On Submaximality in Intuitionistic Topological Spaces

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Abstract—In this study, a minimal submaximal element of LIT(X) (the lattice of all intuitionistic topologies for X, ordered by inclusion) is determined. Afterwards, a new contractive property, intuitionistic mega-connectedness, is defined. We show that the submaximality and mega-connectedness are not complementary intuitionistic topological invariants by identifying those members of LIT(X) which are intuitionistic mega-connected.

Keywords—Intuitionistic set; intuitionistic topology; intuitionistic submaximality and mega-connectedness.

I. INTRODUCTION

The submaximality was defined and characterized by Bourbaki ([2], [11]). It is worth mentioning that the submaximality is a significant condition for maximal topologies of many topological invariants (e.g. connectedness, quasi-H-closure and pseudo-compactness) [4]. Arhangel’skii and Collins have carried out a detailed study of how the submaximality affects the structure of familiar topological spaces and groups [16]. Dontchev has presented several characterizations of submaximality [8]. Also Dontchev and Rose have approached submaximality via topological ideals as characterizations of submaximality [8]. Recently, the submaximality has been studied prominently by many researchers [3,10].

Let LT(X) be the family of all topologies definable for an infinite set X forming a complete atomic and complemented lattice (ordered by inclusion). For a given member τ of LT(X), having property P, if all members of LT(X) weaker (stronger) than τ have the property P, then a topological invariant property P is called contractive (expansive). For a given contractive (expansive) property P, a member τ of LT(X) is called maximal P (minimal P) if τ has the property P but no stronger (weaker) member of LT(X) has not property P.

An expansive property P and a contractive property Q are called complementary when the minimal P members of LIT(X) coincide with the maximal Q members.

“T₁ and ‘all proper closed sets are finite’; door and ‘filter-connected’; T₀ and nested; disconnected and principal of order two” are some examples of complementary topological invariants ([4], [12], [13], [17]).

The main purpose of this article is to identify those members of LIT(X) (the lattice of all intuitionistic topologies for X, ordered by inclusion) which are minimal submaximal by using the definition of intuitionistic submaximality [15] and show that submaximality and megaconnectedness in Intuitionistic topological spaces are not complementary topological invariants.

II. PRELIMINARIES

The idea of “intuitionistic fuzzy set” was suggested by Krassimir T. Atanassov [1]. Later, D. Coker has presented the classical version of this concept [5]. The definitions which are actively used in this paper, are listed below.

Definition 2.1. Let X be a nonempty set. An intuitionistic set (IS for short) A is an object having the form $A = \langle X, A_1, A_2 \rangle$ (or $A = \langle A_1, A_2 \rangle$), where $A_1$ and $A_2$ are disjoint subsets of X. The set $A_1$ is called the set of members of A, while $A_2$ is called the set of nonmembers of A [5,1].

Definition 2.2. Let X be a nonempty set and A and B be in the form $A = \langle A_1, A_2 \rangle$, $B = \langle B_1, B_2 \rangle$, respectively. Furthermore, let $\{A_i, i \in I\}$ be an arbitrary family of IS’s in X, where $A_i = \langle A_{i1}, A_{i2} \rangle$ then;

a) $A = \langle A_1, A_2 \rangle$, $X = \langle X, O \rangle$

b) $A \subseteq B$ if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$

c) $A \supseteq B$ if $A_1 \supseteq B_1$ and $A_2 \subseteq B_2$

d) $\bigwedge_i A_i = \langle A_{1i}, A_{2i} \rangle$ here $A^c$ complementary of $A$

e) $\bigcap_i A_i = \langle \bigcap_i A_{1i}, \bigcup_i A_{2i} \rangle$ and $\bigcup_i A_i = \langle \bigcup_i A_{1i}, \bigcap_i A_{2i} \rangle$ [5].

