

A Novel System of Two Coupled Equations for the Longitudinal Components of the Electromagnetic Field in a Waveguide

Arti Vaish and Harish Parthasarathy

Abstract—In this paper, a novel wave equation for electromagnetic waves in a medium having anisotropic permittivity has been derived with the help of Maxwell's curl equations. The x and y components of the Maxwell's equations are written with the permittivity (ϵ) being a 3×3 symmetric matrix. These equations are solved for E_x , E_y , H_x , H_y in terms of E_z , H_z , and the partial derivatives. The Z components of the Maxwell's curl are then used to arrive to the generalized Helmholtz equations for E_z and H_z .

Keywords—Electromagnetism, Maxwell's Equations, Anisotropic permittivity, Wave equation, Matrix Equation, Permittivity tensor.

I. INTRODUCTION

OVER the several decades, anisotropic materials have been extensively analysed and studied as they can successfully describe the real-world configurations. In addition to this, devices incorporating these materials acquire interesting directionally dependent properties which are useful in their operations [1]. In recent years, there have been increasing interests in the interactions between electromagnetic fields and anisotropic media [2]-[20]. Anisotropic medium has wide applications in the design and analysis of various novel antenna and microwave devices of high performance and utility [21]-[25]. With the advancement of the technology, the reconstruction of anisotropic constructive parameters and principal axis in three dimensional problem is of practical importance in microwave range. In the studies of Fedorov, and Borzdov, the direction of the principal axis of the anisotropic medium was obtained [24]-[27]. The idea of excluding the transverse components in favour of the longitudinal components of the electromagnetic field in a anisotropic medium has been applied, number of times, in connection with the anisotropic dielectric waveguide or optic fibres [28]-[38]. This work extends this idea for a more general model of a medium. In particular, a novel system of two coupled equations has been presented for the longitudinal components of the electromagnetic field in a waveguide filled with a homogeneous medium characterized by a general symmetric permittivity tensor.

II. WAVE EQUATION WITH ANISOTROPIC PERMITTIVITY

The vector wave equation which describes the wave propagation along a rectangular waveguide with a nonhomogeneous

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cross section and anisotropic dielectric material may be obtained from the Maxwell equations. The fields are assumed to propagate in the Z direction and have a harmonic time dependence of the form $e^{j\omega t - \gamma z}$ where $\gamma = \alpha + j\beta$, with ω , α and β being the angular frequency, the attenuation constant and the phase constant, respectively. The magnetic permeability μ is assumed to be constant and equal to the free space permeability, i.e., $\mu = \mu_0$. The Maxwell equations are given as follow:

$$\nabla \times E = -j\omega\mu H \quad (1)$$

$$\nabla \times H = j\omega\epsilon E \quad (2)$$

where the electric and magnetic field vectors E and H are given by

$$E = E_x i + E_y j + E_z k, \quad (3)$$

$$H = H_x i + H_y j + H_z k. \quad (4)$$

Here i , j and k are the unit vectors along the x , y and z directions. The tensor permittivity ϵ is given by,

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \quad (5)$$

Now expanding equation (1) and comparing the coefficients of i , j and k

$$\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{pmatrix} = -j\omega\mu(H_x i + H_y j + H_z k) \quad (6)$$

It can be written as follow:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \quad \text{or} \quad E_{z,y} + \gamma E_y = -j\omega\mu H_x \quad (7)$$

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = j\omega\mu H_y \quad \text{or} \quad E_{z,x} + \gamma E_x = j\omega\mu H_y \quad (8)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad \text{or} \quad E_{y,x} - E_{x,y} = -j\omega\mu H_z \quad (9)$$

where

$$\frac{\partial}{\partial z} = -\gamma \quad \text{and} \quad E_{i,j} = \frac{\partial E_i}{\partial j} \quad i, j = x, y, z$$

Similarly, from equation (2), we have

$$H_{z,y} + \gamma H_y = j\omega(\epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z) \quad (10)$$

$$H_{z,x} + \gamma H_x = -j\omega(\epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z) \quad (11)$$

$$H_{y,x} - H_{x,y} = j\omega(\epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z) \quad (12)$$

After rearranging equations (7), (8), (10) and (11), we have:

$$\gamma E_y + j\omega\mu H_x = -E_{z,y} \quad (13)$$

$$\gamma E_x - j\omega\mu H_y = -E_{z,x} \quad (14)$$

$$j\omega\epsilon_{xx}E_x + j\omega\epsilon_{xy}E_y - \gamma H_y = H_{z,y} - j\omega\epsilon_{xz}E_z \quad (15)$$

$$j\omega\epsilon_{yx}E_x + j\omega\epsilon_{yy}E_y + \gamma H_x = -j\omega\epsilon_{yz}E_z - H_{z,x} \quad (16)$$

