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On Submaximality in Intuitionistic Topological Spaces

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***Abstract*—**In this study, a minimal submaximal element of LIT(X) (the lattice of all intuitionistic topologies for X, ordered by inclusion) is determined. Afterwards, a new contractive property, intuitionistic mega-connectedness, is defined. We show that the submaximality and mega-connectedness are not complementary intuitionistic topological invariants by identifying those members of LIT(X) which are intuitionistic mega-connected.

***Keywords*—**Intuitionistic set; intuitionistic topology; intuitionistic submaximality and mega-connectedness.

I. Introduction

THE submaximality was defined and characterized by Bourbaki ([2], [11]). It is worth mentioning that the submaximality is a significant condition for maximal topologies of many topological invariants (e.g. connectedness, quasi-H-closure and pseudo-compactness) [4]. Arhangel'skii and Collins have carried out a detailed study of how the submaximality affects the structure of familiar topological spaces and groups [16]. Dontchev has presented several characterizations of submaximality [8]. Also Dontchev and Rose have approached submaximality via topological ideals as well as related it with some allied concepts [9]. Recently, the submaximality has been studied prominently by many researchers [3,10].

Let LT(X) be the family of all topologies definable for an infinite set X forming a complete atomic and complemented lattice (ordered by inclusion). For a given member τ of LT(X), having property P, if all members of LT(X) weaker (stronger) than τ have the property P, then a topological invariant property P is called contractive (expansive). For a given contractive (expansive) property P, a member τ of LT(X) is called maximal P (minimal P) if τ has the property P but no stronger (weaker) member of LT(X) has not property P.

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An expansive property P and a contractive property Q are called complementary when the minimal P members of LIT(X) coincide with the maximal Q members.

"T_1 and 'all proper closed sets are finite'; door and 'filter-connected'; T_D and nested; disconnected and principal of order two" are some examples of complementary topological invariants ([4], [12], [13], [17]).

The main purpose of this article is to identify those members of LIT(X) (the lattice of all intuitionistic topologies for X, ordered by inclusion) which are minimal submaximal by using the definition of intuitionistic submaximality [15] and show that submaximality and megaconnectedness in Intuitionistic topological spaces are not complementary topological invariants.

II. Preliminaries

The idea of "intuitionistic fuzzy set" was suggested by Krassimir T. Atanassov [1]. Later, D. Coker has presented the classical version of this concept [5]. The definitions which are actively used in this paper, are listed below.

Definition 2.1. Let X be a nonempty set. An intuitionistic set (IS for short) $\underline{A}$ is an object having the form

$\underline{A}=<X, A_1, A_2>$ (or $\underline{A}=<A_1, A_2>$),

where A_1 and A_2 are disjoint subsets of X. The set A_1 is called the set of members of A, while A_2 is called the set of nonmembers of A [5,1].

Definition 2.2. Let X be a nonempty set and $\underline{A}$ and $\underline{B}$ be in the form $\underline{A}=<A_1, A_2>$, $\underline{B}=<B_1, B_2>$, respectively. Furthermore, let $\{\underline{A_i}: i\in I\}$ be an arbitrary family of IS's in X, where $\underline{A_i}=<A_i^{(1)}, A_i^{(2)}>$ then;

a) $\underline{\emptyset}=<\emptyset,X>$, $\underline{X}=<X,\emptyset>$
b) $\underline{A}\subset\underline{B}$ if $A_1\subset B_1$ and $A_2\supset B_2$
c) $\underline{A}\supset\underline{B}$ if $A_1\supset B_1$ and $A_2\subset B_2$
d) $\underline{A}^c=<A_2,A_1>$ here $\underline{A}^c$ complementary of $\underline{A}$
e) $\cap\underline{A}_i =< \cap A_i^{(1)}, \cup A_i^{(2)}>$ and $\cup\underline{A}_i =< \cup A_i^{(1)},\cap A_i^{(2)}>$ [5].

