

# Two-Stage Compensator Designs with Partial Feedbacks

Kazuyoshi MORI

**Abstract**—The two-stage compensator designs of linear system are investigated in the framework of the factorization approach. First, we give “full feedback” two-stage compensator design. Based on this result, various types of the two-stage compensator designs with partial feedbacks are derived.

**Keywords**—Linear System, Factorization Approach, Two-Stage Compensator Design, Parametrization of Stabilizing Controllers.

## I. INTRODUCTION

The factorization approach to control systems has the advantage that it embraces, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, one-dimensional as well as multidimensional systems, etc.[1], [2], [3]. Hence the result given in this paper will be able to a number of models in addition to the multidimensional systems. In factorization approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. This approach leads to conceptually simple and computationally tractable solutions to many important and interesting problems[4]. A transfer matrix of this approach is considered as the ratio of two stable causal transfer matrices. For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer matrices, which is satisfied by transfer matrices over the principal ideal domains or the Bézout domains.

In some design problems, one uses a so-called *two-state procedure* for selecting an appropriate stabilizing compensator[4]. Given a plant, the first stage consists of selecting a stabilizing compensator for the plant. The second stage consists of selecting a stabilizing controller for the new closed-loop system that also achieves some other design objectives such as decoupling, sensitivity minimization, etc. The rationale behind this procedure is that the design problems are often easier to solve when the plant is stable. So far, the results of the two-stage compensator design use the norm algebras as well as the factorization approach. Because the analysis by the norm algebra is based on a concrete specified model, this reduces the attractiveness of the factorization approach.

First, we present a two-stage compensator design based on “full feedback” (Theorem 3). Using this result, we will present various types of the two-stage compensator designs as subsets of “full feedback” (Theorems 4 to 7).

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## II. PRELIMINARIES

The stabilization problem considered in this paper follows that of [5], and [6], who consider the feedback system  $\Sigma$  [4, Ch.5, Fig. 5.1] as in Fig. 1. For further details the reader is referred to [4], [7], [5], and [6].

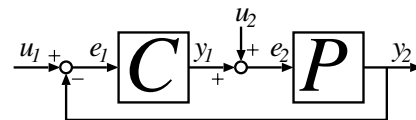


Fig. 1. Feedback system  $\Sigma$ .

We consider that the set of stable causal transfer functions is an integral domain, denoted by  $\mathcal{A}$ . The total ring of fractions of  $\mathcal{A}$  is denoted by  $\mathcal{F}$ ; that is,  $\mathcal{F} = \{n/d \mid n, d \in \mathcal{A}, d \neq 0\}$ . This  $\mathcal{F}$  is considered as the set of all possible transfer functions. Matrices over  $\mathcal{F}$  are transfer matrices. Let  $\mathcal{Z}$  be a prime ideal of  $\mathcal{A}$  with  $\mathcal{Z} \neq \mathcal{A}$ . Define the subsets  $\mathcal{P}$  and  $\mathcal{P}_S$  of  $\mathcal{F}$  as follows:  $\mathcal{P} = \{a/b \in \mathcal{F} \mid a \in \mathcal{A}, b \in \mathcal{A} \setminus \mathcal{Z}\}$ ,  $\mathcal{P}_S = \{a/b \in \mathcal{F} \mid a \in \mathcal{Z}, b \in \mathcal{A} \setminus \mathcal{Z}\}$ . Then, every transfer function in  $\mathcal{P}$  ( $\mathcal{P}_S$ ) is called *causal* (*strictly causal*). Analogously, if every entry of a transfer matrix is in  $\mathcal{P}$  ( $\mathcal{P}_S$ ), the transfer matrix is called *causal* (*strictly causal*).

Throughout the paper, the plant we consider has  $m$  inputs and  $n$  outputs, and its transfer matrix, which is also called a *plant* itself simply, is denoted by  $P$  and belongs to  $\mathcal{P}^{n \times m}$ . We can always represent  $P$  in the form of a fraction  $P = ND^{-1}$  ( $P = \tilde{D}^{-1}\tilde{N}$ ), where  $N \in \mathcal{A}^{n \times m}$  ( $\tilde{N} \in \mathcal{A}^{n \times m}$ ) and  $D \in \mathcal{A}^{m \times m}$  ( $\tilde{D} \in \mathcal{A}^{n \times n}$ ) with nonsingular  $D$  ( $\tilde{D}$ ).

For  $P \in \mathcal{F}^{n \times m}$  and  $C \in \mathcal{F}^{m \times n}$ , a matrix  $H(P, C) \in \mathcal{F}^{(m+n) \times (m+n)}$  is defined as

$$H(P, C) := \begin{bmatrix} (I_n + PC)^{-1} & -P(I_m + CP)^{-1} \\ C(I_n + PC)^{-1} & (I_m + CP)^{-1} \end{bmatrix} \quad (1)$$

provided that  $\det(I_n + PC)$  is a nonzero of  $\mathcal{A}$ . This  $H(P, C)$  is the transfer matrix from  $[u_1^t \ u_2^t]^t$  to  $[e_1^t \ e_2^t]^t$  of the feedback system  $\Sigma$ . If  $\det(I_n + PC)$  is a nonzero of  $\mathcal{A}$  and  $H(P, C) \in \mathcal{A}^{(m+n) \times (m+n)}$ , then we say that the plant  $P$  is *stabilizable*,  $P$  is *stabilized* by  $C$ , and  $C$  is a *stabilizing controller* of  $P$ . In the definition above, we do not mention the causality of the stabilizing controller. However, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant [6].

