s_1\end{cases}. \quad (8)$$

B. Gaussian Noise

The Gaussian PDF with zero mean and variance σ^2 has the form $f(x)=\frac{1}{\sqrt{2\pi\sigma}}\exp(-\frac{x^2}{2\sigma^2})$. Then

$$q_0=F_\eta(u-s_0)=\int_{-\infty}^{u-s_0}f(x)dx=\frac{1}{2}+\frac{1}{2}erf(\frac{u-s_0}{\sqrt{2}\sigma}), \quad (9)$$

$$q_1=F_\eta(u-s_1)=\int_{-\infty}^{u-s_1}f(x)dx=\frac{1}{2}+\frac{1}{2}erf(\frac{u-s_1}{\sqrt{2}\sigma}). \quad (10)$$

C. Laplace Noise

The Laplace PDF with zero mean and variance σ^2 has the form

$f(x)=\frac{1}{\sqrt{2}\sigma}\exp(-\left|\frac{\sqrt{2}x}{\sigma}\right|)$. Then

$$q_0=F_\eta(u-s_0)=\int_{-\infty}^{u-s_0}f(x)dx$$

$$=\begin{cases}\frac{1}{2}\exp(\frac{\sqrt{2}(u-s_0)}{\sigma}) & u-s_0<0\\ 1-\frac{1}{2}\exp(-\frac{\sqrt{2}(u-s_0)}{\sigma}) & u-s_0>0\end{cases}, \quad (11)$$

$$q_1=F_\eta(u-s_1)=\int_{-\infty}^{u-s_1}f(x)dx$$

$$=\begin{cases}\frac{1}{2}\exp(\frac{\sqrt{2}(u-s_1)}{\sigma}) & u-s_1<0\\ 1-\frac{1}{2}\exp(-\frac{\sqrt{2}(u-s_1)}{\sigma}) & u-s_1>0\end{cases}. \quad (12)$$

D. Cauchy Noise

The Cauchy PDF with zero location and finite dispersion σ^2 (but infinite variance) has the form

$f(x) = \dfrac{\sigma}{\pi(x^2+\sigma^2)}$. Then

$$q_0 = F_\eta(u-s_0) = \int_{-\infty}^{u-s_0} f(x)dx = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{u-s_0}{\sigma}, \quad (13)$$

$$q_1 = F_\eta(u-s_1) = \int_{-\infty}^{u-s_1} f(x)dx = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{u-s_1}{\sigma}. \quad (14)$$

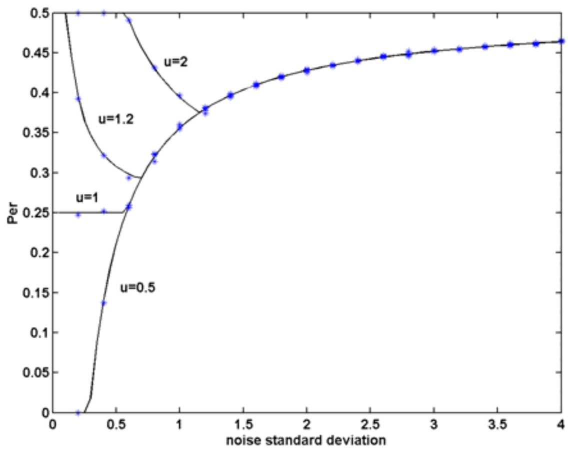

Fig. 1 *Per* is a function of standard deviation for different threshold levels for Uniform noise with zero mean

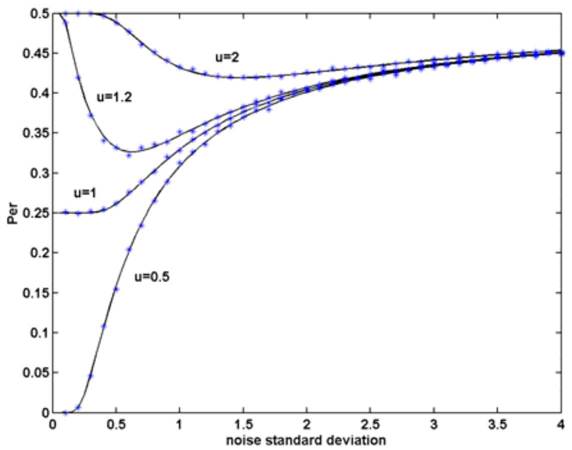

Fig. 2 *Per* is a function of standard deviation for different threshold levels for Gaussian noise with zero mean

