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A preliminary study on the eventual positivity of irreducible tridiagonal sign patterns

Berlin Yu

Abstract—Motivated by Berman et al. [Sign patterns that allow eventual positivity, *ELA*, 19(2010): 108-120], we concentrate on the potential eventual positivity of irreducible tridiagonal sign patterns. The minimal potential eventual positivity of irreducible tridiagonal sign patterns of order less than six is established, and all the minimal potentially eventually positive tridiagonal sign patterns of order ≤ 5 are identified. Our results indicate that if an irreducible tridiagonal sign pattern of order less than six $\mathcal{A}$ is minimal potentially eventually positive, then $\mathcal{A}$ requires the eventual positivity.

Keywords—eventual positivity, potentially positive sign pattern, tridiagnoal sign pattern, minimal potentially positive sign pattern.

I. Introduction

A sign pattern is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in $\{+, -, 0\}$. The set of all real matrices with the $n \times n$ sign pattern $\mathcal{A}$ is called to be the qualitative class of $\mathcal{A}$, denoted by $Q(\mathcal{A})$. A permutation pattern is a sign pattern matrix with exactly one entry in each row and column equal to $+$, and the remaining entries equal to 0. A product of the form $S^T \mathcal{A} S$, where S is a permutation pattern and $\mathcal{A}$ is a sign pattern matrix of the same order as S, is called a permutation similarity. A subpattern of $\mathcal{A}$ is an $n \times n$ sign pattern $\mathcal{B}$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{B}$ is a proper subpattern of $\mathcal{A}$. We also call $\mathcal{A}$ is a proper superpattern of $\mathcal{B}$.

Two sign patterns $\mathcal{A}$ and $\mathcal{B}$ are equivalent if $\mathcal{A} = P^T \mathcal{B} P$, or $\mathcal{A} = P^T \mathcal{B}^T P$, where P is a permutation pattern. A pattern $\mathcal{A}$ is reducible if there is a permutation matrix P such that

$$P^T \mathcal{A} P = \begin{pmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is irreducible if it is not reducible. For a sign pattern $\mathcal{A} = [a_{ij}]$, we define the positive part of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha_{ij}^+]$ and the negative part of $\mathcal{A}$ to be $\mathcal{A}^- = [\alpha_{ij}^-]$, where $\alpha_{ij}^+ = +$(respectively, 0) for $\alpha_{ij} = +$ (respectively, 0 or $-$), and $\alpha_{ij}^- = -$(respectively, 0) for $\alpha_{ij} = -$ (respectively, 0 or $+$).

It is well known that graph theoretical methods are often useful in the study of sign patterns, so we now introduce some graph theoretical concepts.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has signed digraph $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \cdots, n\}$ and for all i and j, a positive (negative) arc from i to j if and only if α_{ij} is positive (negative). A (directed) simple cycle of length k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_1)$ such that the vertices $i_1, \cdots, i_k$ are distinct; see, e.g., [5]. A (signed) digraph $D = (V_D, E_D)$ is primitive if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1; see, for example, [6].

B. Yu is with the Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huaian, Jiangsu, China e-mail: berlin.yu@gmail.com.

A sign pattern $\mathcal{A}$ is primitive if its signed digraph $\Gamma(\mathcal{A})$ is primitive.

A sign pattern matrix $\mathcal{A}$ is said to require a certain property P referring to real matrices if every real matrix $A \in Q(\mathcal{A})$ has the property P and allow P or be potentially P if there is some $A \in Q(\mathcal{A})$ that has property P.

The allow problems of sign patterns have been studied by many researchers; see, for instance, [2, 3, 4]. Cartral, Olesky and Driessche presented a survey about allow problems concerning spectral properties of sign pattern matrices in [1]. Recently, there is an increasing interest in the eventual positivity of sign patterns. In [2], Ellison, Hogben, Tsatsomeros studied the sign patterns that require eventual positivity or require eventual nonnegativity. In [3], Catral, Hogben, Olesky, et al. investigated sign patterns that require or allow power-positivity. Sign patterns that allow eventual positivity have been studied in [4]. A sufficient condition for sign patterns to be potentially eventually positive is that its positive part is primitive. Some necessary conditions are also given in [4]. The Corollary 4.5 in [4] states that for $n \geq 2$, the minimum number of positive entries in an $n \times n$ sign pattern that allows eventual positivity is $n + 1$.