Definition 2.3. An intuitionistic topology (IT for short) on a nonempty set X is a family $\underline{\tau}$ of IS's in X containing $\underline{\emptyset}$ and $\underline{X}$ which is closed under arbitrary unions and finite intersections. In this case the pair ($\underline{X}$, $\underline{\tau}$) is called an intuitionistic topological space (ITS for short) and any IS in $\underline{\tau}$ is known as an intuitionistic open set (IOS for short) in $\underline{X}$, the complement of such an IOS in X is called an intuitionistic closed set (ICS for short) in $\underline{X}$ [6,7].

Example 2.4. Any topological space (X,τ) is obviously an intuitionistic topological space with the form;
$\underline{\tau}=\{\underline{A}: A\in\tau\}$ whenever we identify a subset A in X with its counter pair $\underline{A}=<A,A^c>$.

Example 2.5. Let $X=\{a,b,c,d\}$ and consider the family $\underline{\tau}=\{\underline{\emptyset}, \underline{X}, \underline{T}_1, \underline{T}_2, \underline{T}_3, \underline{T}_4, \underline{T}_5, \underline{T}_6\}$, where $\underline{T}_1=<\{a\},\emptyset>$, $\underline{T}_2=<\{b\},\emptyset>$, $\underline{T}_3=<\{c\},\emptyset>$, $\underline{T}_4=<\{a,b\},\emptyset>$, $\underline{T}_5=<\{b,c\},\emptyset>$, and $\underline{T}_6=<\{a,c\},\emptyset>$. Then $\underline{\tau}$ is IT on $\underline{X}$.

Definition 2.6. Let $\underline{\tau}_1$ and $\underline{\tau}_2$ be two IT on X. Then $\underline{\tau}_1$ is said to be contained in $\underline{\tau}_2$ if $\underline{G}\in\underline{\tau}_2$ for each $\underline{G}\in\underline{\tau}_1$. In this case, we also say that $\underline{\tau}_1$ is coarser than $\underline{\tau}_2$, or $\underline{\tau}_2$ is finer than $\underline{\tau}_1$ [7].

Definition 2.7. Let $(\underline{X}, \underline{\tau})$ be an ITS and $\underline{A}=<A_1,A_2>$ be an IS in $\underline{X}$. Then the interior and closure of $\underline{A}$ are defined by:
$int(\underline{A})=\cup\{\underline{G}_i: \underline{G}_i \text{ is an IOS in } \underline{X} \text{ and } \underline{G}_i\subset\underline{A}\}$
$cl(\underline{A})=\cap\{\underline{K}_i : \underline{K}_i \text{ is an ICS in } \underline{X} \text{ and } \underline{A}\subset\underline{K}_i\}$ [7]

Definition 2.8. Let $(\underline{X}, \underline{\tau})$ be an ITS and $\underline{A}=<A_1,A_2>$ be an IS in $\underline{X}$. The set $\underline{A}$ is called $\underline{\tau}$-dense in $\underline{X}$ if $cl(\underline{A})=\underline{X}$.

Example 2.9. Let us consider the ITS $(\underline{X}, \underline{\tau})$ where $X=\{a,b,c,d,e\}$ and
$\underline{\tau}=\{\underline{\emptyset}, \underline{X},<\{a,b,c\},\{e\}>,<\{c,d\},\{e\}>,<\{c\},\{e\}>, <\{c\},\{d,e\}>, <\{a,b,c,d\},\{e\}>\}$
if we let $\underline{B}=<\{b,c\},\{d\}>$, then
$int(\underline{B})=<\{c\},\{d,e\}>$ and
$cl(\underline{B})=\underline{X}$. Therefore, $\underline{B}$ is a dense IS in X.