It is known that  $W(P, C)$  defined below is over  $\mathcal{A}$  if and

only if  $H(P, C)$  is over  $\mathcal{A}$ :

$$W(P, C) := \begin{bmatrix} C(I_m + PC)^{-1} & -CP(I_m + CP)^{-1} \\ PC(I_n + PC)^{-1} & P(I_m + CP)^{-1} \end{bmatrix}. \quad (2)$$

This  $W(P, C)$  is the transfer matrix from  $u_1$  and  $u_2$  to  $y_1$  and  $y_2$ .

We employ the factorization approach [1], [8], [4], [2] and the symbols used in [9] and [5]. Also we will denote by  $\mathcal{S}(P)$  the set of stabilizing controllers of  $P$  and by  $\mathcal{W}(P)$  the set of all  $W(P, C)$ 's with  $C \in \mathcal{S}(P)$ .

### III. TWO-STAGE COMPENSATOR DESIGN

In some design problems, one uses a so-called *two-stage procedure* for selecting an appropriate stabilizing compensator [4]. Given a plant  $P$ , the first stage consists of selecting a stabilizing compensator for  $P$ . Let  $C_0 \in \mathcal{S}(P)$  denote this compensator (that is, an arbitrary but fixed compensator of  $P$ ) and define  $P_1 = P(I + C_0P)^{-1}$ . The second stage consists of selecting a stabilizing controller for  $P_1$  that also achieves some other design objectives such as decoupling, sensitivity minimization, etc. The rationale behind this procedure is that the design problems are often easier to solve when the plant is stable. The resulting configuration with its inner and outer loops is shown in Figure 2.

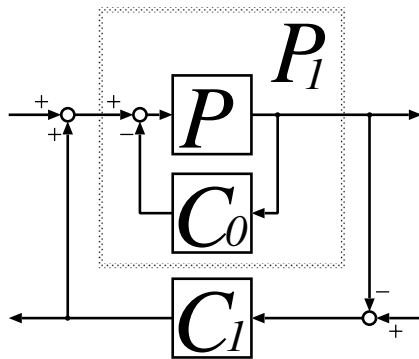


Fig. 2. Two-Stage Compensator Design ( $y_2$  to  $u_2$ ).

Here we give that the two-stage compensator design based on Figure 2 cannot give all stabilizing controllers. The following Theorem 1 is same as Theorem 5.3.10 of [4]. More detailed result can be found in Mori [10], which is shown as Theorem 2.

**Theorem 1:** Let  $P$  denote a causal plant of  $\mathcal{P}^{n \times m}$  and  $C_0$  a causal stabilizing controller of  $P$  ( $C_0 \in \mathcal{P}^{m \times n}$ ). Further let  $P_1$  be  $P(I_m + C_0P)^{-1}$ . Denote by  $C_0 + \mathcal{S}(P_1)$  the following set:

$$\{C_0 + C_1 \mid C_1 \in \mathcal{S}(P_1)\}.$$

Then

$$C_0 + \mathcal{S}(P_1) \subset \mathcal{S}(P), \quad (3)$$

with equality holding if and only if  $C_0 \in \mathcal{A}^{m \times n}$ .

**Theorem 2:** Let  $P, C_0, P_1$  be as in Theorem 1.

Let  $N, D, \tilde{N}, \tilde{D}, Y, X, \tilde{Y}, \tilde{X}$  be matrices over  $\mathcal{A}$  such that

$$\begin{cases} P = ND^{-1} = \tilde{D}^{-1}\tilde{N}, & C_0 = YX^{-1} = \tilde{X}^{-1}\tilde{Y}, \\ \tilde{Y}N + \tilde{X}D = I, & \tilde{N}Y + \tilde{D}X = I. \end{cases} \quad (4)$$

Then we have

$$C_0 + \mathcal{S}(P_1) \quad (5)$$

$$= \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid R = \tilde{X}R_1X, R_1 \in \mathcal{A}^{m \times n}\} \quad (6)$$

$$= \{(Y + DR)(X - NR)^{-1} \mid R = \tilde{X}R_1X, R_1 \in \mathcal{A}^{m \times n}\} \quad (7)$$

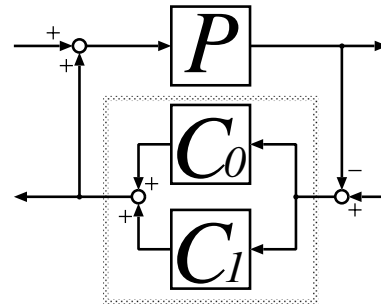


Fig. 3. Composite Stabilized Feedback with  $C_0$  and  $C_1$ .

By Theorem 1, we see that the sum of  $C_0$  and a stabilizing controller of  $P_1$ , say  $C_1$ , is again a stabilizing controller of  $P$ . This sum, a stabilizing controller of  $P$ , is the parallel allocation of  $C_0$  and  $C_1$ , as shown in Figure 3.