This paper is motivated by the idea of [4]. In this paper, we concentrate on the potential eventual positivity of irreducible tridiagonal sign patterns. The minimal potential eventual positivity of irreducible tridiagonal sign patterns of order less than six is established, and all the minimal potentially eventually positive tridiagonal sign patterns of order ≤ 5 are identified. Our results indicate that if an irreducible tridiagonal sign pattern of order less than six $\mathcal{A}$ is minimal potentially eventually positive, then $\mathcal{A}$ requires the eventual positivity.

II. Definitions and Preliminaries

In order to state our results clearly, we need the following definitions and preliminaries.

Definition 1. [2] *An $n \times n$ real matrix A is said to be eventually positive if there exists a nonnegative integer k_0 such that $A^k > 0$ for all $k \geq k_0$.*

Definition 2. [4] *An $n \times n$ sign pattern $\mathcal{A}$ is said to be potentially eventually positive (PEP) if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive.*

We now turn our attention to the minimal PEP sign patterns.

Definition 3. *An $n \times n$ sign pattern $\mathcal{A}$ is said to be minimal potentially eventually positive sign pattern (MPEP sign pattern) if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is potentially eventually positive.*

The following Lemmas 1, 2, 3 and 4 are several necessary or sufficient conditions for a sign pattern to allow eventual positivity established by Berman et al. [4].

Lemma 1. If $\Gamma(\mathcal{A}^+)$ is primitive, then $\mathcal{A}$ is PEP.

Lemma 2. If an $n \times n$ sign pattern $\mathcal{A}$ is PEP, then

(1) Every row and column of $\mathcal{A}$ has at least one $+$ and the minimal number of $+$ entries in $\mathcal{A}$ is $n+1$.

(2) Every superpattern of $\mathcal{A}$ is PEP.

(3) If $\hat{\mathcal{A}}$ is the sign pattern obtained from sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$, then $\hat{\mathcal{A}}$ is PEP.

Lemma 3. If $\mathcal{A}$ is the block sign pattern

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}$$

with $\mathcal{A}_{12} = \mathcal{A}_{12}^-$, $\mathcal{A}_{21} = \mathcal{A}_{21}^+$, and $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ square, then $\mathcal{A}$ is not PEP.

Lemma 4. If $\mathcal{A}$ is the checkerboard block sign pattern

$$\begin{pmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with square diagonal blocks. Then $-\mathcal{A}$ is not PEP, and if $\mathcal{A}$ has a negative entry, then $\mathcal{A}$ is not PEP.

Unfortunately, there is not any sufficient and necessary conditions for a sign pattern to be a potentially eventually positive and the classification of the potentially eventually positive sign patterns of order ≥ 4 is also unsolved. As a preliminary study, we focus on the the minimal eventual positivity of irreducible tridiagonal sign patterns of orders 3, 4 and 5.

III. Main results

Suppose that an $n \times n$ irreducible tridiagonal sign pattern $\mathcal{A}$ is of the form

$$\begin{pmatrix} ? & * & & \\ * & ? & \ddots & \\ & \ddots & \ddots & * \\ & & * & ? \end{pmatrix},$$

where $*$ denotes the nonzero entries, $?$ denotes one of $0, +, -$ and the entries unspecified in the sign pattern are all zeros.

Following [4], we use the notation $\ominus$ to denote one of $0, -$, $\oplus$ to denote one of $0, +$ and $[+]$ (respectively, $[-]$) to denote a sign pattern consisting entirely of positive (respectively, negative) entries.

It is clear that the minimal potentially eventually positive tridiagonal sign patterns of orders 1 and 2 are $[+]$ and

$$\begin{pmatrix} + & + \\ + & 0 \end{pmatrix}.$$

Next, we focus on the irreducible tridiagonal sign patterns of order ≥ 3.

Theorem 1. Let $\mathcal{A}$ be a 3×3 irreducible tridiagonal sign pattern. Then $\mathcal{A}$ is a minimal potentially eventually positive sign pattern if and only if is equivalent to either

$$\mathcal{A}_1 = \begin{pmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{pmatrix},$$

or

$$\mathcal{A}_2 = \begin{pmatrix} 0 & + & 0 \\ + & + & + \\ 0 & + & 0 \end{pmatrix}.$$

Proof. The potentially eventual positivity follows readily from the fact that the signed digraphs $\Gamma(\mathcal{A}_1)$ and $\Gamma(\mathcal{A}_2)$ are primitive respectively. The minimality of $\mathcal{A}$ follows from $\mathcal{A}_1$ and $\mathcal{A}_2$ have the minimum number of positive entries.

