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Equal sharing solutions for bicooperative games

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Abstract—In this paper, we discuss the egalitarianism solution (ES) and center-of-gravity of the imputation-set value (CIV) for bicooperative games, which can be seen as the extensions of the solutions for traditional games given by Dutta and Ray [1] and Driessen and Funaki [2]. Furthermore, axiomatic systems for the given values are proposed. Finally, a numerical example is offered to illustrate the player ES and CTV.

Keywords—bicooperative games, egalitarianism solution, center-of-gravity of the imputation-set value

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I. Introduction

TUTTA and Ray [1] introduced the egalitarian solution as a solution concept for TU-games. Later, Driessen and Funaki [2] further proposed the centre-of-gravity of the imputation-set value (CIS-value) and egalitarian non-separable contribution value (ENSC-value). van den Brink and Funaki [3] researched axiomatizations of the ES, CIS-value and ENSC-value. Dutta and Ray [4] considered a parallel concept, the S-constrained egalitarian solution, which is not a singleton in general. Dutta [5] characterized the egalitarian solution over the class of convex games. The properties used are the reduced game properties due to Hart and Mas-Colell [6] and Davis and Maschler [7]. Arin and Inarra [8] showed that in general the egalitarian solution of Dutta and Ray [1] does not satisfy the Davis-Maschler reduced game property, nor the Hart and Mas-Colell educed game property. Furthermore, Arin and Inarra [8] proved that the egalitarian solution belongs to the egalitarian set, and that for convex games it even coincides with the latter. Klijn et al. [9] researched five characterizations of the egalitarian solution for convex games. Arin et al. [10] studied the characterizations of two classes games. The one is convex games, and the other is large core games. van den Brink [11] gave three axiomatic systems for equal division solution, and proved the uniqueness of it by using the nullifying player property, coalitional standard equivalence and coalitional monotonicity, respectively. van den Brink [11] further showed the equal surplus division solution is the unique value satisfying efficiency, symmetry, additivity, the nullifying player property for zero normalized games and invariance. Salonen [12] researched a new axiomatization of the - egalitarian solutions for n-person bargaining games. Recently, Branzei et al. [13] researched the egalitarianism solution for convex fuzzy games. Peters and Zank [14] discussed the egalitarian solution for multichoice games.

In 2008, Bilbao et al. [15] introduced bicooperative games, and researched the Shapley value. Later, Bilbao et al. [16] studied the Banzhaf value for this kind of games. Grabisch and Labreuche [17] proposed another Shapley value for bicooperative games. Following Grabisch and Labreuche [17], Yu and Zhang [18] gave another characterization for the Shapley value given by Grabisch and Labreuche [17], and further studied a special kind of fuzzy games introduced by Tsurmmi et al. [19].

In this paper, we shall research the ES and CIV for bicooperative games. Based on the existence axiomatic systems, we study characterizations of the given values.

The rest part of this paper is organized as follows. In section 2, some concepts for traditional games are introduced. In section 3, we mainly research the ES and CIV for bicooperative games. Axiomatic systems for the given values are studied. Meantime, some properties are researched. In section 4, a numerical example is given.

II. Preliminaries

Let $N = \{1, 2, ..., n\}$ be a finite set, and $P(N)$ denote the class of all subsets in N. The coalitions $P(N)$ in are denoted by $S, T, ...$ For any $S \in P(N)$, the cardinality of S is denoted by the corresponding lower case s. We will omit braces for singletons, e.g. by writing $S, S \cup (\cap)T, S \setminus T, i$ instead of $\{S\}, \{S\} \cup (\cap)\{T\}, \{S\} \setminus \{T\}, \{i\}$for any $\{S\}, \{T\}, \{i\} \in P(N)$. A function $v : P(N) \to \Re$, such that $v(\emptyset) = 0$, is called a set function. The set of all set functions in N is denoted by $G(N)$.

Dutta and Ray [1] first introduced egalitarianism for traditional games, which is expressed by

$$\phi_i(N, v) = \frac{v(N)}{n} \qquad \forall i \in N. \tag{1}$$

Later, Driessen and Funaki [2] proposed the center-of-gravity of the imputation-set value, which is written as

$$\gamma_i(N, v) = v(i) + \frac{1}{n}\left(v(N) - \sum_{j \in N} v(j)\right) \quad \forall i \in N. \tag{2}$$