The stabilizing controller  $C_0$  is over  $\mathcal{A}$  if and only if the matrices  $\tilde{X}$  and  $X$  are unimodular. That is the equality of (3) holds if and only if  $C_0$  is over  $\mathcal{A}$ .

### IV. "FULL FEEDBACK" TWO-STAGE COMPENSATOR DESIGN

The two-stage compensator design of the previous section was based on the feedback from  $y_2$  to  $u_2$  (cf. Figures 1 and 2). Even so, we note that we have two inputs  $u_1$  and  $u_2$  and two outputs  $y_1$  and  $y_2$ . Thus we consider the feedback as shown in Figure 4. This is a feedback from  $y_1$  and  $y_2$  to  $u_1$  and  $u_2$ .

In Figure 4,  $C_0$  is a stabilizing controller of  $P$  and  $C_1$  is a stabilizing controller of  $W(P, C_0)$ . By noting that  $W(P, C_0)$  is over  $\mathcal{A}$ , the parametrization of  $C_1$  is given as

$$\begin{aligned} C_1 &= (I_{m+n} - R_1W(P, C_0))^{-1}R_1 \\ &= R_1(I_{m+n} - W(P, C_0)R_1)^{-1} \end{aligned} \quad (8)$$

with a parameter  $R_1 \in \mathcal{A}^{(m+n) \times (m+n)}$ .

We next decompose this  $C_1$  as follows:

$$C_1 = \begin{matrix} m & n \\ n & m \end{matrix} \begin{bmatrix} C_{111} & C_{112} \\ C_{121} & C_{122} \end{bmatrix}.$$

where  $C_{111} \in \mathcal{P}^{n \times m}$ ,  $C_{112} \in \mathcal{P}^{n \times n}$ ,  $C_{121} \in \mathcal{P}^{m \times m}$ , and  $C_{122} \in \mathcal{P}^{m \times n}$ . Also we next decompose  $R_1$  as follows:

$$R_1 = \begin{matrix} m & n \\ n & m \end{matrix} \begin{bmatrix} R_{111} & R_{112} \\ R_{121} & R_{122} \end{bmatrix},$$

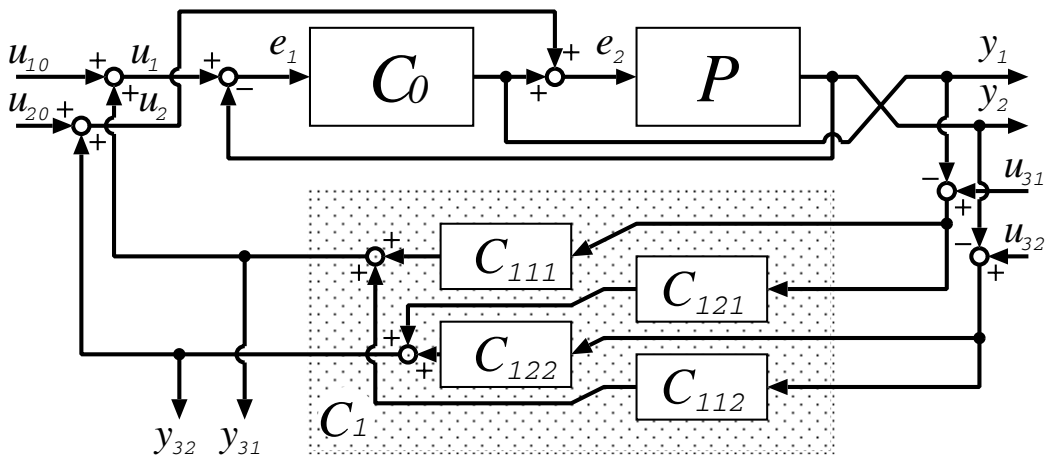


Fig. 4. Composite Stabilized Feedback with  $C_0$  and  $C_1$ .

