System Overflow/Blocking Transients For Queues with Batch Arrivals Using a Family of Polynomials Resembling Chebyshev Polynomials

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Abstract—The paper shows that in the analysis of a queuing system with fixed-size batch arrivals, there emerges a set of polynomials which are a generalization of Chebyshev polynomials of the second kind. The paper uses these polynomials in assessing the transient behaviour of the overflow (equivalently call blocking) probability in the system. A key figure to note is the proportion of the overflow (or blocking) probability resident in the transient component, which is shown in the results to be more significant at the beginning of the transient and naturally decays to zero in the limit of large t. The results also show that the significance of transients is more pronounced in cases of lighter loads, but lasts longer for heavier loads.

Keywords—batch arrivals, blocking probability, generalized Chebyshev polynomials, overflow probability, queue transient analysis

I. INTRODUCTION

THE paper considers a queuing system in which the arrivals occur in fixed-size batches of B packets each, according to a Poisson process of mean rate λ arrivals per unit time. The single-server has exponentially distributed service times, having a mean service rate of μ packets per unit time. In determining the queue statistics, it has been convenient to resort to steady-state analysis, mainly because of mathematical tractability. However, transient analysis has been accepted as being complementary to the steady-state analysis [1-5]. This has been echoed in justifying the analysis presented in [6,7] where use is made of special functions.

The characterization of the transient behaviour of queuing systems has often been complicated by the multidimensional transforms that have to be inverted [8]; it has been difficult to accomplish these inversions in manageable closed form. Accordingly, it has been necessary in some cases to use numerical techniques [9,10] .

It has been shown [6,7] that for fixed-size batch Poisson arrivals and exponential service times it is possible to obtain an expression for the empty state probability in terms of functions that are related to the modified Bessel functions of the first kind. It is further shown [7] that these functions are a generalization of the modified Bessel functions of the first kind, with the batch size B as the generalizing parameter. In the present paper, a set of polynomials arises in the treatment of the same system. These polynomials are seen to resemble Chebyshev polynomials.

The rest of the paper is organized as follows. Section II gives the system model, on which is based the analysis that begins by determining the probability flow balance equations. Section III presents the family of polynomials that are fundamental in the analysis. This section also presents the similarities of these polynomials with the Chebyshev counterparts. Section IV presents the characterizing system probabilities – steady state, transient and overflow probabilities. Section V combines results and discussion. Section VI gives the conclusion

II. SYSTEM MODEL

Packets arrive at a service point in fixed size batches of B packets according to a Poisson process of mean rate λ arrivals per second. The single server completes the service at the rate of μ packets per second. The probability flow balance is shown in Fig.1 in which the top half indicates transitions among states from the empty state (state-0) up to B+1, and the lower half is for a general state-k where k ≥ B.
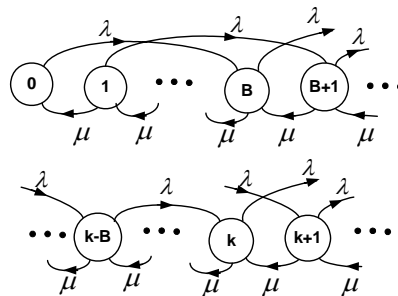

Fig.1. Probability Flow For Arrival Batch size B

Denote by $P_k(t)$ the probability that at time t there are k packets in the system. From the diagram the following equations follow immediately.

$$\frac{d}{dt}P_k(t) + (\lambda + \mu)P_k(t) = \begin{cases} \mu P_1(t) + \mu P_0(t) & k = 0 \\ \mu P_{k+1}(t) & 1 \le k < B \\ \mu P_{k+1}(t) + \lambda P_{k-B}(t) & k \ge B \end{cases} \quad (1)$$

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To simplify the analysis, the following definitions are introduced.

$$P_k(t) = P_k + Q_k(t)\exp[-(\lambda+\mu)t] \qquad (2)$$

where P_k is the steady state probability of there being k packets in the system, inclusive of the one in service, and the term $Q_k(t)exp[-(\lambda+\mu)t]$ represents the transient part of the probability of occupancy.

