

Partial Derivatives and Optimization Problem on Time Scales

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Abstract—The optimization problem using time scales is studied. Time scale is a model of time. The language of time scales seems to be an ideal tool to unify the continuous-time and the discrete-time theories. In this work we present necessary conditions for a solution of an optimization problem on time scales. To obtain that result we use properties and results of the partial diamond-alpha derivatives for continuous-multivariable functions. These results are also presented here.

Keywords—Lagrange multipliers, mathematical programming, optimization problem, time scales.

I. INTRODUCTION

THE calculus on time scales has been initiated by Aulbach and Hilger in order to create a theory that can unify and extend discrete and continuous analysis [1], [2]. One of the main concepts of this theory that is very important to our work is the diamond-alpha derivative, which is a generalization of ordinary (time) derivative. If the time scale is the real set, we get ordinary derivative. Many results of calculus on time scales have been developed, particularly in partial differentiation (see, e.g., [3]–[6]), where were studied properties of partial delta and nabla derivatives and their applications. However, there is no much information about partial diamond-alpha derivatives and the few works that exist are limited to two variables (see [7]). Our work gives us properties and results of partial diamond-alpha derivatives for continuous-multivariable functions with applications in the optimization problem on time scales. The time scale systems are a powerful tool in engineering and economics applications where both of the discrete-time and continuous-time systems are used. The unification of the discrete and continuous theories provides a new perspective and easiness for modeling and solving optimization problems on a general domain. There are very studies related with the problems of the calculus of variations on time scales, that is, functional optimization problems (see, e.g., [8]–[14]). However, to the best of our knowledge, the function optimization problems for continuous-multivariable functions have not been seriously treated. In [15] the authors presented results for the linear and the quadratic programming using convex optimization. In the present work we obtain necessary conditions for a solution of an optimization problem on time scales for a general continuous-multivariable function. This is a generalization of the classical Lagrange multipliers method for continuous-time case. In addition it is also an important result to be applied in optimal control problems on

times scales, because with that result we can obtain necessary conditions to optimal solution for a control system.

This paper is organized as follows. In Section II we introduce the properties and results about partial diamond-alpha derivatives for n -variable functions. The generalized optimization problem and other important results on time scales are given in Section III. In the same section to illustrate the possibility of the developed techniques we consider an example. In Section IV we present the conclusions and the future work that we propose to do.

Throughout this paper we denote by \mathbb{N} , \mathbb{Z} , \mathbb{R} , and \mathbb{R}_0^+ the set of positive integers, the set of integers, the set of real numbers, and the set of nonnegative real numbers, respectively. By \mathbb{R}^n we denote the usual n -dimensional space of vectors $x = (x_1, x_2, \dots, x_n)$, where $x_i \in \mathbb{R}$, $i = \overline{1, n}$. The inner product of two vectors x and y in \mathbb{R}^n is expressed by $\langle x, y \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$. We denote by the symbol \emptyset the empty set. The boundary of a set $A \subset \mathbb{R}^n$ is defined as $\partial A = \{x \in \mathbb{R}^n : U(x) \cap A \neq \emptyset \wedge U(x) \cap A^c \neq \emptyset\}$, where $U(x)$ is any neighborhood of the point x and A^c is the complementary set of A .

II. PARTIAL DERIVATIVES ON TIME SCALES

This section is devoted to the extension of the differentiability of continuous-multivariable functions to time scales using the diamond-alpha notion. Let $n \in \mathbb{N}$ be fixed. Denote by \mathbb{T}_i , $i = \overline{1, n}$, a time scale, that is, a nonempty closed subset of the real numbers \mathbb{R} . Let us set

$$\Lambda_n = \mathbb{T}_1 \times \mathbb{T}_2 \times \dots \times \mathbb{T}_n \\ = \{t = (t_1, t_2, \dots, t_n) : t_i \in \mathbb{T}_i, i = \overline{1, n}\}.$$

We call Λ_n an n -dimensional time scale and it is a subset of the usual n -dimensional space \mathbb{R}^n . The set Λ_n is a complete metric space with the metric d defined by

$$d(t, s) = \left(\sum_{i=1}^n |t_i - s_i|^2 \right)^{1/2} \quad \text{for } t, s \in \Lambda_n.$$

Therefore, for a given number $\delta > 0$, the δ -neighborhood $U_\delta(t^0)$ of a given point $t^0 = (t_1^0, t_2^0, \dots, t_n^0) \in \Lambda_n$ is the set of all points $t \in \Lambda_n$ such that $d(t^0, t) < \delta$. For functions $f : \Lambda_n \rightarrow \mathbb{R}$ we have the concepts of the limit, continuity, and properties of continuous functions on general complete metric spaces. Following standard one-dimensional concepts, we can define jump operators for each time scale \mathbb{T}_i , $i = \overline{1, n}$. For $t_i^0 \in \mathbb{T}_i$, the i -th forward jump operator $\sigma_i : \mathbb{T}_i \rightarrow \mathbb{T}_i$ is defined by

$$\sigma_i(t_i^0) = \begin{cases} \inf\{t_i \in \mathbb{T}_i : t_i > t_i^0\} & \text{if } t_i^0 \neq \max \mathbb{T}_i, \\ \max \mathbb{T}_i & \text{if } t_i^0 = \max \mathbb{T}_i. \end{cases}$$