III. SUBMAXIMALITY AND MEGACONNECTEDNESS IN INTUITIONISTIC TOPOLOGICAL SPACES

A topological space (X, τ) is submaximal if every dense subset of X is τ -open [2]. The concept of submaximal intuitionistic space was defined by Ozcelik and Narlı [15]. In this section the definition of submaximality for an ITS is given. In addition, some characterizations of the submaximality on IT are investigated.

Definition 3.1. Let $(\underline{X}, \underline{\tau})$ be an ITS. Then $\underline{\tau}\in LIT(\underline{X})$ is called the submaximal intuitionistic topology (IT-sub for short) if every $\underline{\tau}$-dense subset of $\underline{X}$ is an IOS in $\underline{X}$ [15].

Example 3.2. Let $X=\{a,b\}$and the family
$\underline{\tau}=\{\underline{\emptyset}, \underline{X}, <\{a\},\emptyset>, <\{b\},\emptyset>, <\{a\},\{b\}>, <\emptyset,\emptyset>, <\emptyset,\{b\}>\}$ is an IT-sub on $\underline{X}$.

Remark 3.3. As the following example indicates, it is not necessary that the intuitionistic form of a submaximal space is submaximal.

Example 3.4. Let $X=\{1,2\}$ and $\tau=\{\emptyset,X,\{1\}\}$, then the topological space (X, τ) is submaximal. The intuitionistic form of τ is $\underline{\tau}=\{\underline{\emptyset}, \underline{X}, <\{1\},\{2\}>\}$ and $\underline{A}=<\{1\},\emptyset>$ is a $\underline{\tau}$-dense IS but $\underline{A}$ is not an IOS. As a result, $\underline{\tau}$ is not an IT-sub on X.

Theorem 3.5. The submaximality is an expansive intuitionistic topological invariant.
Proof. Let $\underline{\tau}_1$, $\underline{\tau}_2$ in LIT(X) such that $\underline{\tau}_1\subset\underline{\tau}_2$ and $\underline{\tau}_1$ is IT-sub. Take $\underline{A}$ as a $\underline{\tau}_2$-dense IS then $\underline{A}$ is a $\underline{\tau}_1$-dense IS. Since $\underline{\tau}_1$ is an IT-sub, $\underline{A}$ is a $\underline{\tau}_1$-IOS. Then $\underline{A}$ is a $\underline{\tau}_2$-IOS so $\underline{\tau}_2$ is a IT-sub.

Theorem 3.6. Let X be a non empty set and $\underline{A}$ be an IS. Then $M(\underline{A})=\{\underline{G}: \underline{A}\subset\underline{G} \text{ or } \underline{G}\subset\underline{A}\}$ is a IT-sub member of LIT(X).
Proof. Let us take $\underline{A}=<A_1 ,A_2>$.
If $\underline{B}=<B_1,B_2>$ is any IS, since $<\emptyset, A_2\cup B_1>$ is an IOS then $cl(\underline{B})\subset< A_2\cup B_1, \emptyset >$.
If $A_2\cup B_1\neq X$ then $cl(\underline{B})\neq\underline{X}$
If $A_2\cup B_1=X$ then $(A_2)^c\subset B_1$, $A_1\subset B_1$, $B_2\subset A_2$ and $\underline{B}$ is an IOS.
Therefore $M(\underline{A})$ is submaximal IT on X

McCARTAN [12] has shown that $M(A)=\{G:A\subset G \text{ or } G\subset A, \emptyset\neq A\subset X\}$ is a minimal submaximal member of LT(X). The following example shows that $M(\underline{A})$ is not a minimal IT –sub member of LIT(X).