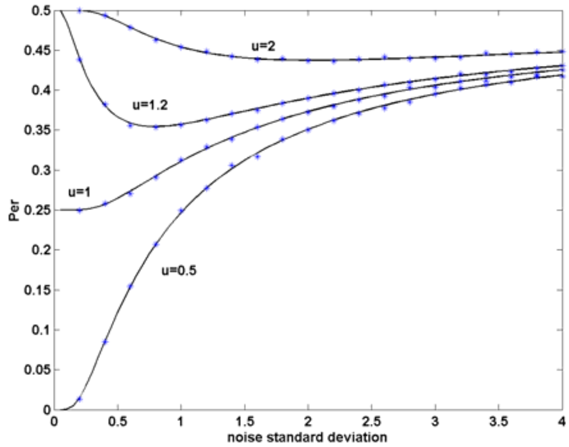

Fig. 3 *Per* is a function of standard deviation for different threshold levels for Laplace noise with zero mean

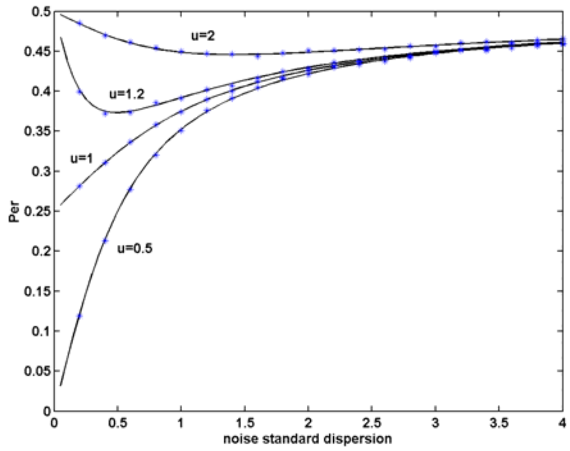

Fig. 4 *Per* is a function of standard dispersion for different threshold levels for Cauchy noise with zero mean

For different noise distributions, Figs. 1-4 show the variation of *Per* for different threshold levels and give the Monte Carlo computer simulation (data points), where the parameters are $s_0 = 0, s_1 = 1$ and $P_0 = P_1 = 0.5$. When the input signal is not subthreshold signal, *Per* increases monotonously from zero at zero noise intensity (standard deviation or dispersion). This show the performance of detection degenerates with the addition of noise. However, when the input signal is subthreshold signal (weak signal), SR is observed, *Per* decreases from initial value 0.5 (because $P_0 = P_1 = 0.5$) at zero noise intensity to a minimum, where the noise intensity is optimal, and then increases. This show the performance of detection is improved with the addition of noise.

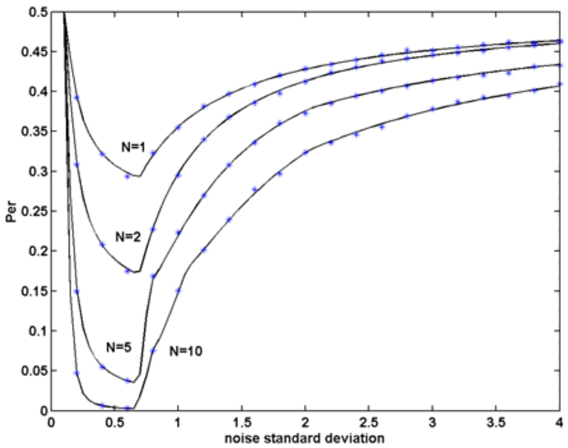

Fig. 5 *Per* is a function of standard deviation for different sample numbers for Uniform noise with zero mean