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The i -th backward jump operator $\rho_i : \mathbb{T}_i \rightarrow \mathbb{T}_i$ is defined by

$$\rho_i(t_i^0) = \begin{cases} \sup\{t_i \in \mathbb{T}_i : t_i < t_i^0\} & \text{if } t_i^0 \neq \min \mathbb{T}_i, \\ \min \mathbb{T}_i & \text{if } t_i^0 = \min \mathbb{T}_i. \end{cases}$$

A point $t^0 \in \Lambda_n$ is called i -right-dense, i -right-scattered, i -left-dense and i -left-scattered if $\sigma_i(t_i^0) = t_i^0$, $\sigma_i(t_i^0) > t_i^0$, $\rho_i(t_i^0) = t_i^0$ and $\rho_i(t_i^0) < t_i^0$, respectively. The corresponding i -th forward graininess and i -th backward graininess functions $\mu_i, \eta_i : \mathbb{T}_i \rightarrow \mathbb{R}$, respectively, are defined by $\mu_i(t_i^0) = \sigma_i(t_i^0) - t_i^0$ and $\eta_i(t_i^0) = t_i^0 - \rho_i(t_i^0)$.

In order to define partial derivatives properly, we introduce the following. If \mathbb{T}_i has a i -left-scattered maximum, then we define $(\mathbb{T}_i)^\kappa = \mathbb{T}_i \setminus \max \mathbb{T}_i$, otherwise $(\mathbb{T}_i)^\kappa = \mathbb{T}_i$. If \mathbb{T}_i has a i -right-scattered minimum, then we define $(\mathbb{T}_i)_\kappa = \mathbb{T}_i \setminus \min \mathbb{T}_i$, otherwise $(\mathbb{T}_i)_\kappa = \mathbb{T}_i$. We also introduce

$$\begin{aligned} (\Lambda_n^i)^\kappa &= \mathbb{T}_1 \times \mathbb{T}_2 \times \dots \times \mathbb{T}_{i-1} \times (\mathbb{T}_i)^\kappa \times \mathbb{T}_{i+1} \times \dots \times \mathbb{T}_n, \\ (\Lambda_n^i)_\kappa &= \mathbb{T}_1 \times \mathbb{T}_2 \times \dots \times \mathbb{T}_{i-1} \times (\mathbb{T}_i)_\kappa \times \mathbb{T}_{i+1} \times \dots \times \mathbb{T}_n, \\ (\Lambda_n^i)^\kappa_\kappa &= (\Lambda_n^i)^\kappa \cap (\Lambda_n^i)_\kappa \\ &= \mathbb{T}_1 \times \mathbb{T}_2 \times \dots \times \mathbb{T}_{i-1} \times (\mathbb{T}_i)^\kappa_\kappa \times \mathbb{T}_{i+1} \times \dots \times \mathbb{T}_n, \\ (\Lambda_n)^\kappa &= (\mathbb{T}_1)^\kappa \times (\mathbb{T}_2)^\kappa \times \dots \times (\mathbb{T}_n)^\kappa, \\ (\Lambda_n)_\kappa &= (\mathbb{T}_1)_\kappa \times (\mathbb{T}_2)_\kappa \times \dots \times (\mathbb{T}_n)_\kappa, \\ (\Lambda_n)^\kappa_\kappa &= (\Lambda_n)^\kappa \cap (\Lambda_n)_\kappa = (\mathbb{T}_1)^\kappa_\kappa \times (\mathbb{T}_2)^\kappa_\kappa \times \dots \times (\mathbb{T}_n)^\kappa_\kappa. \end{aligned}$$

For a function f defined on Λ_n , to provide a shorthand notation we set

$$\begin{aligned} f^{\sigma_i}(t) &= f(t_1, t_2, \dots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \dots, t_n), \\ f^{\rho_i}(t) &= f(t_1, t_2, \dots, t_{i-1}, \rho_i(t_i), t_{i+1}, \dots, t_n), \end{aligned}$$

and

$$f_i^s(t) = f(t_1, t_2, \dots, t_{i-1}, s, t_{i+1}, \dots, t_n).$$

Definition 1 (cf. [3]): Let $f : \Lambda_n \rightarrow \mathbb{R}$ be a function and let $t^0 \in (\Lambda_n^i)^\kappa$. Then define $f^{\Delta_i}(t^0)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood $U_\delta(t_i^0) = (t_i^0 - \delta, t_i^0 + \delta) \cap \mathbb{T}_i$ for $\delta > 0$ such that

$$|[f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)] - f^{\Delta_i}(t^0)[\sigma_i(t_i^0) - t_i]| \leq \varepsilon |\sigma_i(t_i^0) - t_i|$$

for all $t_i \in U_\delta(t_i^0)$. $f^{\Delta_i}(t^0)$ is called the *partial delta derivative* of f at t^0 with respect to the variable t_i (*partial Δ_i -derivative*). Similarly, the *partial nabla derivative* of f at $t^0 \in (\Lambda_n^i)_\kappa$ with respect to the variable t_i (*partial ∇_i -derivative*), denoted by $f^{\nabla_i}(t^0)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood $U_\delta(t_i^0) = (t_i^0 - \delta, t_i^0 + \delta) \cap \mathbb{T}_i$ for $\delta > 0$ such that

$$|[f^{\rho_i}(t^0) - f_i^{t_i}(t^0)] - f^{\nabla_i}(t^0)[\rho_i(t_i^0) - t_i]| \leq \varepsilon |\rho_i(t_i^0) - t_i|$$

for all $t_i \in U_\delta(t_i^0)$.