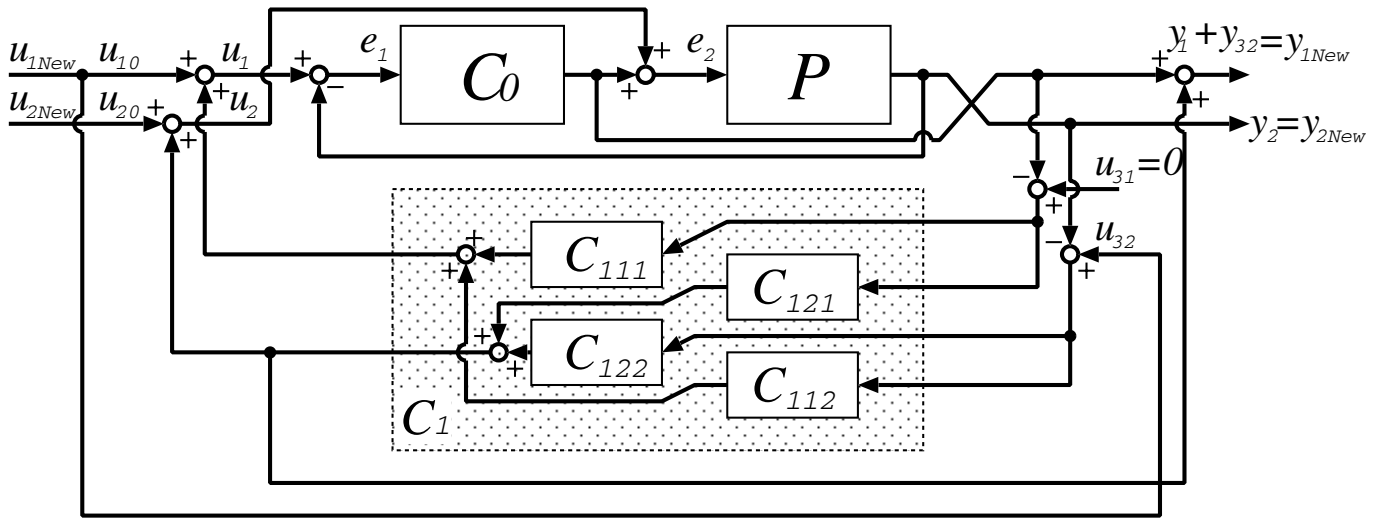


Fig. 5. Feedback System with  $C_0$  and  $C_1$ .

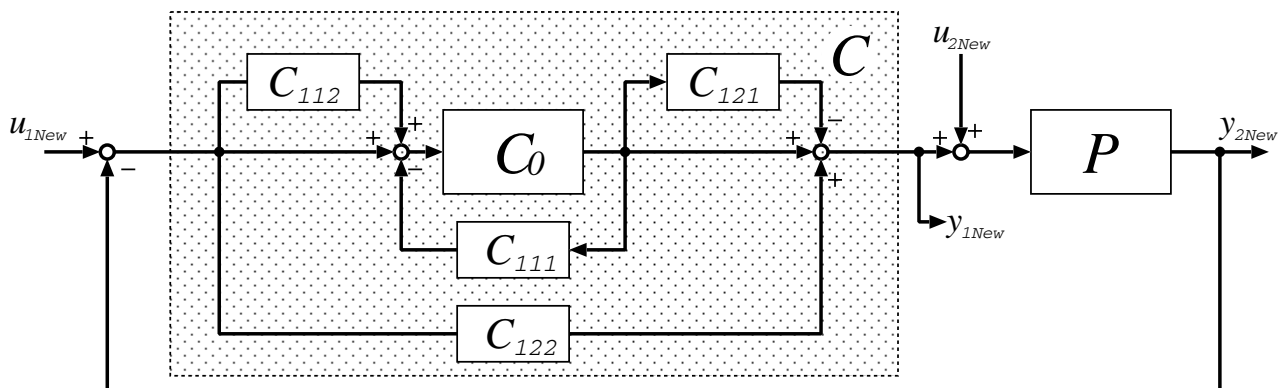


Fig. 6. Reconfigured Feedback System

TABLE I  
 FEEDBACK FROM OUTPUTS TO INPUTS

No.	Input(s)	Output(s)
1	$u_1$	$y_1$
2	$u_1$	$y_2$
3	$u_2$	$y_1$
4	$u_2$	$y_2$
5	$u_1, u_2$	$y_1$
6	$u_1, u_2$	$y_2$
7	$u_1$	$y_1, y_2$
8	$u_2$	$y_1, y_2$
9	$u_1, u_2$	$y_1, y_2$

where  $R_{111} \in \mathcal{A}^{n \times m}$ ,  $R_{112} \in \mathcal{A}^{n \times n}$ ,  $R_{121} \in \mathcal{A}^{m \times m}$ , and  $R_{122} \in \mathcal{A}^{m \times n}$ . Using these parameters, we have the following theorem.

**Theorem 3:** Any stabilizing controller  $C$  of  $P$  is in the form  $C = C_n C_d^{-1}$ , where

$$C_n = \tilde{Y} + (\tilde{Y} R_{112} X + \tilde{X} R_{122} X - \tilde{Y} R_{111} Y - \tilde{X} R_{121} Y) \tilde{D}, \quad (9)$$

$$C_d = \tilde{X} - (\tilde{Y} R_{112} X + \tilde{X} R_{122} X - \tilde{Y} R_{111} Y - \tilde{X} R_{121} Y) \tilde{N}. \quad (10)$$

*Proof:* (Due to the space limitation, we give a brief proof only.) First we consider the feedback system as in Figure 5. This can be reconfigured as in Figure 6.

From the matrix computation with  $C$  in the description of Theorem 3, we have a matrix equation:

$$W(P, C) = \begin{bmatrix} O & I_m & I_m & O \\ O & O & O & I_n \end{bmatrix} W(W(P, C_0), C_1) \begin{bmatrix} O & O \\ I_n & O \\ I_n & O \\ O & I_m \end{bmatrix}.$$

Thus  $C$  is a stabilizing controller of the plant  $P$ .

On the other hand, by letting

$$\begin{aligned} R_{111} &= -NR\tilde{N}, \\ R_{112} &= NR\tilde{D}, \\ R_{121} &= -DR\tilde{N}, \\ R_{122} &= DR\tilde{D}, \end{aligned}$$

we have

$$C = (\tilde{Y} + R\tilde{D})(\tilde{X} - R\tilde{N})^{-1},$$

This means any stabilizing controller can be expressed as in (9) and (10). ■

## V. PARTIAL TWO-STAGE COMPENSATOR DESIGNS

From Theorem 3, we can derive two-stage compensator designs based on various feedback styles. From Figure 1, we have two inputs  $u_1$  and  $u_2$ , and two outputs  $y_1$  and  $y_2$ . Thus we have 9 types of feedbacks, which are shown in Table I. For each type of feedback, we can obtain a two-stage compensator design.