III. The ES and CIV for bicooperative games

Let $N = \{1, 2, ..., n\}$ denote the set of players, and define $3^N = \{(S, T) : S, T \in N, S \cap T = \emptyset\}$. A bicooperative game is a pair (N, v) with the player set N and a function $v : 3^N \to \Re$ satisfying $v(\emptyset, \emptyset) = 0$. Let $BG(N)$ denote the set of all bicooperative games on N. Grabisch and Lebreuche [17] proposed a relation in 3^N as follows:

$$(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C, B \supseteq D$$

Similar to Dutta and Ray [1], we give the ES for bicooperative games as follows:

$$\beta_i(N, v) = \frac{v(N, \emptyset) - v(\emptyset, N)}{n} \quad \forall i \in N. \tag{3}$$

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Similar to Driessen and Funaki [2], we give the CIV for bicooperative games as follows:

$$\varphi_i(N,v) = v(i,\emptyset) - v(\emptyset,i) + \frac{1}{n}\left(v(N,\emptyset) - v(\emptyset,N) - \left(\sum_{j\in N}\left(v(j,\emptyset) - v(\emptyset,j)\right)\right)\right) \quad \forall i \in N. \quad (4)$$

The upper unanimity game $\bar{u}_{(S,T)} : 3^N \to \Re$ is given by

$$\bar{u}_{(S,T)}(A,B) = \begin{cases} 1 & (S,T) \subseteq (A,B) \\ 0 & \text{otherwise} \end{cases},$$

where $(S,T) \in 3^N\emptyset,\emptyset)$.

The lower unanimity game $\underline{u}_{(S,T)} : 3^N \to \Re$ is expressed by

$$\underline{u}_{(S,T)}(A,B) = \begin{cases} 1 & (A,B) \subseteq (S,T) \\ 0 & \text{otherwise} \end{cases},$$

where $(S,T) \in 3^N\emptyset,\emptyset)$.

The identity game $\delta_{(S,T)} : 3^N \to \Re$ is defined by

$$\delta_{(S,T)}(A,B) = \begin{cases} 1 & (A,B) = (S,T) \\ 0 & \text{otherwise} \end{cases},$$

where $(S,T) \in 3^N\emptyset,\emptyset)$.

Let f be a solution on bicooperative games. Inspired by van den Brink and Funaki [3], we give the following properties for bicooperative games.

efficiency Let $v \in BG(N)$, we have

$$\sum_{i\in N} f_i(N,v) = v(N,\emptyset) - v(\emptyset,N);$$

symmetry Let $v \in BG(N)$ and $i,j \in N$, if we have$v(S\cup i, T\cup j) = v(S\cup j, T\cup i)$, $v(S,T\cup i) = v(S,T\cup j)$ and $v(S\cup i,T) = v(S\cup j,T)$ for any $(S,T) \in 3^N$ with $i,j \notin S,T$, then $f_i(N,v) = f_j(N,v)$;

individual rationality Let $v \in BG(N)$, if we have $\sum_{j\in N}(v(j,\emptyset) - v(\emptyset,j)) \le v(N,\emptyset) - v(\emptyset,N)$, then

$$f_i(N,v) \ge v(i,\emptyset) - v(\emptyset,i);$$

additivity Let $v \in BG(N)$, if we have $(v+w)(S,T) = v(S,T) + w(S,T)$ for any $(S,T) \in 3^N$, then

$$f(N,v+w) = f(N,v) + f(N,w);$$

non-negativity Let $v \in BG(N)$ with $v(S,T) \ge 0$ for any $(S,T) \in 3^N$, then we have $f_i(N,v) \ge 0$ for any $i \in N$.

Theorem 1: Solution f satisfies efficiency, symmetry, additivity and non-negativity if and only if $f = \beta$.

Proof: Sufficiency: It is obvious that β satisfies these properties.

Necessity: Let f be a solution that satisfies the above mentioned properties. Since every bicooperative game $v \in BG(N)$ can be expressed by

$$v = \sum_{(S,T)\in 3^N\setminus(\emptyset,\emptyset)} \delta_{(S,T)} v(S,T).$$

From additivity, we only need to show $f = \beta$ for any identity game $\delta_{(S,T)}$.

Consider identity game $((N,\emptyset),\delta_{(S,T)})$, where $S \ne N$. From efficiency, we get $\sum_{i\in N} f_i((N,\emptyset),\delta_{(S,T)}) = 0$.

From non-negativity, we know

$$f_i((N,\emptyset),\delta_{(S,T)}) = 0 = \beta_i((N,\emptyset),\delta_{(S,T)}) \quad \forall i \in N.$$

Consider identity game $((N,\emptyset),\delta_{(S,T)})$. From efficiency, we get

$$\sum_{i\in N} f_i((N,\emptyset),\delta_{(N,\emptyset)}) = 1.$$

From symmetry, we have

$$f_i((N,\emptyset),\delta_{(N,\emptyset)}) = \frac{1}{n} = \beta_i((N,\emptyset),\delta_{(N,\emptyset)}) \quad \forall i \in N.$$

Similarly, we have

$$f_i((\emptyset,N),\delta_{(\emptyset,N)}) = -\frac{1}{n} = \beta_i((\emptyset,N),\delta_{(\emptyset,N)}) \quad \forall i \in N$$

and

$$f_i((\emptyset,N),\delta_{(S,T)}) = 0 = \beta_i((\emptyset,N),\delta_{(S,T)}) \quad \forall i \in N,$$

where $T \ne N$.