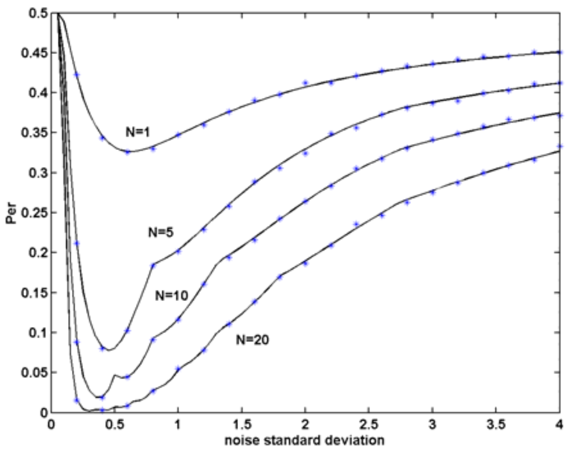

Fig. 6 *Per* is a function of standard deviation for different sample numbers for Gaussian noise with zero mean

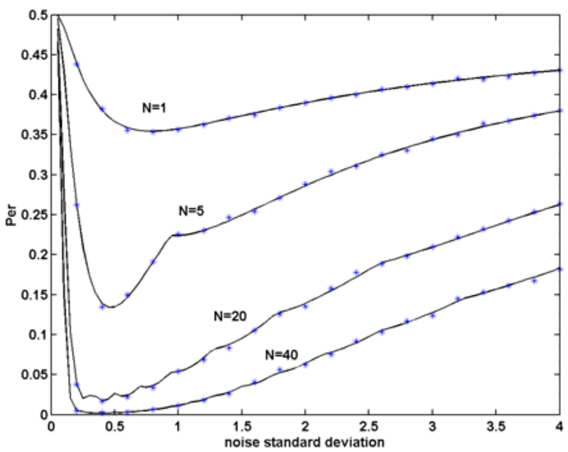

Fig. 7 *Per* is a function of standard deviation for different sample numbers for Laplace noise with zero mean

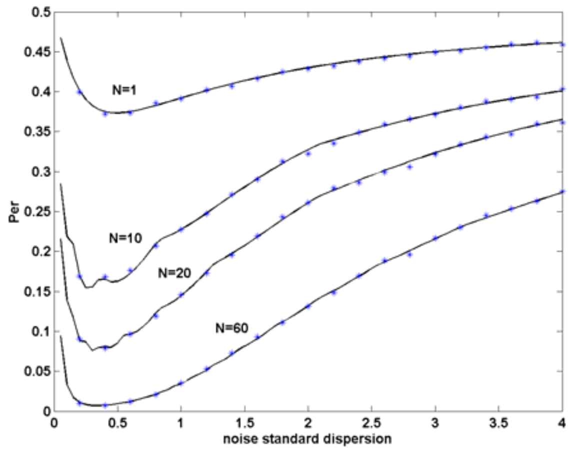

Fig. 8 *Per* is a function of standard deviation for different sample numbers for Cauchy noise with zero mean

For a fixed threshold level $u = 1.2$, Figs. 5-8 give the theoretical results and the Monte Carlo computer simulation results of *Per* for different sample number N. The efficacy of SR is significantly enhanced and the minimum of *Per* can dramatically approach to zero as the sample number increases. The rate of approaching to zero is fastest for Uniform noise and is slowest for impulsive Cauchy noise.

IV. Discussion

In this section, we shall simply discuss why SR is observed or not, and why the rate of approaching to zero is different for four different noises.

When $N = 1$ and $P_0 = P_1 = 0.5$, equation (4) becomes

$$
\begin{aligned}
P_{er} &= \frac{1}{2} - \frac{1}{4}\sum\nolimits_{n\in\{0,1\}} |\Pr(y = n \mid H_0) - \Pr(y = n \mid H_1)| \\
&= \frac{1}{2} - \frac{1}{2} |\Pr(y = 0 \mid H_0) - \Pr(y = 0 \mid H_1)| \\
&= \frac{1}{2} - \frac{1}{2} | q_0 - q_1 | \\
&= \frac{1}{2} - \frac{1}{2} F_\eta (u - s_1 < \eta < u - s_0) . \qquad (15)
\end{aligned}
$$

For fixed threshold level $u = 0.5$ and $u = 1.2$, Figs. 9-12 present a similar change: the areas under the noise PDF between line a and line b decreases monotonously, which results that the *Per* increases monotonously and SR is not observed by (13). However, the areas under the noise PDF between line c and line d firstly increases and then decreases as the noise intensity increases, which result *Per* varies nonmonotonously and SR is observed by (13).