From the works [4]–[6] we have that $f^{\Delta_i}(t^0)$ and $f^{\nabla_i}(t^0)$ are equal to

$$\lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} \quad \text{and} \quad \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \rho_i(t_i^0)}} \frac{f^{\rho_i}(t^0) - f_i^{t_i}(t^0)}{\eta_i(t_i^0, t_i)}, \quad (1)$$

respectively, where $\mu_i(t_i^0, t_i) = \sigma_i(t_i^0) - t_i$ and $\eta_i(t_i^0, t_i) = \rho_i(t_i^0) - t_i$. For $n = 1$ we obtain the one-dimensional time scales delta and nabla derivatives

$$\lim_{\substack{t \rightarrow t^0 \\ t \neq \sigma(t^0)}} \frac{f(\sigma(t^0)) - f(t)}{\sigma(t^0) - t} \quad \text{and} \quad \lim_{\substack{t \rightarrow t^0 \\ t \neq \rho(t^0)}} \frac{f(\rho(t^0)) - f(t)}{\rho(t^0) - t},$$

respectively, where σ and ρ are forward jump operator and backward jump operator, respectively. See [16], [17].

Similarly to the one-dimensional time scale Λ_1 , [16], [17], we obtain the partial delta and nabla derivatives of sums, products, and quotients of functions that have partial delta and nabla derivatives.

Theorem 1: If $f, g : \Lambda_n \rightarrow \mathbb{R}$ have partial delta and nabla derivatives at $t^0 \in (\Lambda_n^i)^\kappa_\kappa$ with respect to the variable t_i , then

- (i) $f + g$ has partial delta and nabla derivatives at t^0 with respect to the variable t_i , and

$$(f + g)^{\Delta_i}(t^0) = f^{\Delta_i}(t^0) + g^{\Delta_i}(t^0),$$

$$(f + g)^{\nabla_i}(t^0) = f^{\nabla_i}(t^0) + g^{\nabla_i}(t^0).$$

- (ii) For any constant c , cf has partial delta and nabla derivatives at t^0 with respect to the variable t_i , and

$$(cf)^{\Delta_i}(t^0) = cf^{\Delta_i}(t^0),$$

$$(cf)^{\nabla_i}(t^0) = cf^{\nabla_i}(t^0).$$

- (iii) fg has partial delta and nabla derivatives at t^0 with respect to the variable t_i , and

$$\begin{aligned} (fg)^{\Delta_i}(t^0) &= f^{\Delta_i}(t^0)g(t^0) + f^{\sigma_i}(t^0)g^{\Delta_i}(t^0) \\ &= g^{\Delta_i}(t^0)f(t^0) + g^{\sigma_i}(t^0)f^{\Delta_i}(t^0), \end{aligned}$$

$$\begin{aligned} (fg)^{\nabla_i}(t^0) &= f^{\nabla_i}(t^0)g(t^0) + f^{\rho_i}(t^0)g^{\nabla_i}(t^0) \\ &= g^{\nabla_i}(t^0)f(t^0) + g^{\rho_i}(t^0)f^{\nabla_i}(t^0). \end{aligned}$$

- (iv) If $g(t^0)g^{\sigma_i}(t^0) \neq 0$ and $g(t^0)g^{\rho_i}(t^0) \neq 0$, then f/g has partial delta and nabla derivatives at t^0 with respect to the variable t_i , and

$$\left(\frac{f}{g}\right)^{\Delta_i}(t^0) = \frac{f^{\Delta_i}(t^0)g(t^0) - f(t^0)g^{\Delta_i}(t^0)}{g(t^0)g^{\sigma_i}(t^0)},$$

$$\left(\frac{f}{g}\right)^{\nabla_i}(t^0) = \frac{f^{\nabla_i}(t^0)g(t^0) - f(t^0)g^{\nabla_i}(t^0)}{g(t^0)g^{\rho_i}(t^0)}.$$

Proof: Using Definition 1 we can prove these properties similarly to the proofs for the one-dimensional time scales presented in [16] and [17]. Another method to prove this, is to use (1). Let us prove the first equality of (iii). The other results are proved by the same way. If t^0 is i -right-dense, then

we have

$$\begin{aligned} (fg)^{\Delta_i}(t^0) &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{(fg)^{\sigma_i}(t^0) - (fg)^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} \\ &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{f^{\sigma_i}(t^0)g^{\sigma_i}(t^0) - f_i^{t_i}(t^0)g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} \\ &\quad + \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} (f^{\sigma_i}(t^0) - f_i^{t_i}(t^0))g^{\Delta_i}(t^0) \\ &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} g(t^0) \\ &\quad + \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{g^{\sigma_i}(t^0) - g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} f^{\sigma_i}(t^0) \\ &= f^{\Delta_i}(t^0)g(t^0) + f^{\sigma_i}(t^0)g^{\Delta_i}(t^0), \end{aligned}$$

because

$$\begin{aligned} g^{\Delta_i}(t^0) &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{g^{\sigma_i}(t^0) - g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} \\ &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{g(t^0) - g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} \end{aligned}$$

and $f(t^0) = f^{\sigma_i}(t^0)$. If t^0 is i -right-scattered, then we have

$$\begin{aligned} (fg)^{\Delta_i}(t^0) &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{(fg)^{\sigma_i}(t^0) - (fg)^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} \\ &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{f^{\sigma_i}(t^0)g^{\sigma_i}(t^0) - f_i^{t_i}(t^0)g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} \\ &\quad + \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{g(t^0) - g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} (f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)) \\ &= \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} g(t^0) \\ &\quad + \lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{g^{\sigma_i}(t^0) - g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} f^{\sigma_i}(t^0) \\ &= f^{\Delta_i}(t^0)g(t^0) + f^{\sigma_i}(t^0)g^{\Delta_i}(t^0), \end{aligned}$$

because

$$\lim_{\substack{t_i \rightarrow t_i^0 \\ t_i \neq \sigma_i(t_i^0)}} \frac{g(t^0) - g_i^{t_i}(t^0)}{\mu_i(t_i^0, t_i)} = \frac{g(t^0) - g(t^0)}{\mu_i(t_i^0, t_i^0)} = 0.$$