Example 3.7. Let $X=\{1,2,3\}$ and $\underline{A}=<\{1\},\{3\}>$ be an IS. Consider the IT-sub members of LIT(X)

$M(\underline{A})=\{<\{1\},\{3\}>,<\{1\},\{2,3\}>,<\emptyset,\{3\}>,<\emptyset,\{2,3\}>, <\emptyset,\{1,3\}>,<\{1\},\emptyset>,<\{1,2\},\{3\}>,<\{1,2\},\emptyset>,<\{1,3\},\emptyset>, <\{1,2,3\},\emptyset>,<\emptyset,\{1,2,3\}>\}$ and

$\underline{\tau}=\{<\emptyset,\{3\}>,<\emptyset,\{2,3\}>,<\emptyset,\{1,3\}>,<\{1,2\},\{3\}>,<\{1,2\}, \emptyset>,<\{1,2,3\},\emptyset>,<\emptyset,\{1,2,3\}>\}$

Since $\underline{\tau}$ is coarser than $M(\underline{A})$, $M(\underline{A})$ is not minimal submaximal IT member of LIT(X).

Theorem 3.8. Let X be a nonempty set and A is any subset of X. The family

$\delta(A)=\{<\emptyset,B>: A\subset B\subset X\}\cup\{<C,D>: (A)^c\subset C, D\subset(C)^c\}$ is a minimal IT-sub member of LIT(X).

Proof. First of all we show that $\delta(A)$ is IT-sub on X . We look at the following cases:

Let $\underline{E}=<E_1,E_2>$ be an IS .

i) If $E_1\subset A$ then $<\emptyset,A>\in\delta(A)$ so $cl(\underline{E})\subset<\emptyset,A>\neq\underline{X}$

ii) If $E_2\subset A$ then either $\underline{E}\in\delta(A)$ or $<\emptyset,A\cup E_1>\in\delta(A)$. This implies that $cl(\underline{E})\subset <A\cup E_1,\emptyset>\neq\underline{X}$

iii) If $A \subset E_1$ then either $E_1 = X$ in which case $\underline{E} = \underline{X} \in \delta(A)$ or $cl(\underline{E}) \subset \langle E_1, \varnothing \rangle \neq \underline{X}$

iv) If $A \not\subset E_1$ and $E_1 \not\subset A$ then either $\underline{E} \in \delta(A)$ or $cl(\underline{E}) \subset \langle A \cup E_1, \varnothing \rangle \neq \underline{X}$

These cases show that $\delta(A)$ is an IT-sub.

Let $\underline{\tau}$ be an IT-sub and $\underline{\tau} \subset \delta(A)$. If any IS $\underline{E}$ in the family $\{\langle C,D \rangle : A^c \subset C, D \subset C^c\}$ is $\delta(A)$-dense then $\underline{E}$ is $\underline{\tau}$-dense. Since $\underline{\tau}$ is IT-sub, $\underline{E}$ is an $\underline{\tau}$-IOS. On the other hand, all the IS in the form $\langle \varnothing, X \backslash \{x\} \rangle$ where $x \in A^c$ must be $\underline{\tau}$-IOS, because if $\langle \varnothing, X \backslash \{x\} \rangle$ is not an $\underline{\tau}$-IOS then the IS $\langle X \backslash \{x\}, \varnothing \rangle$ is $\underline{\tau}$-dense, this contradicts whit the submaximality of $\underline{\tau}$. This implies that all the IS in the family $\{\langle \varnothing, B \rangle : A \subset B, B \subset X\}$are $\underline{\tau}$-IOS. From this $\delta(A) = \underline{\tau}$

Definition 3.9. $\underline{\tau} \in$ LIT(X) is called megaconnected if there exist no IS of X which is
both IOS and is "sandwiched" (that is there exist non-empty proper ICSs $\underline{F}_1$, $\underline{F}_2$ such that $\underline{F}_1 \subset A \subset \underline{F}_2$) between non-empty proper $\underline{\tau}$-ICS of X.

Corallary 3.10. The mega-connectedness is a contractive invariant.
Proof. This immediately follows from the definition 3.9.

Corallary 3.11. $\delta(A)$ is a megaconnected member of LIT(X).
Proof: In the family $\{\langle \varnothing, B \rangle : A \subset B, B \subset X\}$, there is no IOS which contains any ICS different from $\underline{\varnothing}$ and in the family $\{\langle C,D \rangle : A^c \subset C, D \subset C^c\}$, there is no IOS which is contained by any ICS different from $\underline{X}$. This means there is not any IOS in $\delta(A)$ which is sandwiched. Therefore, $\delta(A)$ is megaconnected.