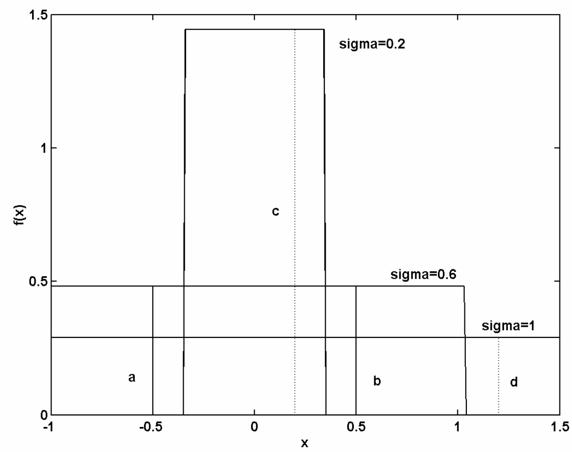

Fig. 9 Uniform noise PDF between $a = -0.5$ and $b = 0.5$, and between $c = 0.2$ and $d = 1.2$ for different noise intensity (sigma)

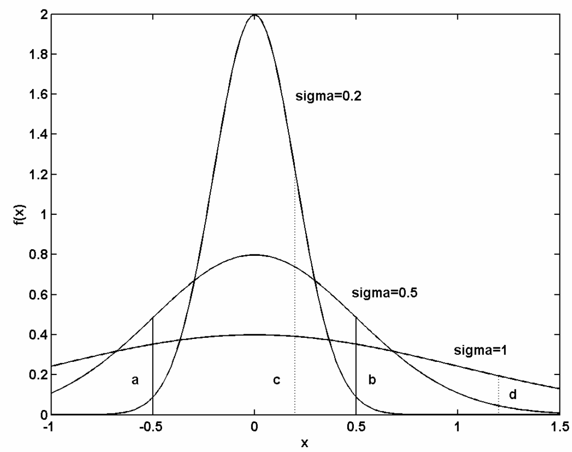

Fig. 10 Gaussian noise PDF between $a = -0.5$ and $b = 0.5$, and between $c = 0.2$ and $d = 1.2$ for different noise intensity (sigma)

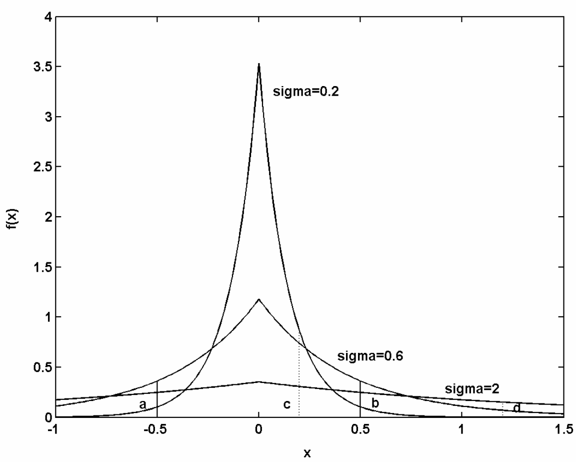

Fig. 11 Laplace noise PDF between $a = -0.5$ and $b = 0.5$, and between $c = 0.2$ and $d = 1.2$ for different noise intensity (sigma)

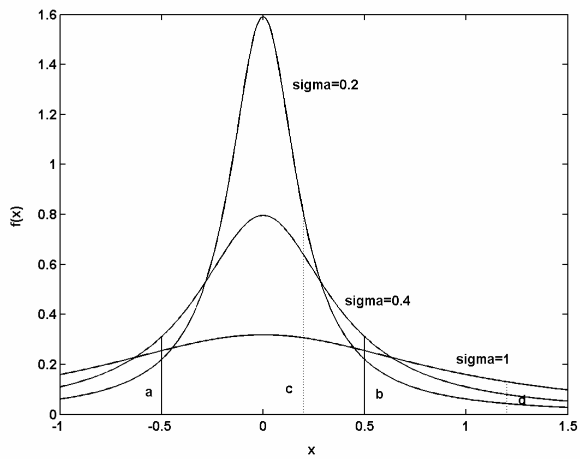

Fig. 12 Cauchy noise PDF between $a = -0.5$ and $b = 0.5$, and between $c = 0.2$ and $d = 1.2$ for different noise intensity (sigma)