Definition 2: Let $f : \Lambda_n \rightarrow \mathbb{R}$ be a function and let $t^0 \in (\Lambda_n)_{\kappa}^i$. Then define $f^{\diamond_{\alpha_i}}(t^0)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood $U_{\delta}(t_i^0) = (t_i^0 - \delta, t_i^0 + \delta) \cap \mathbb{T}_i$ for $\delta > 0$ such that

$$|\alpha_i[f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)]\eta_i(t_i^0, t_i) + (1 - \alpha_i)[f^{\rho_i}(t^0) - f_i^{t_i}(t^0)]$$

$$\begin{aligned} &\times \mu_i(t_i^0, t_i) - f^{\diamond_{\alpha_i}}(t^0)\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \\ &\leq \varepsilon|\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \end{aligned}$$

for all $t_i \in U_{\delta}(t_i^0)$. $f^{\diamond_{\alpha_i}}(t^0)$ is called the *partial diamond- α_i derivative* of f at t^0 with respect to the variable t_i (*partial \diamond_{α_i} -derivative*).

Easily, using the same process that Rogers Jr. and Sheng used for one-dimension case in [18], we obtain that the partial diamond- α_i derivative is well-defined.

Now, we define the partial \diamond_{α_i} -derivative in relation to the partial Δ_i -derivative and partial ∇_i -derivative.

Theorem 2: Let $0 \leq \alpha_i \leq 1$, $i = \overline{1, n}$. If f has both partial Δ_i -derivative and partial ∇_i -derivative at $t^0 \in (\Lambda_n)_{\kappa}^i$, then f has partial \diamond_{α_i} -derivative at t^0 and

$$f^{\diamond_{\alpha_i}}(t^0) = \alpha_i f^{\Delta_i}(t^0) + (1 - \alpha_i) f^{\nabla_i}(t^0).$$

Proof: The proof mimics the one given in [18] for the one-dimensional time scales. Assume that $f^{\Delta_i}(t^0)$ and $f^{\nabla_i}(t^0)$ exist at $t^0 \in (\Lambda_n)_{\kappa}^i$. Then, for all $\varepsilon > 0$, there exists a neighborhood $U_{\delta_1}(t_i^0) = (t_i^0 - \delta_1, t_i^0 + \delta_1) \cap \mathbb{T}_i$ for $\delta_1 > 0$ such that

$$|[f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)] - f^{\Delta_i}(t^0)\mu_i(t_i^0, t_i)| \leq \varepsilon|\mu_i(t_i^0, t_i)|$$

for all $t_i \in U_{\delta_1}(t_i^0)$ and there exists a neighborhood $U_{\delta_2}(t_i^0) = (t_i^0 - \delta_2, t_i^0 + \delta_2) \cap \mathbb{T}_i$ for $\delta_2 > 0$ such that

$$|[f^{\rho_i}(t^0) - f_i^{t_i}(t^0)] - f^{\nabla_i}(t^0)\eta_i(t_i^0, t_i)| \leq \varepsilon|\eta_i(t_i^0, t_i)|$$

for all $t_i \in U_{\delta_2}(t_i^0)$. Then, for all $t_i \in U_{\delta_1}(t_i^0)$, we get

$$\begin{aligned} |\alpha_i[f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)]\eta_i(t_i^0, t_i) - \alpha_i f^{\Delta_i}(t^0)\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \\ \leq \alpha_i \varepsilon |\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \end{aligned}$$

and, for all $t_i \in U_{\delta_2}(t_i^0)$, we obtain

$$\begin{aligned} |(1 - \alpha_i)[f^{\rho_i}(t^0) - f_i^{t_i}(t^0)]\mu_i(t_i^0, t_i) \\ - (1 - \alpha_i)f^{\nabla_i}(t^0)\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \\ \leq (1 - \alpha_i)\varepsilon|\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)|. \end{aligned}$$

Thus, for all $t_i \in U_{\delta_1}(t_i^0) \cap U_{\delta_2}(t_i^0)$, we have

$$\begin{aligned} |\alpha_i[f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)]\eta_i(t_i^0, t_i) + (1 - \alpha_i)[f^{\rho_i}(t^0) - f_i^{t_i}(t^0)] \\ \times \mu_i(t_i^0, t_i) - [\alpha_i f^{\Delta_i}(t^0) + (1 - \alpha_i) f^{\nabla_i}(t^0)]\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \\ \leq |\alpha_i[f^{\sigma_i}(t^0) - f_i^{t_i}(t^0)]\eta_i(t_i^0, t_i) \\ - \alpha_i f^{\Delta_i}(t^0)\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \\ + |(1 - \alpha_i)[f^{\rho_i}(t^0) - f_i^{t_i}(t^0)]\mu_i(t_i^0, t_i) \\ - (1 - \alpha_i) f^{\nabla_i}(t^0)\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \\ \leq \alpha_i \varepsilon |\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| + (1 - \alpha_i)\varepsilon |\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)| \\ = \varepsilon |\mu_i(t_i^0, t_i)\eta_i(t_i^0, t_i)|. \end{aligned}$$

Therefore $f^{\diamond_{\alpha_i}}(t^0)$ exists and

$$f^{\diamond_{\alpha_i}}(t^0) = \alpha_i f^{\Delta_i}(t^0) + (1 - \alpha_i) f^{\nabla_i}(t^0).$$

Remark 1: It is clear that $f^{\diamond_{\alpha_i}}(t^0)$ reduces to $f^{\Delta_i}(t^0)$ for $\alpha_i = 1$ and $f^{\nabla_i}(t^0)$ for $\alpha_i = 0$.