Definition 2.3. An intuitionistic topology (IT for short) on a nonempty set X is a family $\tau$ of IS’s in X, where $\bigcap_i A_i$ is closed under arbitrary unions and finite intersections. In this case the pair $(X, \tau)$ is called an intuitionistic topological space (ITS for short) and any IS in $\tau$ is known as an intuitionistic open set (IOS for short) in $X$, the
Example 2.4. Any topological space \((X, \tau)\) is obviously an intuitionistic topological space with the form; 
\[
\exists \{A: A \in \tau\} \quad \text{whenever we identify a subset A in X with its counter pair A}=\langle A, A^c \rangle.
\]

Example 2.5. Let \(X=\{a,b,c,d\}\) and consider the family \(\tau=\{\emptyset, X, \{a\}, \{a,b\}, \{a,\}, \{a\}, \{c\}, \{c,d\}, \{c\}, \{d\}, \{a,c\}\}\), where \(\{a\}\), \(\{a,b\}\), \(\{a\}\), \(\{c\}\), \(\{c,d\}\), \(\{c\}\), \(\{d\}\), \(\{a,c\}\) are defined by:
\[
\text{int}(\{a\}) = \emptyset, \quad \text{int}(\{a,b\}) = \{a\}, \quad \text{int}(\{a\}) = \{a\}, \quad \text{int}(\{c\}) = \{c\}, \quad \text{int}(\{c,d\}) = \{c\}, \quad \text{int}(\{c\}) = \{c\}, \quad \text{int}(\{d\}) = \{d\}, \quad \text{int}(\{a,c\}) = \{a,c\}.
\]

Example 2.6. Let \(A\) be an IS. Then \(A\) is not an IOS. As a result, \(A\) \(\neq \emptyset\). Then \(A\) is not an IOS in X. Since \(A\) is an IS, \(A\) is a \(\tau\)-dense IS. Then \(A\) is an \(\tau\)-IOS. Then \(A\) is an \(\tau\)-IOS so \(A\) is an IS-sub.

Example 2.7. Let \(A\) be contained in \(\emptyset\) if we let \(B=\langle B_1, B_2, B \rangle\) be an IS in X. Then \(A\) is coarser than \(\emptyset\) if \(\emptyset\) is finer than \(\emptyset\) [7].

Example 2.8. Let \(\emptyset\) be an IS and \(\emptyset \subset A_{1}, A_{2}\) be an IS in X. Then \(\emptyset\) is a \(\emptyset\)-dense IS in X if \(\emptyset\) is a \(\emptyset\)-dense IS member of \(\emptyset\) [7].

Example 2.9. Let us consider the ITS \((X, \tau)\) where \(X=\{a,b,c,d,e\}\) and
\[
\tau=\{\emptyset, X, \{a\}, \{a,b\}, \{a\}, \{c\}, \{c,d\}, \{c\}, \{d\}, \{a,c\}\}\]
if we let \(B=\langle b, d, e \rangle\), then
\[
\text{int}(\{b\}) = \emptyset, \quad \text{int}(\{d\}) = \{d\}, \quad \text{int}(\{e\}) = \emptyset.
\]

Example 2.10. Let \((X, \tau)\) be an ITS. Then \(\tau\) is an IT-sub on X. Therefore, \(\tau\) is a \(\tau\)-dense IS in X. Since \(\tau\) is an IS, \(\tau\) is a \(\tau\)-dense IS. Then \(\tau\) is an \(\tau\)-IOS. Then \(\tau\) is an \(\tau\)-IOS so \(\tau\) is an IS-sub.

Example 3.2. Let \(X=\{a,b\}\) and the family
\[
\tau=\{\emptyset, X, \{a\}, \emptyset, \emptyset, \{a\}, \emptyset, \{b\}, \emptyset, \{a\}, \emptyset, \{b\}\}
\]
is an IT-sub on X.

Remark 3.3. As the following example indicates, it is not necessary that the intuitionistic form of a submaximal space is submaximal.

Example 3.4. Let \(X=\{1,2\}\) and \(\tau=\{\emptyset, X, \{1\}\}\), then the topological space \((X, \tau)\) is submaximal. The intuitionistic form of \(\tau\) is \(\tau=\{\emptyset, X, \{1\}, \{2\}\}\) and \(\tau=\{\emptyset, X, \{1,2\}\}\) is a \(\tau\)-dense IS in X. Then \(\tau\) is not an IOS. As a result, \(\tau\) is not an IT-sub on X.