The above four equations can be written as a matrix equation for the variables E_x , E_y , H_x and H_y in terms of E_z and H_z and their partial derivatives as follow:

$$\begin{pmatrix} 0 & \gamma & j\omega\mu & 0 \\ \gamma & 0 & 0 & -j\omega\mu \\ j\omega\epsilon_{xx} & j\omega\epsilon_{xy} & 0 & -\gamma \\ j\omega\epsilon_{yx} & j\omega\epsilon_{yy} & \gamma & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} = \begin{pmatrix} -j\omega\epsilon_{yz}E_z - H_{z,x} - E_{z,y} \\ -E_{z,x} \\ H_{z,y} - j\omega\epsilon_{xz}E_z \\ -j\omega\epsilon_{yz}E_z - H_{z,x} \end{pmatrix} \quad (17)$$

Or

$$\begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} = \begin{pmatrix} 0 & \gamma & j\omega\mu & 0 \\ \gamma & 0 & 0 & -j\omega\mu \\ j\omega\epsilon_{xx} & j\omega\epsilon_{xy} & 0 & -\gamma \\ j\omega\epsilon_{yx} & j\omega\epsilon_{yy} & \gamma & 0 \end{pmatrix}^{-1} \begin{pmatrix} -j\omega\epsilon_{yz}E_z - H_{z,x} - E_{z,y} \\ -E_{z,x} \\ H_{z,y} - j\omega\epsilon_{xz}E_z \\ -j\omega\epsilon_{yz}E_z - H_{z,x} \end{pmatrix} \quad (18)$$

Letting A denote the coefficient matrix [39] and $\Delta = |A|$, we have

$$A^{-1} = \frac{C}{\Delta} = B \quad (19)$$

Now the determinant of matrix A is given by

$$\Delta = |A| = -\gamma^4 - \gamma^2\omega^2\mu(\epsilon_{xx} + \epsilon_{yy}) - \omega^4\mu^2(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx}) \quad (20)$$

We write down explicit expression for all the matrix elements of B

$$b_{ij} = \frac{(-1)^{i+j} D_{ji}}{\Delta} \quad (21)$$

Here,

$$\Delta = -\omega^4 \left[\frac{1}{c^4} - \frac{\mu(\epsilon_{xx} + \epsilon_{yy})}{c^2} - \mu^2(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx}) \right] = -\omega^4 k \quad (22)$$

$$k = \left[\frac{1}{c^4} - \frac{\mu(\epsilon_{xx} + \epsilon_{yy})}{c^2} - \mu^2(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx}) \right] \quad (23)$$

Here c is the speed of light and is equal to 3×10^8 meter/second and k is an arbitrary constant. The solution to the transverse components of the electric and magnetic fields is given by

$$\begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} = B \begin{pmatrix} -j\omega\epsilon_{yz}E_z - H_{z,x} - E_{z,y} \\ -E_{z,x} \\ H_{z,y} - j\omega\epsilon_{xz}E_z \\ -j\omega\epsilon_{yz}E_z - H_{z,x} \end{pmatrix} \quad (24)$$

Now the components of matrix B and b_{ij} are given by

$$b_{[i=1-4, j=1]} = \begin{pmatrix} \omega^2\mu\gamma\epsilon_{xy} \\ -\gamma(\gamma^2 + \omega^2\mu\epsilon_{yy}) \\ j\omega\gamma^2\epsilon_{yy} + j\omega^3\mu(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx}) \\ 0 \end{pmatrix}$$

$$b_{[i=1-4, j=2]} = \begin{pmatrix} -\gamma(\gamma^2 + \omega^2\mu\epsilon_{yy}) \\ \omega^2\mu\gamma\epsilon_{yx} \\ j\omega\gamma^2\epsilon_{yx} \\ -j\omega\gamma^2\epsilon_{xx} - j\omega^3\mu(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx}) \end{pmatrix}$$

$$b_{[i=1-4, j=3,4]} = \begin{pmatrix} j\omega\mu(\gamma^2 + \omega^2\mu\epsilon_{yy}) & j\omega^3\mu^2\epsilon_{xy} \\ -j\omega^3\mu^2\epsilon_{yx} & j\omega\mu(\gamma^2 + \omega^2\mu\epsilon_{yy}) \\ \omega^2\mu\gamma\epsilon_{yx} & -\gamma(\gamma^2 + \omega^2\mu\epsilon_{xx}) \\ \gamma(\gamma^2 + \omega^2\mu\epsilon_{yy}) & -\omega^2\mu\gamma\epsilon_{xy} \end{pmatrix}$$