From Definition 2, Theorem 1 and Theorem 2 we obtain the following properties.

Theorem 3: If $f, g : \Lambda_n \rightarrow \mathbb{R}$ have partial diamond- α_i derivative at $t^0 \in (\Lambda_n)_{\kappa}^i$ with respect to the variable t_i , then

- (i) $f + g$ has partial diamond- α_i derivative at t^0 with respect to the variable t_i , and

$$(f + g)^{\diamond_{\alpha_i}}(t^0) = f^{\diamond_{\alpha_i}}(t^0) + g^{\diamond_{\alpha_i}}(t^0).$$

- (ii) For any constant c , cf has partial diamond- α_i derivative at t^0 with respect to the variable t_i , and

$$(cf)^{\diamond_{\alpha_i}}(t^0) = cf^{\diamond_{\alpha_i}}(t^0).$$

- (iii) fg has partial diamond- α_i derivative at t^0 with respect to the variable t_i , and

$$(fg)^{\diamond_{\alpha_i}}(t^0) = f^{\diamond_{\alpha_i}}(t^0)g(t^0) + \alpha_i f^{\sigma_i}(t^0)g^{\Delta_i}(t^0) + (1 - \alpha_i)f^{\rho_i}(t^0)g^{\nabla_i}(t^0).$$

- (iv) If $g(t^0)g^{\sigma_i}(t^0)g^{\rho_i}(t^0) \neq 0$, then f/g has partial diamond- α_i derivative at t^0 with respect to the variable t_i , and

$$\left(\frac{f}{g}\right)^{\diamond_{\alpha_i}}(t^0) = \frac{f^{\diamond_{\alpha_i}}(t^0)g^{\sigma_i}(t^0)g^{\rho_i}(t^0)}{g(t^0)g^{\sigma_i}(t^0)g^{\rho_i}(t^0)} - \frac{\alpha_i f^{\sigma_i}(t^0)g^{\rho_i}(t^0)g^{\Delta_i}(t^0)}{g(t^0)g^{\sigma_i}(t^0)g^{\rho_i}(t^0)} - \frac{(1 - \alpha_i)f^{\rho_i}(t^0)g^{\sigma_i}(t^0)g^{\nabla_i}(t^0)}{g(t^0)g^{\sigma_i}(t^0)g^{\rho_i}(t^0)}.$$

Definition 3: Suppose that there exist partial \diamond_{α_i} -derivatives at $t^0 \in (\Lambda_n)_{\kappa}^i$ for all $i = \overline{1, n}$. The *diamond- α gradient* of a function $f : \Lambda_n \rightarrow \mathbb{R}$ at point $t^0 \in (\Lambda_n)_{\kappa}$ is the vector whose coordinates are equal to partial \diamond_{α_i} -derivatives at t^0 for $i = \overline{1, n}$. We denoted it by

$$\diamond_{\alpha} f(t^0) = (f^{\diamond_{\alpha_1}}(t^0), f^{\diamond_{\alpha_2}}(t^0), \dots, f^{\diamond_{\alpha_n}}(t^0)).$$

The *delta* and *nabla* gradients of f at t^0 , denoted by $\Delta f(t^0)$ and $\nabla f(t^0)$, respectively, are defined equivalently.

III. OPTIMIZATION PROBLEM ON TIME SCALES

In this section we present the main result. Similarly to definition of continuity on time scales to the one-dimensional case we obtain the following.

Definition 4: A function $f : \Lambda_n \rightarrow \mathbb{R}$ is *continuous* at t^0 if for all $\varepsilon > 0$ there is some $\delta > 0$ such that $|f(t) - f(t^0)| \leq \varepsilon$ whenever $d(t, t^0) < \delta$, $t \in \Lambda_n$. We say that the function f is continuous on Λ_n if it is continuous for all $t^0 \in \Lambda_n$ and we write $f \in C(\Lambda_n, \mathbb{R})$.

Suppose that $\max\{d(t) : t \in \Lambda_n\} = \infty$, where

$$d(t) = \left(\sum_{i=1}^n |t_i|^2\right)^{1/2}.$$

Consider an optimization problem on time scales

$$\begin{aligned} f(t) &\rightarrow \min, \\ g_j(t) &= 0, \quad j = \overline{1, m}, \\ h_k(t) &\leq 0, \quad k = \overline{1, l}, \end{aligned} \quad (2)$$

where $f : \Lambda_n \rightarrow \mathbb{R}$, $g_j : \Lambda_n \rightarrow \mathbb{R}$, $j = \overline{1, m}$, $h_k : \Lambda_n \rightarrow \mathbb{R}$, $k = \overline{1, l}$, are continuous and have partial \diamond_{α_i} -derivatives, $i = \overline{1, n}$, for all $t \in \Lambda_n$.

Definition 5: A point $t^* \in \Lambda_n$ is a *local solution* to problem (2) if there exists $\varepsilon > 0$ such that for all t that verifies the conditions $d^2(t, t^*) \leq \varepsilon$, $g_j(t) = 0$, $j = \overline{1, m}$, $h_k(t) \leq 0$, $k = \overline{1, l}$, we have $f(t) \geq f(t^*)$.