Theorem 3.5. The submaximality is an expansive intuitionistic topological invariant.

Proof. Let \(\tau_1, \tau_2\) in \(\text{LIT}(X)\) such that \(\tau_1 \subset \tau_2\) and \(\tau_2\) is IT-sub. Then \(\tau_1\) is a \(\tau_2\)-IOS. Then \(\tau_1\) is a \(\tau_2\)-IOS.

Theorem 3.6. Let \(X\) be a non empty set and \(A\) be an IS. Then \(M(A)=\{G: A \subset G \text{ or } G \subset A\}\) is an IT-sub member of \(\text{LIT}(X)\).

Proof. Let us take \(A=\langle A_1, A_2 \rangle\).
If \(B=\langle B_1, B_2, B \rangle\) is any IS, since \(\emptyset, A_1 \cup B_1\) is an IOS then \(\text{cl}(B) \subset A_1 \cup B_1\).
If \(A_1 \cup B_1 \subset X\) then \(\text{cl}(B) \subset X\).
If \(A_2 \cup B_2 \subset X\) then \(\text{cl}(B) \subset X\).
If \(A_1 \cup B_1 \subset X\) and \(A_2 \cup B_2 \subset X\) and \(B\) is an IOS.
Therefore \(M(A)\) is submaximal IT on X.

McCARTAN [12] has shown that \(M(A)=\{G: A \subset G \text{ or } G \subset A\}\) is a minimal submaximal member of \(\text{LIT}(X)\).

Example 3.7. Let \(X=\{1,2,3\}\) and \(A=\{1\}\) be an IS. Consider the IT-sub members of \(\text{LIT}(X)\).

\[
M(A)=\{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}
\]
and
\[
\exists \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}\]

Since \(\tau\) is coarser than \(M(A)\), \(M(A)\) is not a minimal IT member of \(\text{LIT}(X)\).

Theorem 3.8. Let \(X\) be a nonempty set and \(A\) is any subset of \(X\). Then \(\delta(A)\) is a IT-sub member of \(\text{LIT}(X)\).

Proof. First of all we show that \(\delta(A)\) is a IT-sub on X. We look at the following cases:

i) If \(E=\{E_1, E_2\}\) be an \(\delta(A)\) then \(\text{cl}(E) \subset \{A \subset E_1, \emptyset \}

ii) If \(E \subset A\) then either \(E \in \delta(A)\) or \(\emptyset, A \subset E \subset \emptyset \). This implies that \(\text{cl}(E) \subset \emptyset, A \subset E \subset \emptyset\).
iii) If $A \subset E_1$ then either $E_1 = X$ in which case $E_1 = X \in \delta(A)$ or $cl(E_1) \subset \langle E_1, \emptyset \rangle = X$

iv) If $A \not\subset E_1$ and $E \not\subset A$ then either $E \in \delta(A)$ or $cl(E) \subset \langle A \cup E_1, \emptyset \rangle = X$

These cases show that $\delta(A)$ is an IT-sub.

Let $\tau$ be an IT-sub and $\tau \subset \delta(A)$. If any IS $E$ in the family $\{\langle C, D \rangle \colon A \subset C, D \subset C \}$ is $\delta(A)$-dense then $E$ is $\tau$-dense. Since $\tau$ is IT-sub, $E$ is an $\tau$-IOS. On the other hand, the IS in the form $\langle \emptyset, X \setminus \{x\} \rangle$ where $x \in A'$ must be $\tau$-IOS, because if $\langle \emptyset, X \setminus \{x\} \rangle$ is not an $\tau$-IOS then the IS $\langle X \setminus \{x\}, \emptyset \rangle$ is $\tau$-dense, this contradicts the submaximality of $\tau$. This implies that all the IS in the family $\{\langle \emptyset, B \rangle \colon A \subset B, B \subset X\}$ are $\tau$-IOS. From this $\delta(A) = \tau$.