The representation (2) enables (1) to be decoupled into two sets of equations, one for the steady-state and the other for the transient component. From (1) it can shown that the quantities $Q_k(t)$, for the transient component, satisfy the relations

$$\frac{d}{dt}Q_k(t) = \begin{cases} \mu Q_1(t) + \mu Q_0(t) & k=0 \\ \mu Q_{k+1}(t) & 1 \le k < B \\ \mu Q_{k+1}(t) + \lambda Q_{k-B}(t) & k \ge B \end{cases} \qquad (3)$$

The usual procedure at this point would be to obtain for $Q_k(t)$ a multidimensional transform, a suitable one in this case being a two-dimensional Laplace-Stieltjes transform, and attempt to invert the transform. Owing to the difficulty with transform inversion already alluded to, this paper relies on the expression for the empty state probability obtained in [7] and uses the properties of the functions presented in [6,7] together with a family of polynomials introduced in the sequel to obtain the occupancy probabilities. These are then used to determine the transient behaviour of the overflow (call blocking) probability. It is important at this point to digress and discuss the family of polynomials that will prove convenient in the subsequent analysis.

III. THE FAMILY OF POLYNOMIALS

The steady-state probabilities P_k are just the result of the traditional queuing analysis. In much of what follows an attempt is made to express in closed form both the steady state probabilities and the transient terms in (2) using a set of functions already discussed [6,7] and a set of polynomials to be introduced here. These polynomials are denoted $T_k^{(B)}(x)$, signifying that they are parameterized by B, and for each B, they are indexed by k. They are are defined as follows

$$T_k^{(B)}(x) = \sum_{m=0}^{[k/(B+1)]} (-1)^m \binom{k-mB}{m} (x)^{k-m(B+1)} \qquad (4)$$

where [k/(B+1)] is the integer part of k/(B+1). For B=1,2,3 the first few of these polynomials are shown in Table I for k=0,1,2,…,9. From (4) and also evident in the table, these polynomials satisfy the recurrence relations

$$T_k^{(B)}(x) = \begin{cases} x^k & 0 \le k \le B \\ x T_{k-1}^{(B)}(x) - T_{k-(B+1)}^{(B)}(x) & k > B \end{cases} \qquad (5)$$

The even polynomials are plotted in Fig.2, while the odd polynomials are given Fig.3. The odd polynomials have odd symmetry and the even polynomials have even symmetry about the origin, as expected.

TABLE I   POLYNOMIALS $T_k^{(B)}(x)$ FOR B=2 AND K=0,1,2,…,9

k	B=1	B=2	B=3
0	1	1	1
1	x	x	x
2	x^2-1	x^2	x^2
3	x^3-2x	x^3-1	x^3
4	x^4-3x^2+1	x^4-2x	x^4-1
5	x^5-4x^3+3x	x^5-3x^2	x^5-2x
6	$x^6-5x^4+6x^2-1$	x^6-4x^3+1	x^6-3x^2
7	$x^7-6x^5+10x^3-4x$	x^7-5x^4+3x	x^7-4x^3
8	$x^8-7x^6+15x^4-10x^2+1$	$x^8-4x^5+6x^2$	x^8-5x^4+1
9	$x^9-8x^7+21x^5-20x^3+5x$	$x^9-5x^6+10x^3-1$	x^9-6x^5+3x

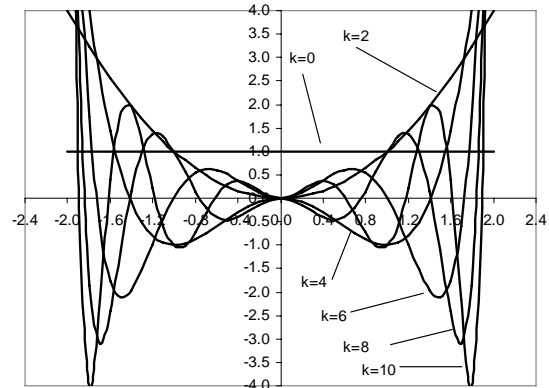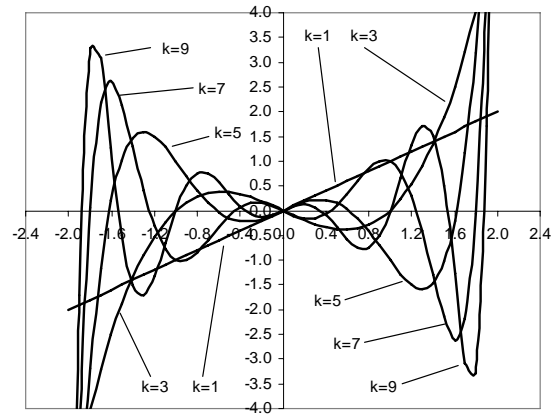Fig. 2. Plot of the polynomials $T_k^{(B)}(x)$ for k even , up to k = 10