Lemma 1: Let $X \subset \Lambda_n$ be a closed set and let $f : \Lambda_n \rightarrow \mathbb{R}$ be a continuous function that verifies

$$\lim_{\substack{d(t) \rightarrow \infty \\ t \in \Lambda_n}} f(t) = +\infty.$$

Then there exists $t^* \in X$ such that $f(t^*) \leq f(t)$ for all $t \in X$.

Proof: In order to prove that result we need an auxiliary function which extends f to $\prod_{i=1}^n [\min \mathbb{T}_i, \max \mathbb{T}_i]$,

$$\hat{f} : \prod_{i=1}^n [\min \mathbb{T}_i, \max \mathbb{T}_i] \rightarrow \mathbb{R},$$

defined as

$$\hat{f}(t) := \begin{cases} f(t) & \text{if } t \in \Lambda_n, \\ g(t) & \text{if } t \in X^l, \end{cases}$$

for some $l \in I$, with $I \subset \mathbb{N}$ and t^l a point i -right-scattered for some $i = \overline{1, n}$, where

$$X^l = \prod_{\substack{i=1 \\ \sigma_i(t_i^l) \neq t_i^l}}^n (t_i^l, \sigma_i(t_i^l)) \times \prod_{\substack{i=1 \\ \sigma_i(t_i^l) = t_i^l}}^n \mathbb{T}_i,$$

$g(t)$ is a continuous function on X^l ,

$$\lim_{\substack{t \rightarrow \hat{t} \\ t \in X^l}} g(t) = f(\hat{t}) < g(t), \quad \hat{t} \in \partial X^l \cap \Lambda_n, \quad (3)$$

and \prod is the cartesian product. By continuity of f and by definition of \hat{f} we deduce that \hat{f} is continuous on $\prod_{i=1}^n [\min \mathbb{T}_i, \max \mathbb{T}_i]$. We have also that

$$\lim_{\substack{d(t) \rightarrow \infty \\ t \in \prod_{i=1}^n [\min \mathbb{T}_i, \max \mathbb{T}_i]}} \hat{f}(t) = \lim_{\substack{d(t) \rightarrow \infty \\ t \in \Lambda_n}} f(t) = +\infty.$$

Consider $X_i \subset \mathbb{T}_i$ such that $X = \prod_{i=1}^n X_i$. Since X is closed then there exist $\min X_i$ and $\max X_i$, $i = \overline{1, n}$. As a direct consequence of the Weierstrass theorem for the continuous-time case, there exists $t^* \in \hat{X} = \prod_{i=1}^n [\min X_i, \max X_i]$ such that $\hat{f}(t^*) \leq \hat{f}(t)$ for all $t \in \hat{X}$. See, e.g., [19].

Suppose that $t^* \notin X$. Then there is an $l^* \in I \subset \mathbb{N}$ such that $\hat{f}(t^*) = g(t^*)$ for $t^* \in X^{l^*}$. By (3) we have $g(t^*) > f(\hat{t})$, $\hat{t} \in \partial X^{l^*} \cap \Lambda_n$, and $\hat{f}(t^*) > f(\hat{t}) = \hat{f}(\hat{t})$ that is a contradiction. Thus $t^* \in X$ and $f(t^*) \leq \hat{f}(t)$, $t \in \hat{X}$. This implies that $f(t^*) \leq f(t)$ for all $t \in X$. ■

Definition 6: A function $f : \Lambda_n \rightarrow \mathbb{R}$ has a *local extremum* at $t^* \in (\Lambda_n)_{\kappa}^i$ if there is a neighborhood $U(t^*)$ of the point t^* such that either $f(t) \geq f(t^*)$ or $f(t) \leq f(t^*)$ for all $t \in U(t^*)$. For the case $f(t) \geq f(t^*)$, the image of t^* by f is defined by *local minimum*. For another case, $f(t) \leq f(t^*)$, $f(t^*)$ is defined by *local maximum*.

Lemma 2: Suppose that a function $f : \Lambda_n \rightarrow \mathbb{R}$ assumes its local extremum at $t^* \in (\Lambda_n)_{\kappa}^i$ and f has partial Δ_i and ∇_i -derivatives at t^* . Then, there exist $\alpha_i \in [0, 1]$, $i = \overline{1, n}$, such that $\diamond_{\alpha} f(t^*) = 0$.

Proof: We prove this result for a local minimum. The proof for a local maximum is done to a similar way. Suppose that f assumes its local minimum at $t^* \in (\overline{\Lambda_n})_k^c$. Using (1) we obtain $f^{\Delta_i}(t^*) \geq 0$ and $f^{\nabla_i}(t^*) \leq 0, i = \overline{1, n}$. If $f^{\Delta_i}(t^*) = 0$ or $f^{\nabla_i}(t^*) = 0$, then we put $\alpha_i = 1$ or $\alpha_i = 0$, respectively. If $f^{\Delta_i}(t^*) > 0$ and $f^{\nabla_i}(t^*) < 0$ we make

$$\alpha_i = \frac{f^{\nabla_i}(t^*)}{f^{\nabla_i}(t^*) - f^{\Delta_i}(t^*)}.$$

Since $f^{\nabla_i}(t^*) - f^{\Delta_i}(t^*) < 0$ and $f^{\Delta_i}(t^*) - f^{\nabla_i}(t^*) > -f^{\nabla_i}(t^*)$ we have that $0 < \alpha_i < 1$ and we obtain the result. ■

We can see that result for one-dimensional case in [20].