For the necessity, Theorem 6.4 in [4] indicates that if a 3×3 sign pattern $\mathcal{A}$ is PEP, then either $\Gamma(\mathcal{A}^+)$ is primitive or $\mathcal{A}$ is equivalent to the sign pattern of the form

$$\begin{pmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{pmatrix}.$$

It follows that the 3×3 minimal PEP sign pattern is equivalent to one of the following two sign patterns

$$\mathcal{A}_1 = \begin{pmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{pmatrix},$$

and

$$\mathcal{A}_2 = \begin{pmatrix} 0 & + & 0 \\ + & + & + \\ 0 & + & 0 \end{pmatrix}.$$

Theorem 2. Let $\mathcal{A}$ be a 4×4 irreducible tridiagonal sign pattern. Then $\mathcal{A}$ is a minimal potentially eventually positive sign pattern if and only if is equivalent to either

$$\mathcal{A}_1 = \begin{pmatrix} + & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{pmatrix},$$

or

$$\mathcal{A}_2 = \begin{pmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \end{pmatrix}.$$

To state our proof clearly, let α_{ij} denote the (i, j) entry of sign pattern $\mathcal{A} = [\alpha_{ij}]$.

Proof. For the necessity, we prove it by four steps. $\alpha_{ii+1}\alpha_{i+1i} = +$, for $i = 1, 2, 3$.

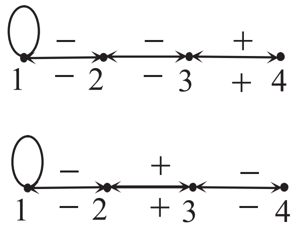

Fig. 1. Signed digraphs of the tridiagonal sign pattern of order 4.

Step 1. Assume that $\alpha_{kk+1}\alpha_{k+1k} = -$ for some k such that $1 \le k \le 3$. Without loss of generality, let $\alpha_{kk+1} = -$ and $\alpha_{k+1k} = +$. Then

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix},$$

where $\mathcal{A}_{12} = \mathcal{A}_{12}^{-}$, $\mathcal{A}_{21} = \mathcal{A}_{21}^{+}$, and $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ have orders k and $4-k$, respectively. By Lemma 3, $\mathcal{A}$ is not potentially eventually positive; a contradiction.

Step 2. we show that $\alpha_{12} = \alpha_{21} = +$.

By a way of contradiction, assume that $\alpha_{12} = \alpha_{21} = -$. Then we have

$$\mathcal{A} = \begin{pmatrix} + & - & 0 & 0 \\ - & ? & * & 0 \\ 0 & * & ? & * \\ 0 & & * & ? \end{pmatrix}.$$

By the digraph isomorphism of the signed digraphs $\Gamma(\mathcal{A})$ shown in Fig. 1., it is necessary to consider the following two cases:

$$\mathcal{B} = \begin{pmatrix} + & - & 0 & 0 \\ - & ? & - & 0 \\ 0 & - & ? & + \\ 0 & 0 & + & ? \end{pmatrix}.$$

and

$$\mathcal{C} = \begin{pmatrix} + & - & 0 & 0 \\ - & ? & + & 0 \\ 0 & + & ? & - \\ 0 & 0 & - & ? \end{pmatrix}.$$

Case 1. *$\mathcal{A}$ is equivalent to $\mathcal{B}$.*

By changing all the diagonal entries to $+$ and letting $\alpha_{13} = \alpha_{31} = \alpha_{14} = \alpha_{41} = +$ and $\alpha_{24} = \alpha_{42} = -$, we can get a checkerboard block sign pattern

$$\left(\begin{array}{c|c|cc} + & - & + & + \\ \hline - & + & - & - \\ \hline + & - & + & + \\ + & - & + & + \end{array}\right).$$

It follows that $\mathcal{A}$ is not potentially eventually positive; a contradiction.

Case 2. *$\mathcal{A}$ is equivalent to $\mathcal{C}$.*

Similarly, we can a block sign pattern

$$\left(\begin{array}{c|cc|c} + & - & - & + \\ \hline - & + & + & - \\ - & + & + & - \\ \hline + & - & - & + \end{array}\right).$$

It follows that $\mathcal{A}$ is not potentially eventually positive; a contradiction.

Step 3. we show $\alpha_{13} = \alpha_{31} = +$ by a similar discussion.

Therefore, we have $\alpha_{ij} = \alpha_{ji} = +$, for all $i \neq j$, $i, j = 1, 2, 3, 4$.