■

Theorem 2: Solution f satisfies efficiency, symmetry, additivity and individual rationality if and only if $f = \varphi$.

Proof: Sufficiency: It is obvious that φ satisfies these properties.

Necessity: Let f be a solution that satisfies the above mentioned properties. Consider the upper unanimity game $((N,\emptyset),\bar{u}_{(S,\emptyset)})$, where $s = 1$.

From individual rationality, we get $f_i((N,\emptyset),\bar{u}_{(S,\emptyset)}) \ge 0$ for any $i \in N\setminus S$, and $f_i((N,\emptyset),\bar{u}_{(S,\emptyset)}) \ge 1$ for any $i \in S$.

Form efficiency, we get

$$f_i((N,\emptyset),\bar{u}_{(S,\emptyset)}) = 0 = \varphi_i((N,\emptyset),\bar{u}_{(S,\emptyset)})$$

for any $i \in N\setminus S$, and

$$f_i((N,\emptyset),\bar{u}_{(S,\emptyset)}) = 1 = \varphi_i((N,\emptyset),\bar{u}_{(S,\emptyset)})$$

for any $i \in S$.

Similarly, when $t = 1$, we have

$$f_i((\emptyset,N),\underline{u}_{(\emptyset,T)}) = 0 = \varphi_i((\emptyset,N),\underline{u}_{(\emptyset,T)}) \quad \forall i \in N\setminus T$$

and

$$f_i((\emptyset,N),\underline{u}_{(\emptyset,T)}) = -1 = \varphi_i((\emptyset,N),\underline{u}_{(\emptyset,T)}) \quad \forall i \in T.$$

Next, consider the identity game $((N,\emptyset),\delta_{(S,T)})$ with $2 \le s+t \le n-1$.

From individual rationality, we get $f_i((N,\emptyset),\delta_{(S,T)}) \ge 0$ for all $i \in N$.

From efficiency, we obtain $\sum_{i\in N} f_i((N,\emptyset),\delta_{(S,T)}) = 0$.

Thus, we have

$$f_i((N,\emptyset),\delta_{(S,T)}) = 0 = \varphi_i((N,\emptyset),\delta_{(S,T)})$$

for all $i \in N$.

Similarly, we have $f_i((\emptyset,N),\delta_{(S,T)}) = 0 = \varphi_i((\emptyset,N),\delta_{(S,T)})$ for all $i \in N$, where $2 \le s+t \le n-1$.

For the identity games $((N,\emptyset),\delta_{(N,\emptyset)})$ and $((\emptyset,N),\delta_{(\emptyset,N)})$, from efficiency and symmetry, we get

$$f_i((N,\emptyset),\delta_{(N,\emptyset)}) = \frac{1}{n} = \varphi_i((N,\emptyset),\delta_{(N,\emptyset)}) \quad \forall i \in N$$

and

$$f_i((\emptyset,N),\delta_{(\emptyset,N)}) = -\frac{1}{n} = \varphi_i((\emptyset,N),\delta_{(\emptyset,N)}) \quad \forall i \in N.$$

Since every bicooperative game $v \in BG(N)$ can be expressed by

$$\begin{aligned} v = &\sum_{i\in N}(v(i,\emptyset)\bar{u}_{(i,\emptyset)} - v(\emptyset,i)u_{(\emptyset,i)}) + \\ &\sum_{(S,\emptyset)\in 3^N, s\geq 2}\left(v(S,\emptyset) - \sum_{i\in S} v(i,\emptyset)\right)\delta_{(S,\emptyset)} + \\ &\sum_{(\emptyset,T)\in 3^N, t\geq 2}\left(v(\emptyset,T) + \sum_{i\in T} v(\emptyset,i)\right)\delta_{(\emptyset,T)} + \\ &\sum_{(S,T)\in 3^N, s+t\geq 2, s,t\geq 1} v(S,T)\delta_{(S,T)}. \end{aligned}$$

From additivity, we get the conclusion. ■

Definition 1: Let $v \in BG(N)$, if we have $v(S\cup T,K) \geq v(S,K)+v(T,K)$ and $v(K,S\cup T) \leq v(K,S)+v(K,T)$ for any $S,T,K \subseteq N$, where $S\cap T, S\cap K, T\cap K = \emptyset$. Then, $v \in BG(N)$ is said to be a superadditive bicooperative game.