Let $a \in \mathbb{R}$ be a number. By a_+ we denote the $\max\{0, a\}$ and we take

$$\text{sign}(a) = \begin{cases} a/|a| & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Now, we present the main result that consists to obtain necessary conditions for a solution of problem (2).

Theorem 4: Let t^* be a local solution of problem (2). Then there exist $\lambda \in \mathbb{R}_0^+$, $\mu_j = \text{sign}(g_j(t))a_j, a_j \in \mathbb{R}_0^+, j = \overline{1, m}$, $\nu_k \in \mathbb{R}_0^+, k = \overline{1, l}$, and $\alpha_i \in [0, 1], i = \overline{1, n}$, such that

1.

$$\lambda \diamond_{\alpha} f(t^*) + \langle \mu, \diamond_{\alpha} g(t^*) \rangle + \langle \nu, \diamond_{\alpha} h(t^*) \rangle = 0, \quad (4)$$

2.

$$\nu_k h_k(t^*) = 0, \quad k = \overline{1, l}, \quad (5)$$

3.

$$\lambda^2 + \sum_{j=1}^m \mu_j^2 + \sum_{k=1}^l \nu_k^2 \neq 0, \quad (6)$$

where

$$\mu = (\mu_1, \mu_2, \dots, \mu_m), \quad \nu = (\nu_1, \nu_2, \dots, \nu_l),$$

$$\diamond_{\alpha} g(t^*) = (\diamond_{\alpha} g_1(t^*), \diamond_{\alpha} g_2(t^*), \dots, \diamond_{\alpha} g_m(t^*)),$$

and

$$\diamond_{\alpha} h(t^*) = (\diamond_{\alpha} h_1(t^*), \diamond_{\alpha} h_2(t^*), \dots, \diamond_{\alpha} h_l(t^*)).$$

Proof: Set $\gamma < f(t^*)$ and $\lambda^{\gamma}, \mu_j^{\gamma}, \nu_k^{\gamma} \in \mathbb{R}, j = \overline{1, m}, k = \overline{1, l}$. Consider the functions

$$\Phi(t, \gamma) = \lambda^{\gamma}(f(t) - \gamma) + \sum_{j=1}^m \mu_j^{\gamma} g_j(t) + \sum_{k=1}^l \nu_k^{\gamma} h_k(t) +$$

and

$$F(t, \gamma) = \Phi(t, \gamma) + \sum_{i=1}^n p_i(t_i),$$

where $p_i : \mathbb{T}_i \rightarrow \mathbb{R}$ are continuous functions such that

$$p_i(t_i) \geq (t_i - t_i^*)^2, \quad t_i \in \mathbb{T}_i \setminus \{t_i^*\},$$

$$p_i(t_i^*) = 0,$$

and

$$0 \leq p_i^{\diamond_{\alpha_i}}(t_i) \leq q_i(\alpha_i, t_i^*)(t_i - t_i^*)^2$$

for functions q_i of $[0, 1] \times \mathbb{T}_i$ into \mathbb{R} and $i = \overline{1, n}$. If $\lambda^{\gamma} \geq 0$, $\mu_j^{\gamma} = \text{sign}(g_j(t))a_j^{\gamma}, a_j^{\gamma} \geq 0$, and $\nu_k^{\gamma} \geq 0$, then we have

$\Phi(t, \gamma) \geq 0$ for all $t \in \Lambda_n$ and $F(t, \gamma) \rightarrow +\infty$ if $d(t) \rightarrow \infty$. From Lemma 1 we get that $F(t, \gamma)$ has a global minimum at a point t^{γ} . Since the functions $f : \Lambda_n \rightarrow \mathbb{R}, g_j : \Lambda_n \rightarrow \mathbb{R}$ and $h_k : \Lambda_n \rightarrow \mathbb{R}$ have partial \diamond_{α_i} -derivatives, $i = \overline{1, n}$, for all $t \in \Lambda_n$, from (i)-(ii) of Theorem 3 we have that $\Phi(t, \gamma)$ has also partial \diamond_{α_i} -derivatives for all $t \in \Lambda_n$. Then $F(t, \gamma)$ has also partial \diamond_{α_i} -derivatives for all $t \in \Lambda_n$ and from Lemma 2 there exist $\alpha_i \in [0, 1], i = \overline{1, n}$, such that $\diamond_{\alpha} F(t^{\gamma}, \gamma) = 0$. From (i)-(ii) of Theorem 3 we obtain

$$\lambda^{\gamma} \diamond_{\alpha} f(t^{\gamma}) + \sum_{j=1}^m \mu_j^{\gamma} \diamond_{\alpha} g_j(t^{\gamma}) + \sum_{k=1}^l \nu_k^{\gamma} \diamond_{\alpha} h_k(t^{\gamma}) = 0. \quad (7)$$

Set $\nu_k^{\gamma} = b_k^{\gamma} h_k(t^{\gamma})_+, b_k^{\gamma} \geq 0, k = \overline{1, l}$, and $\gamma \uparrow f(t^*)$. Without loss of generality we consider that $\lambda^{\gamma} \rightarrow \lambda, \mu_j^{\gamma} \rightarrow \mu_j$, that is, $a_j^{\gamma} \rightarrow a_j$, and $\nu_k^{\gamma} \rightarrow \nu_k$. Since

$$\begin{aligned} d^2(t^{\gamma}, t^*) &\leq \sum_{i=1}^n p_i(t_i^{\gamma}) \\ &\leq \Phi(t^{\gamma}, \gamma) + \sum_{i=1}^n p_i(t_i^{\gamma}) = F(t^{\gamma}, \gamma) \\ &\leq F(t^*, \gamma) = \lambda^{\gamma}(f(t^*) - \gamma) \end{aligned}$$

we have $t^{\gamma} \rightarrow t^*$. Therefore, passing to the limit in (7) we obtain (4) and we have (5) because if $h_k(t^*) < 0$ then $\nu_k = 0$. The condition (6) is very important, because otherwise the condition (4) is always verified. ■