For a fixed threshold level $u = 1.2$ and noise intensity (standard deviation or dispersion) $sigma = 0.6$, from Uniform noise, Gaussian noise, Laplace noise to Cauchy noise, Fig.13 presents that the areas under the noise PDF between line a and line b decreases monotonously. The area variation brings that the efficacy of SR decreases by (13). This result in the rate of approaching to zero is different. The area under the noise PDF between line a and line b is biggest for Uniform noise and is smallest for Cauchy noise, so the rate of approaching to zero is fastest for Uniform noise and is slowest for Cauchy noise.

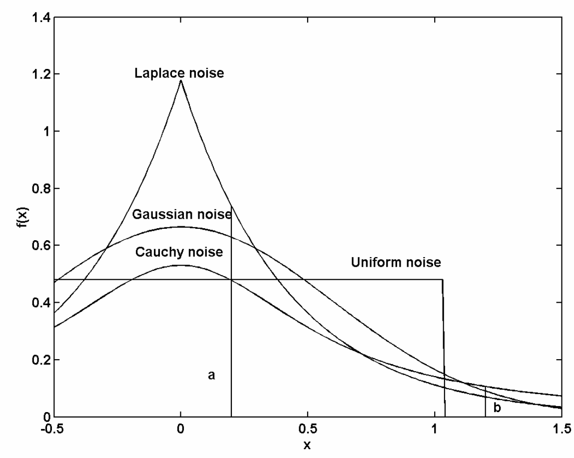

Fig. 13 PDFs between $a = 0.2$ and $b = 1.2$, the noise intensity $sigma = 0.6$

REFERENCES

[1] B. McNamara, K. Wiesenfeld, *Theory of stochastic resonance*, Physical review A, 39(1989) 4854-4869.

[2] A. Restrepo, L. F. Zuluaga, L. E. Pino, *Optimal noise levels for stochastic resonance*, 1997 IEEE International Conference on Acoustics, Speech, and Signal Processing, 3(1997) 1617-1620.

[3] B. Kosko, S. Mitaim, *Stochastic resonance in noisy threshold neurons*, Neural networks 16(2003) 755-761.

[4] A. Saha, G.V. Anand, *Design of detectors based on stochastic resonance*, Signal processing 83(2003) 1193-1212.

[5] S. Zozor, P.-O.Amblard, *On the use of stochastic in signal detection*, Signal Processing 82(3) (2002) 353-367.

[6] S. Kay, *Can detecability be improved by adding noise*? IEEE Signal processinging letters 7(2000) 8-10.

[7] Hu Gang, Gong De-chun, Wen Xiao-dong, Yang Chun-yun, Qing Guang-rong, and Li Rong, *Stochastic resonance in a nonlinear system driven by an a-periodic force*, Physical review A, 46(1992) 3250-3254.

[8] M. E. Inchiosa, A. R. Bulsara, *Signal detection statistics of stochastic resonators*, Physical review E, 53(1996) 2021-2024.

[9] F. Chapeau-Blondeau, *Stochastic resonance for an optimal detector with phase noise*, Signal processing, 83(2003) 665-670.

[10] B. Kosko, S. Mitain; *Robust stochastic resonance: signal detection and adaptation in impulsive noise*, Physical review E 64(2001) 051110,1-11.

[11] V.Galdi, V. Pierro, I. M. Pinto, *Evaluation of stochastic-resonance-based detectors of weak harmonic signals in additive white Gaussian noise*, Physical review E 57(1998) 6470-6479.

[12] F.Chapeau-Blondeau, *Nonlinear test statistic to improve signal detection in non-Gaussian noise*, IEEE Signal Processing Letters, 7(2000) 205-207.