Definition 3.9. $\tau \in LIT(X)$ is called megaconnected if there exist no IS of $X$ which is both IOS and is "sandwiched" (that is there exist non-empty proper ICSs $E_1, E_2$ such that $E_1 \subset A \subset E_2$) between non-empty proper $\tau$-ICS of $X$.

Corollary 3.10. The mega-connectedness is a contractive invariant.

Proof. This immediately follows from the definition 3.9.

Corollary 3.11. $\delta(A)$ is a megaconnected member of $LIT(X)$.

Proof: In the family $\{\langle \emptyset, B \rangle \colon A \subset B, B \subset X\}$, there is no IOS which contains any ICS different from $\emptyset$ and in the family $\{\langle C, D \rangle \colon A \subset C, D \subset C \}$, there is no IOS which is contained by any ICS different from $\emptyset$. This means there is not any IOS in $\delta(A)$ which is sandwiched. Therefore, $\delta(A)$ is megaconnected.

Following example shows that $\delta(A)$ is not a maximal megaconnected member of $LIT(X)$.

Example 3.12. Let $X = \{1, 2, 3, 4\}$ and $A = \{4\}$ which is a subset of $X$. Consider the following families:

$$\delta(A) = \langle \emptyset, \{4\}, \langle \emptyset, \{1, 4\}, \langle \emptyset, \{2, 4\}, \langle \emptyset, \{3, 4\}, \langle \emptyset, \{1, 2, 4\}, \langle \emptyset, \{1, 3, 4\}, \langle \emptyset, \{2, 3, 4\}, \langle \emptyset, X, \langle \emptyset, \{1, 2, 3\}, \emptyset, \emptyset, \{1, 2, 3, 4\} \rangle \rangle \rangle \rangle \rangle \rangle \rangle$$

and $\tau = \delta(A) \cup \langle \{1\}, \{2, 3, 4\}, \langle \{1\}, \{4\}, \langle \{1\}, \{2, 4\}, \langle \{1\}, \{3, 4\} \rangle \rangle$ where $\tau$ is the megaconnected member of $LIT(X)$.

Result 3.12. From theorem 3.8, corollary 3.11 and example 3.12, the submaximality and megaconnectedness are not complementary intuitionistic topological invariants.

Theorem 3.13. Let $X$ be a non-empty set and $A$ a non-empty subset of $X$. The family $N(A) = \{\langle B, C \rangle \colon B \subset A, C \subset X \setminus \{x\} \cup \{D, E\}, A \subset D, E \subset D \}$ is a maximal megaconnected member of $LIT(X)$.

Proof. Let $\tau \in LIT(X)$ such that $N(A) \subset \tau$. Then $\tau$ must contain at least one of the following sets:

$D = \emptyset < F_3, G_2 >$, where $A \subset F_2$

We should look at the following cases:

i) If $B \in \tau$ then either $B$ is an IOS or since $\tau$-IOS’s $\langle A \setminus G_1, \emptyset \rangle$ and $\langle A \setminus G_1, \emptyset \rangle$ are $\tau$-open, then these IOS’s are $\tau$-open. This implies that $\langle \emptyset, A \setminus G_1, \emptyset \rangle$ and $B \subset \langle F_1, A, \emptyset \rangle$ are $\tau$-ICS’s which means that IOS $B$ is sandwiched between these ICS’s.

ii) If $C \in \tau$ then either $\emptyset \subset X$ or with the above argument C is sandwiched between ICS’s $\langle \emptyset, A \setminus G_2, \emptyset \rangle$ and $B \subset \langle F_2, C, \emptyset \rangle$.

iii) If $D \in \tau$ then either $D \subset F_3$ or $D$ is sandwiched between ICS’s $\langle \emptyset, A \setminus G_1, \emptyset \rangle$ and $D \subset \langle F_1, A \rangle$.

This show that $\tau \in LIT(X)$ is not megaconnected and $N(A)$ is a maximal megaconnected member of $LIT(X)$.

IV. Conclusion

In this study, we have shown that in intuitionistic topological spaces the submaximality and mega-connectedness are not complementary. Further studies can define contractive topological invariant property which is complementary with submaximality in intuitionistic topological spaces.

REFERENCES