Now, with the help of above components and the equations 9 and 12, we get the following set of equations

$$\begin{aligned} & \gamma(\gamma^2 + \omega^2\mu\epsilon_{xx}) \frac{\partial^2 E_z}{\partial x \partial y} - j\omega^2\mu\gamma\epsilon_{yx} \frac{\partial^2 E_z}{\partial x^2} - j\omega^3\mu^2\epsilon_{yx} \\ & \left(\frac{\partial^2 H_z}{\partial x \partial y} - j\omega\epsilon_{xz} \frac{\partial E_z}{\partial x} \right) - j\omega\mu(\gamma^2 + \omega^2\mu\epsilon_{xx}) \\ & \left(\frac{\partial^2 H_z}{\partial x^2} + j\omega\epsilon_{yz} \frac{\partial E_z}{\partial x} \right) + \omega^2\mu\epsilon_{xy} \frac{\partial^2 E_z}{\partial y^2} + \gamma(\gamma^2 + \omega^2\mu\epsilon_{yy}) \\ & \frac{\partial^2 E_z}{\partial y \partial x} - j\omega\mu(\gamma^2 + \omega^2\mu\epsilon_{yy}) \left(\frac{\partial^2 H_z}{\partial y^2} + j\omega\epsilon_{yz} \frac{\partial E_z}{\partial y} \right) \\ & - j\omega^3\mu\epsilon_{xy} \left(\frac{\partial^2 E_z}{\partial y \partial x} + j\omega\epsilon_{yz} \frac{\partial E_z}{\partial y} \right) = -j\omega\mu H_z \Delta \end{aligned} \quad (25)$$

$$\begin{aligned} & \omega^2\mu \left[\frac{\partial H_z}{\partial x} (\epsilon_{zy}(\gamma^2 + \omega^2\mu\epsilon_{xx}) + \omega^2\mu\epsilon_{xy}\epsilon_{zx}) \right] + \\ & \omega^2\mu \left[\frac{\partial H_z}{\partial y} (-\epsilon_{zx}(\gamma^2 + \omega^2\mu\epsilon_{yy}) + \mu\epsilon_{yx}\epsilon_{zy}) \right] + \\ & j\omega\gamma [\epsilon_{zy}(\gamma^2 + \omega^2\mu\epsilon_{xx}) - \omega^2\mu\epsilon_{xy}\epsilon_{zx}] \frac{\partial E_z}{\partial y} + j\omega E_z \\ & [\epsilon_{zz} + \omega^2\mu\epsilon_{zy}\epsilon_{yz}(\gamma^2 + \omega^2\mu\epsilon_{xx}) + \gamma^2\omega^2\mu\epsilon_{xz}\epsilon_{zx} + \\ & \omega^4\mu\epsilon_{zx}\epsilon_{yy}\epsilon_{xz}] = -j\omega\gamma^2\epsilon_{xx} - j\omega^3\mu(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx}) \\ & \frac{\partial^2 E_z}{\partial x^2} + \gamma(\gamma^2 + \omega^2\mu\epsilon_{yy}) \frac{\partial^2 H_z}{\partial x \partial y} + \omega^2\mu\gamma \frac{\partial^2 H_z}{\partial x^2} \end{aligned}$$

$$\begin{aligned}
 & +j\omega [\gamma^2\epsilon_{yy} + \omega^2\mu(\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx})] \frac{\partial^2 E_z}{\partial y^2} + \\
 & j\omega\gamma^2\epsilon_{yx} \frac{\partial^2 E_z}{\partial y\partial x} - \omega^2\mu\gamma\epsilon_{yx} \frac{\partial^2 H_z}{\partial y^2} \\
 & -\gamma(\gamma^2 + \omega^2\mu\epsilon_{xx}) \frac{\partial^2 H_z}{\partial y\partial x} - j\omega\gamma \\
 & [\epsilon_{yz}(\gamma^2 + \omega^2\mu\epsilon_{xx}) - \omega^2\mu\epsilon_{yx}\epsilon_{xz}]
 \end{aligned} \tag{26}$$

These are the two wave equations for a rectangular waveguide with anisotropic permittivity.

III. CONCLUSION

We write down the x and y components of the Maxwell's equations

$$\nabla \times E = -j\omega\mu_0 H \tag{27}$$

And

$$\nabla \times H = j\omega\epsilon E \tag{28}$$

Where ϵ is a 3×3 symmetric matrix, given by,

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \tag{29}$$

We impose the condition that the fields depend on z as $e^{j\omega t - \gamma z}$ where $\gamma = \alpha + j\beta$, ω , α and β being the angular frequency, the attenuation constant and the phase constant respectively. Then result is a set of four linear algebraic equations for E_x , E_y , H_x , H_y in terms of E_z , H_z , $E_{z,x}$, $E_{z,y}$, $H_{z,x}$ and $H_{z,y}$. These are solved yielding E_x , E_y , H_x , H_y in terms of E_z , H_z , $E_{z,x}$, $E_{z,y}$, $H_{z,x}$ and $H_{z,y}$. These expressions are then substituted into the z components of the $\nabla \times E$ and $\nabla \times H$ equations giving thereby a pair of second order coupled linear partial differential equations for (E_z, H_z) . It is to be noted that E_z or H_z can not be set to 0 because one of the pair of equations so obtained may not be satisfied. This is in contrast with the isotropic case.

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