Fig. 3. Plot of the polynomials $T_k^{(B)}(x)$ for k odd, up to k = 9

A. Similarities with Chebyshev Polynomials

It is useful to point out some of the similarities with the Chebyshev polynomials. The first similarity of these polynomials is seen in their plots of Fig.2 and Fig.3. The second similarity is seen in the expression for the generating function $T^{(B)}(z,x)$ of these polynomials, which is defined as

$$T^{(B)}(z,x) = \sum_{k=0}^{\infty} z^k T_k^{(B)}(x) \qquad (6)$$

Applying the defining equations (5), gives

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$$T^{(B)}(z,x) = \sum_{k=0}^{B} (zx)^k + zx \sum_{k=B+1}^{\infty} z^{k-1} T_{k-1}^{(B)}(x) \qquad (7)$$
$$- z^{B+1} \sum_{k=0}^{\infty} z^k T_k^{(B)}(x)$$

After some algebra, this results in

$$T^{(B)}(z,x) = \frac{1}{z^{B+1} - zx + 1} \qquad (8)$$

In this equation when B=1 and x is replaced by 2x, this is just the generating function of the Chebyshev polynomials of the second kind [11-14]. It is noteworthy that whereas the functions emerging in [7] are related to the modified Bessel functions of the first kind, and reduce to the latter when the batch size parameter takes the value of unity (B=1) and x is replaced by 2x, there now emerges here a set of polynomials which are related to the Chebyshev polynomials of the second kind, and reduce to the latter polynomials when B=1 and x is replaced by 2x.

The generating function obtained so far states that the polynomials $T_k^{(B)}(x)$ are the coefficients of z^k in the infinite series expansion

$$\frac{1}{z^{B+1} - zx + 1} = \sum_{k=0}^{\infty} z^k T_k^{(B)}(x) \qquad (9)$$

This generating function looks like the one for the Chebyshev polynomials of the second kind $U_k(x)$ when we set B = 1, and replace x above with 2x. That is

$$\frac{1}{z^2 - 2xz + 1} = \sum_{k=0}^{\infty} z^k U_k(x) \qquad (10)$$

The polynomial $U_k(x)$ can also be expressed as [11-14]

$$U_k(x) = \sum_{m=0}^{[k/2]} (-1)^m \binom{k-m}{m} (2x)^{k-2m} \qquad (11)$$

which is really the exact expression (4) for $T_k^{(B)}(x)$ when B=1, and x is replaced by 2x. These similarities suggest here that the polynomials emerging in this analysis are indeed more general than the Chebyshev counterparts, with the batch size B as the generalizing parameter.

A study of these polynomials, exploring their similarities with the Chebyshev polynomials, and more, can be an interesting subject in itself; here the intention is to see how they can be used in the analysis of the transient behaviour of the queuing system with batch arrivals. In the sequel, they are used together with solutions already found in previous works to obtain results from which conclusions are drawn.

IV. SYSTEM PROBABILITES

This section presents the key system probabilities such as the steady state, transient, and system overflow (blocking) probabilities. It also shows the possible implementations of the steady state probabilities.

A. Steady-State Probabilities

The steady state probabilities are obtained by considering that in the limit of large t, the transient terms go to zero, as do the time derivatives, yielding the equations below.

$$P_k = \begin{cases} \dfrac{\lambda}{\mu} P_0 & k = 1 \\[2ex] \left(1 + \dfrac{\lambda}{\mu}\right) P_{k-1} & 2 \le k \le B \\[2ex] \left(1 + \dfrac{\lambda}{\mu}\right) P_{k-1} - \dfrac{\lambda}{\mu} P_{k-(B+1)} & k \ge B+1 \end{cases} \qquad (12)$$

It is shown in [7] that for k > 0, these steady state probabilities satisfy the moving average relations

$$P_k = \left(\frac{\lambda}{\mu}\right) \sum_{m=1}^{\min(k,B)} P_{k-m} \qquad (13)$$

where the averaging window size equals the fixed sized B of the batch. In fact (13) is seen as a scaled down moving average when the definition $\rho = B\lambda/\mu$ of the offered load is used.