In the following, we first, review the two-stage compensator design based on the feedback from  $y_2$  to  $u_2$  (No. 4). Then we give the two-stage compensator designs based on the feedbacks of Nos. 5 to 8. The feedback of No. 9 is just Theorem 3.

### (Review) Feedback from $y_2$ to $u_2$ (No. 4)

Let us review the two-stage compensator design of Section III, which is based on the feedback from  $y_2$  to  $u_2$ . In this case, the feedback system is as in Figure 2. Under the current situation, we have used new symbol  $C_{122}$  instead of  $C_1$ , which is shown in Figure 7.

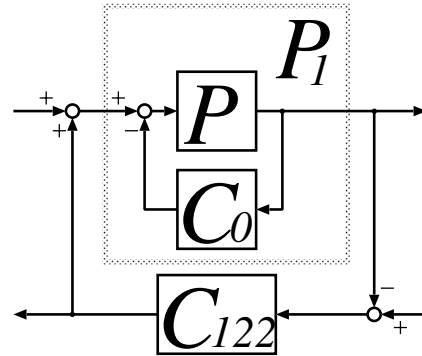


Fig. 7. Feedback from  $y_2$  to  $u_2$ .

This is the case where  $C_{111}$ ,  $C_{112}$ , and  $C_{121}$  of Figures 5 and 6 are zero matrices. From  $(I_{m+n} - R_1 W(P, C_0))^{-1} R_1$  of (8),  $R_{111}$  and  $R_{121}$  are zero matrices. Analogously, from  $R_1 (I_{m+n} - W(P, C_0) R_1)^{-1}$  of (8),  $R_{111}$  and  $R_{112}$  are zero matrices. Thus only  $R_{122}$  can be nonzero matrix and other  $R_{111}$ ,  $R_{112}$ , and  $R_{121}$  are zero matrices. This implies Theorem 2.

The procedure to obtain a stabilizing controller based on this feedback (No. 4) is summarized below.

### Input

- $P$ : A plant to be stabilized ( $\in \mathcal{P}^{n \times m}$ ).
- $C_0$ : A stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).
- $R_1$ : A parameter matrix ( $\in \mathcal{A}^{m \times n}$ ).

### Output

- $C_{\text{New}}$ : A new stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).

### Procedure

- (i) Let  $P_1 = P(I_m + C_0 P)^{-1}$ .
- (ii) Assume  $I_m - R_1 P_1$  is nonsingular.
- (iii) Let  $C_1 = (I_m - R_1 P_1)^{-1} R_1$ .
- (iv) Let  $C_{\text{New}} = C_0 + C_1$ .
- (v) Return  $(C_{\text{New}})$ .

### Feedback from $y_1$ to $u_1$ and $u_2$ (No. 5)

Let us consider the two-stage compensator design based on the feedback from  $y_1$  to  $u_1$  and  $u_2$ . That is, we do not consider the output  $u_2$ . In this case, the feedback system is as in Figure 8.

This is the case where  $C_{112}$  and  $C_{122}$  of Figures 5 and 6 are zero matrices. From  $(I_{m+n} - R_1 W(P, C_0))^{-1} R_1$  of (8),  $R_{112}$  and  $R_{122}$  are zero matrices. Thus only  $R_{111}$  and  $R_{121}$  can be nonzero matrix. Hence the configuration is as in Figure 9.

The following is a derivative result of Theorem 3 based on the feedback from  $y_1$  to  $u_1$  and  $u_2$ .

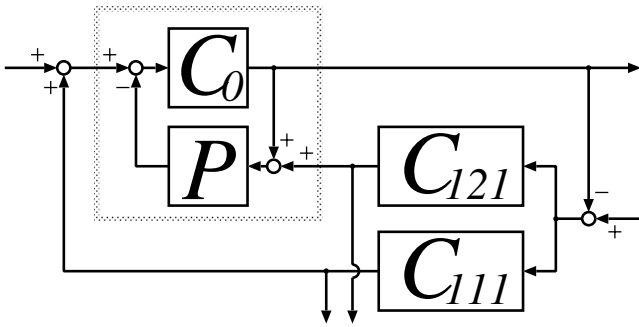


Fig. 8. Feedback from  $y_1$  to  $u_1$  and  $u_2$ .

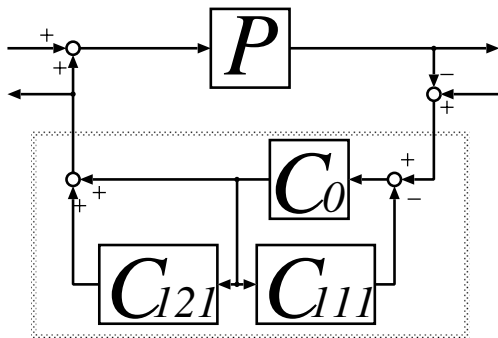


Fig. 9. Composite Stabilized Feedback based on Feedback from  $y_1$  to  $u_1$  and  $u_2$ .