Following example shows that $\delta(A)$ is not a maximal megaconnected member of LIT(X).
Example 3.12. Let X ={1,2,3,4} and A={4}which is a subset of X. Consider the following families:

$\delta(A) = \langle \varnothing,\{4\} \rangle, \langle \varnothing,\{1,4\} \rangle, \langle \varnothing,\{2,4\} \rangle, \langle \varnothing,\{3,4\} \rangle, \langle \varnothing,\{1,2,4\} \rangle, \langle \varnothing,\{1,3,4\} \rangle, \langle \varnothing,\{2,3,4\} \rangle, \langle \varnothing,X \rangle, \langle \{1,2,3\},\varnothing \rangle, \langle X,\varnothing \rangle, \langle \{1,2,3\},\{4\} \rangle\}$

and$\underline{\tau} = \delta(A) \cup \{\langle \{1\},\{2,3,4\} \rangle, \langle \{1\},\{4\} \rangle, \langle \{1\},\{2,4\} \rangle, \langle \{1\},\{3,4\} \rangle\}$ where $\underline{\tau}$ is the megaconnected member of LIT(X).

Result 3.12. From theorem 3.8, corollary 3.11 and example 3.12, the submaximality and megaconnectedness are not complementary intuitionistic topological invariants.

Theorem 3.13. Let X be a non-empty set and A, a non-empty subset of X. The family $N(A) = \{\langle B,C \rangle : B \subset A^c, A \subset C, B \cap C = \varnothing\} \cup \{\langle D,E \rangle : A^c \subset D, E \subset D^c\}$ is a maximal megaconnected member of LIT(X).
Proof. Let $\underline{\tau} \in$LIT(X) such that $N(A) \subset \underline{\tau}$. Then $\underline{\tau}$ must contain at least one of the following sets:
$\underline{B} = \langle F_1, G_1 \rangle$, where $A \not\subset G_1$
$\underline{C} = \langle F_2, G_2 \rangle$, where $A \subset F_2$
$\underline{D} = \langle F_3, G_3 \rangle$, where $F_3 \subset A$
We should look at the following cases:

i) If $B \in \underline{\tau}$ then either $\underline{B} \in N(A)$ or since IOS's $\langle A^c \cup G_1, \varnothing \rangle$ and $\langle \varnothing, F_1 \cup A \rangle$ are N(A)-open, then these IOS's are $\underline{\tau}$-open. This implies that IS's $\langle \varnothing, A^c \cup G_1 \rangle \subset \underline{B}$ and $\underline{B} \subset \langle F_1 \cup A, \varnothing \rangle$ are $\underline{\tau}$-ICS's which means that IOS $\underline{B}$ is sandwiched between these ICS's.

ii) If $C \in \underline{\tau}$ then either $C = \underline{X}$ or with the above argument C is sandwiched between ICS's $\langle \varnothing, A^c \cup G_2 \rangle \subset \underline{C}$ and $\underline{C} \subset \langle F_2, \varnothing \rangle$.

iii) If $\underline{D} \in \underline{\tau}$ then either $\underline{D} = \varnothing$ or $\underline{D}$ is sandwiched between ICS's $\langle \varnothing, A^c \cup G_3 \rangle \subset \underline{D}$ and $\underline{D} \subset \langle \varnothing, A \rangle$.

This show that $\underline{\tau} \in$ LIT(X) is not megaconnected and N(A) is a maximal megaconnected member of LIT(X).

IV. Conclusion

In this study, we have shown that in intuitionistic topological spaces the submaximality and mega-connectedness are not complementary. Further studies can define contractive topological invariant property which is complementary with submaximality in intuitionistic topological spaces.

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