Remark 2: If optimization problem (2) has not any restrictions, the Theorem 4 becomes into Lemma 2. If $\Lambda_n = \mathbb{R}^n$, then $\diamond_{\alpha} f(t^*) = \nabla_t f(t^*), \diamond_{\alpha} g_j(t^*) = \nabla_t g_j(t^*), j = \overline{1, m}$, and $\diamond_{\alpha} h_k(t^*) = \nabla_t h_k(t^*), k = \overline{1, l}$, where ∇_t represents the classical gradient in \mathbb{R}^n with respect to the variable t , and we obtain the classical conditions of the Lagrange multipliers rule. See, e.g., [19].

Example 1: Consider the optimization problem

$$\begin{aligned} t_1^2 + t_2^2 + t_3^2 &\rightarrow \min, \\ t_1 + t_2 &= 1, \\ t_2 &\leq 0, \end{aligned} \quad (8)$$

on time scale $\Lambda_3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Since

$$\Delta(t_1^2 + t_2^2 + t_3^2) = (2t_1 + 1, 2t_2 + 1, 2t_3 + 1),$$

$$\nabla(t_1^2 + t_2^2 + t_3^2) = (2t_1 - 1, 2t_2 - 1, 2t_3 - 1),$$

$$\Delta(t_1 + t_2 - 1) = \nabla(t_1 + t_2 - 1) = (1, 1, 0),$$

$$\Delta t_2 = \nabla t_2 = (0, 1, 0),$$

by (4) we obtain

$$\lambda \begin{pmatrix} 2t_1 + 2\alpha_1 - 1 \\ 2t_2 + 2\alpha_2 - 1 \\ 2t_3 + 2\alpha_3 - 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\lambda \geq 0, \mu = \text{sign}(t_1 + t_2 - 1)a, a \geq 0, \nu \geq 0$, and $\alpha_i \in [0, 1], i = \overline{1, 3}$. Set $\lambda = 0$. Thus we obtain $\mu = \nu = 0$. This contradicts the condition (6). Then we can choose $\lambda = 1$.

We obtain

$$t_1 = \frac{1 - 2\alpha_1 - \mu}{2},$$

$$t_2 = \frac{1 - 2\alpha_2 - \mu - \nu}{2},$$

and

$$t_3 = \frac{1 - 2\alpha_3}{2}.$$

Since $\alpha_3 \in [0, 1]$, we get $t_3 \in [-1/2, 1/2]$. Then we obtain $t_3 = 0$ because $t_3 \in \mathbb{Z}$. If $\nu = 0$ we have

$$t_2 = \frac{1 - 2\alpha_2 - \mu}{2}$$

and by the first restriction of problem (8) we obtain $\mu = -\alpha_1 - \alpha_2$. First, we make $\mu = 0$. We obtain $\alpha_1 = -\alpha_2$. That implies

$$t_1 = \frac{1 + 2\alpha_2}{2} \quad \text{and} \quad t_2 = \frac{1 - 2\alpha_2}{2}.$$

Since we also have $\alpha_2 \in [0, 1]$, we obtain $t_1 \in [1/2, 3/2]$ and $t_2 \in [-1/2, 1/2]$. Then we get the critical point $(1, 0, 0)$. Now, put $\mu \neq 0$. We obtain $t_1 = 0$ and $t_2 = 1$ or $t_1 = 1$ and $t_2 = 0$. The first case contradicts the last restriction of (8) and the second case is the same critical point $(1, 0, 0)$. Consider $\nu > 0$. By condition (5) we get

$$t_2 = \frac{1 - 2\alpha_2 - \mu - \nu}{2} = 0.$$

If $\mu = 0$, we obtain $\nu = 1 - 2\alpha_2$ and

$$t_1 = \frac{1 - 2\alpha_1}{2}.$$

Thus we have $t_1 = t_2 = t_3 = 0$ that contradicts the first restriction of (8). Finally, taking $\mu \neq 0$ and using again the first restriction of (8) we obtain the same critical point $(1, 0, 0)$. Therefore, the image of the point $(1, 0, 0)$ is the only local extremum. Since the function $f(t_1, t_2, t_3) = t_1^2 + t_2^2 + t_3^2$ tends to infinity when $d((t_1, t_2, t_3))$ tends to infinity, from Lemma 1 we have that $f(1, 0, 0)$ is the global minimum.

IV. CONCLUSION

In this work we obtained necessary conditions for a solution of an optimization problem on time scales that are a generalization of the results to the discrete and continuous cases. The obtained result allows to solve optimization problems on mixed domains. We also obtained important properties and results about partial diamond-alpha derivatives of continuous-multivariable functions. We presented the properties of the partial delta, nabla and diamond-alpha derivatives of an n -dimensional function. We defined continuity to an n -dimensional function, local extremum, local minimum and local maximum, and we presented conditions to local extremum. Based on this work, it is possible to obtain results about necessary conditions for a solution of an optimal control problem on time scales. That will be our future work.

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