**Theorem 4:** Let  $P_{y_1 u_1 u_2}$  denote

$$[C_0(I_n + PC_0)^{-1} \quad -C_0P(I_m + C_0P)^{-1}].$$

Then we have

$$\begin{aligned} & \{(I_n + C_{121})(I_m + C_0C_{111})^{-1}C_0 \mid \\ & \quad \begin{bmatrix} C_{111} \\ C_{121} \end{bmatrix} \in \mathcal{S}(P_{y_1 u_1 u_2}), \\ & \quad |I_m + C_0C_{111}| \neq 0\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = -\tilde{Y}R_{111}Y - \tilde{X}R_{121}Y, \\ & \quad R_{111} \in \mathcal{A}^{n \times m}, R_{121} \in \mathcal{A}^{m \times m}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = R'Y, R' \in \mathcal{A}^{m \times m}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\}. \end{aligned} \quad (11)$$

The procedure to obtain a stabilizing controller based on this feedback is summarized below.

**Input**

- $P$ : A plant to be stabilized ( $\in \mathcal{P}^{n \times m}$ ).
- $C_0$ : A stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).
- $R_1$ : A parameter matrix ( $\in \mathcal{A}^{(m+n) \times m}$ ).

**Output**

- $C_{New}$ : A new stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).

**Procedure**

- (i) Let  $P_1 = [C_0(I_n + PC_0)^{-1} \quad -C_0P(I_m + C_0P)^{-1}]$ .

- (ii) Let  $C_1 = (I_{m+n} - R_1P_1)^{-1}R_1$ .
- (iii) Decompose  $[C_{111}^t \quad C_{121}^t]^t := C_1$  with  $C_{111} \in \mathcal{F}^{n \times m}$  and  $C_{121} \in \mathcal{F}^{m \times m}$ .
- (iv) Assume  $I_m + C_0C_{111}$  is nonsingular.
- (v) Let  $C_{New} = (I_m + C_{121})(I_m + C_0C_{111})^{-1}C_0$ .
- (vi) Return ( $C_{New}$ ).

**Feedback from  $y_2$  to  $u_1$  and  $u_2$  (No. 6)**

The feedback of No.5 described above has employed the output  $y_1$ . Alternatively we now consider to use the output  $y_2$ . That is, consider the two-stage compensator design based on the feedback from  $y_2$  to  $u_1$  and  $u_2$ . In this case, the feedback system is as in Figure 10.

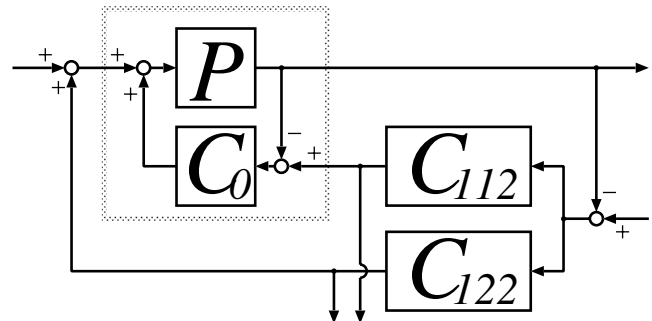


Fig. 10. Feedback from  $y_2$  to  $u_1$  and  $u_2$ .

This is the case where  $C_{111}$  and  $C_{121}$  of Figures 5 and 6 are zero matrices. From  $(I_{m+n} - R_1W(P, C_0))^{-1}R_1$  of (8),  $R_{111}$  and  $R_{121}$  are zero matrices. Thus only  $R_{112}$  and  $R_{122}$  can be nonzero matrix. Hence the configuration is as in Figure 11.

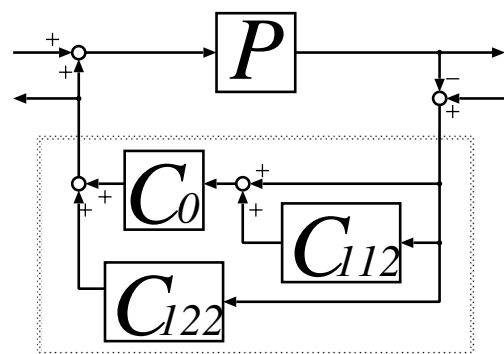


Fig. 11. Composite Stabilized Feedback based on Feedback from  $y_2$  to  $u_1$  and  $u_2$ .

The following is a derivative result of Theorem 3 based on the feedback from  $y_1$  to  $u_1$  and  $u_2$ .

**Theorem 5:** Let  $P_{y_2 u_1 u_2}$  denote

$$[PC_0(I_n + PC_0)^{-1} \quad P(I_m + C_0P)^{-1}].$$

Then we have

$$\begin{aligned} & \{C_0(I_n + C_{112}) + C_{122} \mid \begin{bmatrix} C_{112} \\ C_{122} \end{bmatrix} \in \mathcal{S}(P_{y_2 u_1 u_2})\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = \tilde{Y}R_{112}X + \tilde{X}R_{122}X, \\ & \quad R_{112} \in \mathcal{A}^{n \times n}, R_{122} \in \mathcal{A}^{m \times n}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = R'X, R' \in \mathcal{A}^{m \times n}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\}. \end{aligned} \quad (12)$$

The procedure to obtain a stabilizing controller based on this feedback is summarized below.

