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A neighborhood condition for fractional k-deleted graphs

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Abstract—Let $k \geq 3$ be an integer, and let G be a graph of order n with $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. In this paper, it is proved that G is a fractional k-deleted graph if G satisfies $\delta(G) \geq k+1$ and $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices x, y of G.

Keywords—graph, minimum degree, neighborhood union, fractional k-factor, fractional k-deleted graph.

I. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, $N_G(S) = \cup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. We say that S is independent if $N_G(S) \cap S = \emptyset$. Let S and T be disjoint subsets of $V(G)$. We use $e_G(S,T)$ to denote the number of edges joining S and T in G. The minimum vertex degree of G is denoted by $\delta(G)$.

Let k be a positive integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. If $k = 1$, then a k-factor is simply called a 1-factor. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. If $k = 1$, then a fractional k-factor is a fractional 1-factor. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. If $k = 1$, then a fractional k-deleted graph is a fractional 1-deleted graph. Some other terminologies and notations can be found in [1,2].

Many authors have studied graph factors [3-8]. Many authors have investigated fractional k-factors [9–12] and fractional k-deleted graphs [13,14]. The following results on k-factors, fractional k-factors and fractional k-deleted graphs are known.

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Theorem 1[15] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, kn is even, and the minimum degree is at least k. If G satisfies $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a k-factor.

Theorem 2[11] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, and the minimum degree $\delta(G) \geq k$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.

Theorem 3[16] Let $k \geq 2$ be an integer. Let G be a connected graph of order n with $n \geq 13k+1-4\sqrt{2(k-1)^2+2}$, $\delta(G) \geq k+2$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices x, y of G, then G is a fractional k-deleted graph.

The purpose of this paper is to weaken the conditions on the order, minimum degree and connectivity of G in Theorem 3. The main result is the following theorem.

Theorem 4 Let $k \geq 3$ be an integer. Let G be a graph of order n with $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$, $\delta(G) \geq k+1$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional k-deleted graph.

II. The Proof of Theorem 4

The following result is essential to the proof of our main theorem.

Lemma 2.1[17] A graph G is a fractional k-deleted graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S,T),$$

where $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $\varepsilon(S,T)$ is defined as follows,

$$\varepsilon(S,T) = \begin{cases} 2, & if\ T\ is\ not\ independent, \\ 1, & if\ T\ is\ independent,\ and \\ & e_G(T, V(G) \setminus (S \cup T)) \geq 1, \\ 0, & otherwise. \end{cases}$$

Proof of Theorem 4. Let G be a graph satisfying the hypothesis of Theorem 4, we prove the theorem by contradiction.

Suppose that G is not a fractional k-deleted graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ such that

$$\delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\leq\varepsilon(S,T)-1, \quad (1)$$

where $T=\{x: x\in V(G)\setminus S, d_{G-S}(x)\leq k\}$. Firstly, we prove the following claims.

Claim 1. $S\neq\emptyset$.

Proof. Note that $\varepsilon(S,T)\leq|T|$. If $S=\emptyset$, then by (1) we have

$$\begin{aligned}\varepsilon(S,T)-1 &\geq \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\\ &= d_G(T)-k|T|\geq(\delta(G)-k)|T|\\ &\geq |T|\geq\varepsilon(S,T).\end{aligned}$$

It is a contradiction. This completes the proof of Claim 1.

Claim 2. $|T|\geq k+1$.

Proof. Assume that $|T|\leq k$. Then from (1) and $|S|+d_{G-S}(x)-k\geq d_G(x)-k\geq\delta(G)-k\geq 1$, we get

$$\begin{aligned}\varepsilon(S,T)-1 &\geq \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\\ &\geq |T||S|+d_{G-S}(T)-k|T|\\ &= \sum_{x\in T}(|S|+d_{G-S}(x)-k)\\ &\geq |T|\geq\varepsilon(S,T).\end{aligned}$$

That is a contradiction. This completes the proof of Claim 2.

Claim 3. $|T|\geq|S|+1$.

Proof. Let $|T|\leq|S|$. Then by (1), we obtain

$$\varepsilon(S,T)-1\geq k|S|+d_{G-S}(T)-k|T|\geq d_{G-S}(T). \quad (2)$$

On the other hand, according to the definition of $\varepsilon(S,T)$, we have

$$d_{G-S}(T)\geq\varepsilon(S,T),$$

which contradicts (2). The proof of Claim 3 is complete.

Claim 4. $|S|\leq\frac{n-1}{2}$.

Proof. In terms of Claim 3 and $|S|+|T|\leq n$, we have

$$n\geq|S|+|T|\geq 2|S|+1,$$

that is,

$$|S|\leq\frac{n-1}{2}.$$

The proof of Claim 4 is complete.

In terms of Claim 2, $T\neq\emptyset$. Now we define

$$h_1=\min\{d_{G-S}(x): x\in T\}$$

and choose $x_1\in T$ such that $d_{G-S}(x_1)=h_1$. Clearly, we have $0\leq h_1\leq k$. In the following, we consider two cases.

Case 1. $T=N_T[x_1]$.

Using Claim 2, $T=N_T[x_1]$ and $0\leq h_1\leq k$, we obtain

$$k\geq h_1=d_{G-S}(x_1)\geq|T|-1\geq k,$$

which implies

$$h_1=k. \quad (3)$$

In terms of (3) and Claim 1, we get

$$\begin{aligned}\delta_G(S,T) &= k|S|+d_{G-S}(T)-k|T|\\ &\geq k|S|+h_1|T|-k|T|=k|S|\\ &\geq k>2\geq\varepsilon(S,T).\end{aligned}$$

That contradicts (1).

Case 2. $T\setminus N_T[x_1]\neq\emptyset$.