Definition 2: Let $v \in BG(N)$, if we have $\frac{v(S,T)}{s} \geq \frac{v(K,T)}{k}$ and $\frac{v(T,S)}{s} \leq \frac{v(T,K)}{k}$ for any $S,T,K \in N$, where $S\cap T, S\cap K, T\cap K = \emptyset$ and $s \geq k$. Then, $v \in BG(N)$ is said to be an average monotonic bicooperative game.

Definition 3: Let $v \in BG(N)$. The vector $x = (x_1, x_2, ..., x_n)$ is called an imputation for $v \in BG(N)$ if it satisfies

(1)$x_i \geq v(i,\emptyset) - v(\emptyset,i) \quad \forall i \in N$;

(2) $\sum_{i\in N} x_i = v(N,\emptyset) - v(\emptyset,N)$.

Theorem 3: Let $v \in BG(N)$. If v is average monotonic, then $(\varphi_i(N,v))_{i\in N}$ is an imputation.

Proof: From Definition 2, we get

$$\frac{v(N,\emptyset) - v(\emptyset,N)}{n} \geq v(i,\emptyset) - v(\emptyset,i).$$

From Eq.(3), we have $(\varphi_i(N,v))_{i\in N}$ is an imputation. ■

Theorem 4: Let $v \in BG(N)$. If v is superadditive, then $(\varphi_i(N,v))_{i\in N}$ is an imputation.

Proof: From Definition 3, we have

$$v(N,\emptyset) - v(\emptyset,N) \geq \sum_{i\in N}(v(i,\emptyset) - v(\emptyset,i)).$$

From Eq.(4), we get

$$\begin{aligned} \varphi_i(N,v) &= v(i,\emptyset) - v(\emptyset,i) + \frac{1}{n}(v(N,\emptyset) - v(\emptyset,N) \\ &\quad -(\sum_{j\in N}(v(j,\emptyset) - v(\emptyset,j)))) \\ &\geq v(i,\emptyset) - v(\emptyset,i) \quad \forall i \in N. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \sum_{i\in N}\varphi_i(N,v) &= \sum_{i\in N} v(i,\emptyset) - v(\emptyset,i) + \sum_{i\in N}\frac{1}{n}(v(N,\emptyset) - \\ &\quad v(\emptyset,N) - (\sum_{j\in N}(v(j,\emptyset) - v(\emptyset,j)))) \\ &= \sum_{i\in N} v(i,\emptyset) - v(\emptyset,i) + v(N,\emptyset) - \\ &\quad v(\emptyset,N) - \left(\sum_{j\in N}(v(j,\emptyset) - v(\emptyset,j))\right) \\ &= v(N,\emptyset) - v(\emptyset,N). \end{aligned}$$

■

IV. Numerical Example

There are three companies who product the same products. Denoted by the player $1,2,3$, respectively. Namely, the player set is $N = \{1,2,3\}$. If they compete, their incomes will reduce. If they cooperate, all of them are not willing to get the incomes by cooperation. There exist both competition and cooperation among them. The payoffs got by cooperation are $v(1,\emptyset) = 2$, $v(2,\emptyset) = 3$, $v(3,\emptyset) = 2$, $v(1,2,3,\emptyset) = 12$, and the payoffs obtained by competition are $v(\emptyset,1) = 1$, $v(\emptyset,2) = 1$, $v(\emptyset,3) = 1$, $v(\emptyset,1,2,3) = 3$.

From Eq.(3), we obtain the player ES are $\beta_i(N,v) = 3 (i = 1,2,3)$.

From Eq.(4), we get the player CIV are $\varphi_1(N,v) = \varphi_3(N,v) = 8/3$ and $\varphi_2(N,v) = 11/3$.

V. Conclusion

We have studied the ES and CTV for bicooperative games, which can be seen as the extensions for the ES and CIS-value for traditional games. Like other payoff induces, there exist other axiomatic systems for the ES and CTV for bicooperative games.

Furthermore, similar to the ENSC-value given by Driessen and Funaki [2], we can get the ENSC-value for bicooperative games, which can be expressed by

$$\begin{aligned} \alpha_i(N,v) = v(\emptyset,N\backslash i) - v(N\backslash i,\emptyset) + \frac{1}{n}(v(N,\emptyset) - v(\emptyset,N) + \\ (\sum_{j\in N}(v(N\backslash j,\emptyset) - v(\emptyset,N\backslash j)))) \quad \forall i \in N. \end{aligned}$$

By establishing axiomatic system, we can show the above given value is unique.

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