**Input**

- $P$ : A plant to be stabilized ( $\in \mathcal{P}^{n \times m}$ ).
- $C_0$ : A stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).
- $R_1$ : A parameter matrix ( $\in \mathcal{A}^{(m+n) \times n}$ ).

**Output**

- $C_{New}$ : A new stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).

**Procedure**

- (i) Let  $P_1 = [PC_0(I_n + PC_0)^{-1} \quad P(I_m + C_0P)^{-1}]$ .
- (ii) Let  $C_1 = (I_{m+n} - R_1P_1)^{-1}R_1$ .
- (iii) Decompose  $\begin{bmatrix} C_{112} & C_{122}^t \end{bmatrix}^t := C_1$  with  $C_{112} \in \mathcal{F}^{n \times n}$  and  $C_{122} \in \mathcal{F}^{m \times n}$ .
- (iv) Let  $C_{New} = C_0(I_n + C_{112}) + C_{122}$ .
- (v) Return ( $C_{New}$ ).

*Feedback from  $y_1$  and  $y_2$  to  $u_1$  (No. 7)*

The feedbacks of Nos.5 and 6 described above has employed two inputs. In the following, we consider one inputs and two outputs. First, we consider the feedback from  $y_1$  and  $y_2$  to  $u_1$ .

In this case, the feedback system is as in Figure 12.

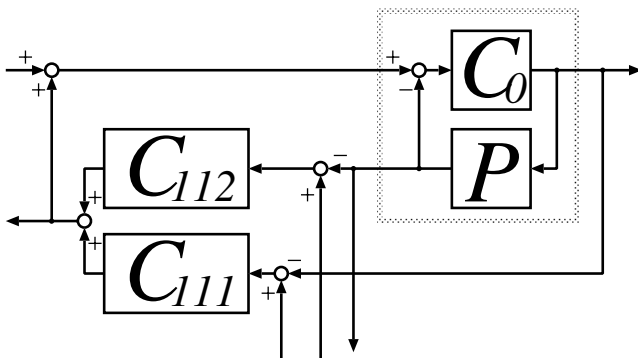


Fig. 12. Feedback from  $y_1$  and  $y_2$  to  $u_1$ .

This is the case where  $C_{121}$  and  $C_{122}$  of Figures 5 and 6 are zero matrices. From  $R_1(I_{m+n} - W(P, C_0)R_1)^{-1}$  of (8),  $R_{121}$  and  $R_{122}$  are zero matrices. Thus only  $R_{111}$  and  $R_{112}$  can be nonzero matrix. Hence the configuration is as in Figure 13.

The following is a derivative result of Theorem3 based on the feedback from  $y_1$  and  $y_1$  to  $u_1$ .

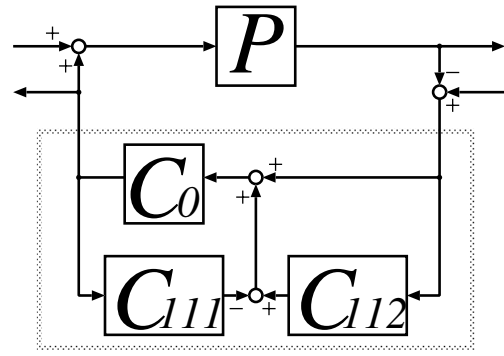


Fig. 13. Composite Stabilized Feedback based on Feedback from  $y_1$  and  $y_2$  to  $u_1$ .

*Theorem 6:* Let  $P_{y_1 y_2 u_1}$  denote

$$\begin{bmatrix} C_0(I_n + PC_0)^{-1} \\ PC_0(I_n + PC_0)^{-1} \end{bmatrix}.$$

Then we have

$$\begin{aligned} & \{(I_m + C_0C_{111})^{-1}C_0(I_n + C_{112}) \mid \\ & \quad [C_{111} \quad C_{112}] \in \mathcal{S}(P_{y_1 y_2 u_1})\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = \tilde{Y}R_{112}X - \tilde{Y}R_{111}Y, \\ & \quad R_{112} \in \mathcal{A}^{n \times n}, R_{111} \in \mathcal{A}^{n \times m}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = \tilde{Y}R', R' \in \mathcal{A}^{n \times n}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\}. \end{aligned} \quad (13)$$

The procedure to obtain a stabilizing controller based on this feedback is summarized below.

**Input**

- $P$ : A plant to be stabilized ( $\in \mathcal{P}^{n \times m}$ ).
- $C_0$ : A stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).
- $R_1$ : A parameter matrix ( $\in \mathcal{A}^{n \times (m+n)}$ ).

**Output**

- $C_{New}$ : A new stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).

**Procedure**

- (i) Let  $P_1 = \begin{bmatrix} C_0(I_n + PC_0)^{-1} \\ PC_0(I_n + PC_0)^{-1} \end{bmatrix}$ .
- (ii) Let  $C_1 = R_1(I_{m+n} - P_1R_1)^{-1}$ .
- (iii) Decompose  $\begin{bmatrix} C_{111} & C_{112} \end{bmatrix} := C_1$  with  $C_{111} \in \mathcal{F}^{n \times m}$  and  $C_{112} \in \mathcal{F}^{n \times n}$ .
- (iv) Assume  $I_m + C_0C_{111}$  is nonsingular.
- (v) Let  $C_{New} = (I_m + C_0C_{111})^{-1}C_0(I_n + C_{112})$ .
- (vi) Return ( $C_{New}$ ).