Define

$$h_2=\min\{d_{G-S}(x): x\in T\setminus N_T[x_1]\}.$$

We choose $x_2\in T\setminus N_T[x_1]$ such that $d_{G-S}(x_2)=h_2$. Obviously, $0\leq h_1\leq h_2\leq k$ and $x_1x_2\notin E(G)$. According to the hypothesis of Theorem 4, we have

$$\begin{aligned}\frac{n+k-2}{2} &\leq |N_G(x_1)\cup N_G(x_2)|\\ &\leq d_{G-S}(x_1)+d_{G-S}(x_2)+|S|\\ &= h_1+h_2+|S|,\end{aligned}$$

which implies

$$|S|\geq\frac{n+k-2}{2}-h_1-h_2. \quad (4)$$

By (4) and Claim 4, we obtain

$$\frac{n-1}{2}\geq\frac{n+k-2}{2}-h_1-h_2,$$

that is,

$$h_1+h_2\geq\frac{k-1}{2}. \quad (5)$$

In terms of (5), $k\geq 3$, $0\leq h_1\leq h_2\leq k$ and the integrity of h_2, we get

$$h_2\geq 1. \quad (6)$$

Claim 5. $0\leq h_1\leq k-1$.

Proof. If $h_1=k$, then by (1) and Claim 1 we get

$$\begin{aligned}\varepsilon(S,T)-1 &\geq \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T|\\ &\geq k|S|+h_1|T|-k|T|=k|S|\geq k\\ &> 2\geq\varepsilon(S,T),\end{aligned}$$

which is a contradiction. This completes the proof of Claim 5.

Note that

$$|N_T[x_1]|\leq d_{G-S}(x_1)+1=h_1+1. \quad (7)$$

From (4), (7), $0\leq h_1\leq h_2\leq k$ and $|S|+|T|\leq n$, we have

$$\begin{aligned}\delta_G(S,T) &= k|S|+d_{G-S}(T)-k|T|\\ &\geq k|S|+h_1|N_T[x_1]|+h_2(|T|-|N_T[x_1]|)\\ &\quad -k|T|\\ &= k|S|-(h_2-h_1)|N_T[x_1]|-(k-h_2)|T|\\ &\geq k|S|-(h_2-h_1)(h_1+1)\\ &\quad -(k-h_2)(n-|S|)\\ &= (2k-h_2)|S|-(h_2-h_1)(h_1+1)\\ &\quad -(k-h_2)n\\ &\geq (2k-h_2)(\frac{n+k-2}{2}-h_1-h_2)\\ &\quad -(h_2-h_1)(h_1+1)-(k-h_2)n,\end{aligned}$$

that is,

$$\begin{aligned}\delta_G(S,T) \geq\ & (2k-h_2)(\frac{n+k-2}{2}-h_1-h_2)\\ & -(h_2-h_1)(h_1+1)-(k-h_2)n. \qquad (8)\end{aligned}$$

Let $F(h_1,h_2) = (2k-h_2)(\frac{n+k-2}{2}-h_1-h_2)-(h_2-h_1)(h_1+1)-(k-h_2)n$. Then by Claim 5, we have

$$\begin{aligned}F'_{h_1}(h_1,h_2) &= -(2k-h_2)+(h_1+1)-(h_2-h_1)\\ &= 2h_1-2k+1 \leq 2(k-1)-2k+1\\ &= -1<0.\end{aligned}$$

Combining this with $h_1 \leq h_2$, we obtain

$$F(h_1,h_2) \geq F(h_2,h_2). \qquad (9)$$

Using (8) and (9), we get

$$\delta_G(S,T) \geq (2k-h_2)(\frac{n+k-2}{2}-2h_2)-(k-h_2)n. \quad (10)$$

According to (1), (10) and $\varepsilon(S,T) \leq 2$, we get

$$\begin{aligned}1 &\geq \varepsilon(S,T)-1 \geq \delta_G(S,T)\\ &\geq (2k-h_2)(\frac{n+k-2}{2}-2h_2)-(k-h_2)n\\ &= \frac{1}{2}(4h_2^2+(n-9k+2)h_2+2k^2-4k),\end{aligned}$$

which implies

$$4h_2^2+(n-9k+2)h_2+2k^2-4k-2 \leq 0. \qquad (11)$$

Claim 6. For $k \geq 3$, we have $\sqrt{\frac{(k-1)^2+1}{2}}-1 > \frac{1}{2}$.
Proof. Since $k \geq 3$, we have

$$\frac{(k-1)^2+1}{2} \geq \frac{5}{2} > \frac{9}{4},$$

that is,

$$\sqrt{\frac{(k-1)^2+1}{2}} > \frac{3}{2}.$$

Thus, we obtain

$$\sqrt{\frac{(k-1)^2+1}{2}}-1 > \frac{1}{2}.$$

The proof of Claim 6 is complete.

According to (6), (11), $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$, $k \geq 3$ and Claim 6, we obtain

$$\begin{aligned}0 &\geq 4h_2^2+(n-9k+2)h_2+2k^2-4k-2\\ &\geq 4h_2^2+(-4\sqrt{2(k-1)^2+2}+5)h_2+2k^2-4k-2\\ &\geq 4h_2^2-8\sqrt{\frac{(k-1)^2+1}{2}}h_2+2(k-1)^2+2+5h_2-6\\ &= 4(\sqrt{\frac{(k-1)^2+1}{2}}-h_2)^2+5h_2-6\\ &\geq 4(\sqrt{\frac{(k-1)^2+1}{2}}-1)^2-1\\ &> 4(\frac{1}{2})^2-1 \geq 0,\end{aligned}$$

which is a contradiction.

From all the cases above, we deduce the contradictions. Hence, G is a fractional k-deleted graph. This completes the proof of Theorem 4.

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