Step 4. $\mathcal{A}$ must have a positive diagonal entry. If all the diagonal entries are non-positive, then we get a checkerboard block sign pattern from $\mathcal{A}$ by changing the zero entries to be positive or negative appropriately. It is a contradiction. So the positive entry is possibly the $(1,1)$ entry or the $(2,2)$ entry. Since $\mathcal{A}$ is a minimal PEP sign pattern, $\mathcal{A}$ has only one positive entry and the other diagonal entries must be 0. It follows that Theorem 2 holds.

For the sufficiency, since the signed digraphs shown in Fig. 1 are primitive, both $\mathcal{B}$ and $\mathcal{C}$ are potentially eventually positive. It can be verified directly that all their proper subpatterns are not potentially eventually positive. It follows that both $\mathcal{B}$ and $\mathcal{C}$ are minimal potentially eventually positive sign patterns.

Theorem 3. Let $\mathcal{A}$ be a 5×5 irreducible tridiagonal sign pattern. Then $\mathcal{A}$ is a minimal potentially eventually positive sign pattern if and only if is equivalent to

$$\mathcal{A}_1 = \begin{pmatrix} + & + & 0 & 0 & 0 \\ + & 0 & + & 0 & 0 \\ 0 & + & 0 & + & 0 \\ 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & + & 0 \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} 0 & + & 0 & 0 & 0 \\ + & + & + & 0 & 0 \\ 0 & + & 0 & + & 0 \\ 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & + & 0 \end{pmatrix},$$

or

$$\mathcal{A}_3 = \begin{pmatrix} 0 & + & 0 & 0 & 0 \\ + & 0 & + & 0 & 0 \\ 0 & + & + & + & 0 \\ 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & + & 0 \end{pmatrix}.$$

Proof. The sufficiency can be verified directly. For the converse,by a similar discussion, we have $\alpha_{ij} = \alpha_{ji} = +$, for all $i \neq j$, $i, j = 1, 2, 3, 4, 5$, and there exists at least one positive diagonal entry. If $\alpha_{11} = +$, then sign pattern $\mathcal{A}$ is PEP. By the minimality of PEP pattern $\mathcal{A}$, $\alpha_{22} = 0$, $\alpha_{33} = 0$ and $\alpha_{44} = 0$. It follows that $\mathcal{A}$ is equivalent to $\mathcal{A}_1$. Similarly, we can show that $\mathcal{A}$ is equivalent to $\mathcal{A}_2$ for $\alpha_{22} = +$ and $\mathcal{A}$ is equivalent to $\mathcal{A}_3$ for $\alpha_{33} = +$. By digraph

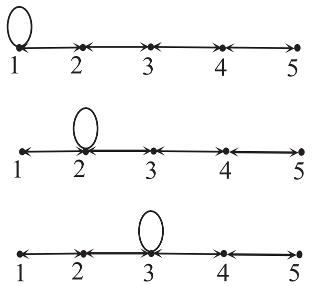

Fig. 2. Signed digraphs of the tridiagonal sign pattern of order 5.

isomorphism of $\Gamma(\mathcal{A})$ shown in Fig. 2, there are no other cases to be considered. So Theorem 3 holds.

Recall that a sign pattern $\mathcal{A}$ is said to require the eventual positivity, if A is a eventually positive matrix, for every real matrix $A \in Q(\mathcal{A})$. Ellison etal. [2] have shown that $\mathcal{A}$ requires the eventual positivity if and only if $\mathcal{A}$ is nonnegative and primitive. It is clear that if $\mathcal{A}$ requires the eventual positivity, then $\mathcal{A}$ is potentially eventually positive. In general, the converse is not true. But for the irreducible tridiagonal sign patterns of order < 6, the converse is valid.

Corollary 4. Let $\mathcal{A}$ be a irreducible tridiagonal sign pattern of order < 6. If $\mathcal{A}$ is a minimal potentially eventually positive sign pattern, the $\mathcal{A}$ requires the eventual positivity.

Proof. Corollary 4 follows readily from the fact the minimal potentially eventually positive tridiagonal sign patterns of orders 3, 4 and 5 are nonnegative and primitive by Theorems 1, 2 and 3.

IV. Conclusion

As a preliminary study, we have shown that, up to equivalence, there exists only two minimal PEP tridiagonal sign patterns of order 3, two minimal PEP tridiagonal sign patterns of order 4 and for the tridiagonal sign patterns of order 5, there are only three minimal PEP sign patterns. As we know, with the increasing of order of the tridiagonal sign patterns, the classifiction and identification of minimal PEP tridiagonal sign patterns are becoming very difficult. In a follow-up paper, we will consider these questions.

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