*Feedback from  $y_1$  and  $y_2$  to  $u_2$  (No. 8)*

The final feedback is that from from  $y_1$  and  $y_2$  to  $u_2$ .

In this case, the feedback system is as in Figure 14.

This is the case where  $C_{111}$  and  $C_{112}$  of Figures 5 and 6 are zero matrices. From  $R_1(I_{m+n} - W(P, C_0)R_1)^{-1}$  of (8),  $R_{111}$

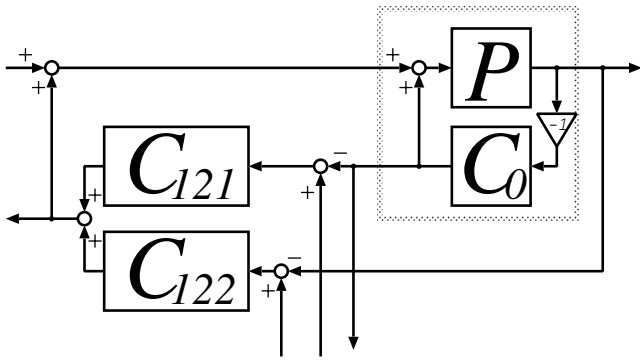


Fig. 14. Feedback from  $y_1$  and  $y_2$  to  $u_2$ .

and  $R_{112}$  are zero matrices. Thus only  $R_{121}$  and  $R_{122}$  can be nonzero matrix. Hence the configuration is as in Figure 15.

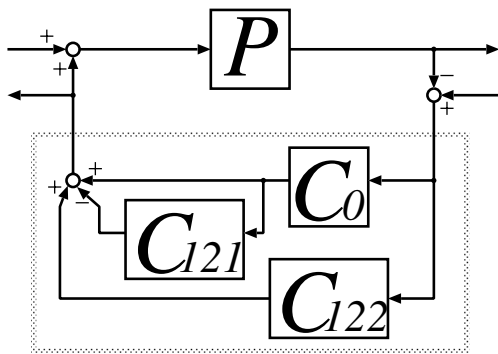


Fig. 15. Composite Stabilized Feedback based on Feedback from  $y_1$  and  $y_2$  to  $u_2$ .

The following is a derivative result of Theorem 3 based on the feedback from  $y_1$  and  $y_2$  to  $u_1$ .

**Theorem 7:** Let  $P_{y_1 y_2 u_2}$  denote

$$\begin{bmatrix} -C_0 P (I_m + C_0 P)^{-1} \\ P (I_m + C_0 P)^{-1} \end{bmatrix}.$$

Then we have

$$\begin{aligned} & \{(I_m + C_{121})C_0 + C_{122} \mid \\ & \quad [C_{121} \quad C_{122}] \in \mathcal{S}(P_{y_1 y_2 u_2})\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = \tilde{X}R_{122}X - \tilde{X}R_{121}Y, \\ & \quad R_{121} \in \mathcal{A}^{m \times m}, R_{122} \in \mathcal{A}^{m \times n}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\} \\ & = \{(\tilde{X} - R\tilde{N})^{-1}(\tilde{Y} + R\tilde{D}) \mid \\ & \quad R = \tilde{X}R', R' \in \mathcal{A}^{m \times n}, \\ & \quad |\tilde{X} - R\tilde{N}| \neq 0\}. \end{aligned} \quad (14)$$

The procedure to obtain a stabilizing controller based on this feedback is summarized below.

**Input**

- $P$ : A plant to be stabilized ( $\in \mathcal{P}^{n \times m}$ ).
- $C_0$ : A stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).

$R_1$ : A parameter matrix ( $\in \mathcal{A}^{m \times (m+n)}$ ).

**Output**

$C_{New}$ : A new stabilizing controller ( $\in \mathcal{F}^{m \times n}$ ).

**Procedure**

- (i) Let  $P_1 = \begin{bmatrix} -C_0 P (I_m + C_0 P)^{-1} \\ P (I_m + C_0 P)^{-1} \end{bmatrix}$ .
- (ii) Let  $C_1 = R_1 (I_{m+n} - P_1 R_1)^{-1}$ .
- (iii) Decompose  $[C_{121} \quad C_{122}] := C_1$  with  $C_{121} \in \mathcal{F}^{m \times m}$  and  $C_{122} \in \mathcal{F}^{m \times n}$ .
- (iv) Let  $C_{New} = (I_m + C_{121})C_0 + C_{122}$ .
- (v) Return ( $C_{New}$ ).

**VI. CONCLUSION**

In this paper, we have investigated two-stage compensator designs. We have given five two-stage compensator designs with partial feedbacks. All results are given based on the factorization approach, so that the results can be applied to numerous linear systems.

As future works, we will consider to obtain the optimized stabilizing controller within the framework of two-stage compensator designs. Our result of this paper have shown that the two-stage compensator designs with partial feedbacks cannot give, in general, all stabilizing controllers. Even so, the optimized stabilizing controller may be included in the set parameterized as in Theorems 4 to 7. We will need to investigate these criteria.

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