The steady state probabilities can be implemented using at least three alternatives. The first two are based on signal flow diagrams using delays, multipliers and adders. The third method uses the polynomials discussed here
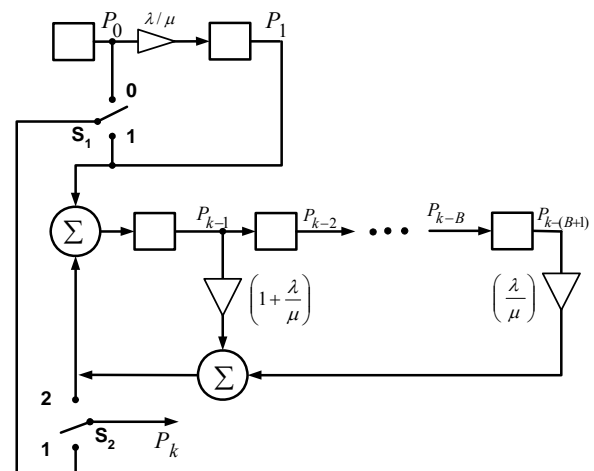

Fig. 4. Recursive implementation of the steady-state probabilities (12)

The first method is based on (12) and uses the signal-flow diagram of Fig.4. Initially the value of P_0 is loaded into the delay stage whose output is labelled P_0, with the other delay elements set to zero. The first switch S1 is initially in position 0, and is moved to position 1 after one clock period, and stays there for the rest of the time. The second switch S2 is initially in position 1 where it stays through the first clock period, and is then moved to the operating position 2, where it remains for the remainder of the system.

The second implementation uses the signal-flow diagram of Fig.5, in which the value of P_0 is loaded in the delay element whose output is labelled P_0. All the delay elements are cleared (set to zero). The switch S is initially at position 0. It is moved to position 1 as the system clock is started. Data is then passed through the B delay stages. At every tick of the

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clock the value of P_k is indeed the desired steady-state probability of there being k packets (customers) in the system.
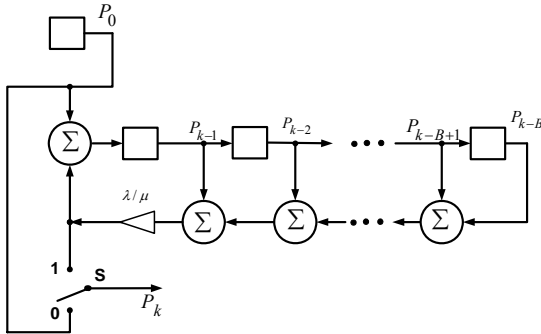

Fig. 5. Moving average implementation of the steady-state probabilities (13)

The recurrence relations (12) can be applied repeatedly to express the steady state probabilities P_k in terms of P_0 and the polynomials already introduced. That is, by repeatedly setting k=2,3, …, in (12) it is found that leads to

$$P_k = \left[\left(\frac{\rho}{B} \right)^{k/(B+1)} T_k^{(B)}(x) - \left(\frac{\rho}{B} \right)^{(k-1)/(B+1)} T_{k-1}^{(B)}(x) \right] P_0 \qquad (14)$$

where $\rho = B\lambda/\mu$ is the offered load, and

$$x = \left(1 + \frac{\rho}{B} \right) \left(\frac{\rho}{B} \right)^{1/(B+1)} \qquad (15)$$

It has already been found [7] that the empty system probability P_0 is given by

$$P_0 = 1 - \rho \qquad (16)$$

B. Transient Probabilities

It is convenient to introduce the differential operator $D = d/dt$ and the parameter α defined as

$$\alpha = \left(\mu^B \lambda \right)^{1/(B+1)} = \mu \left(\frac{\lambda}{\mu} \right)^{1/(B+1)} \qquad (17)$$

Then $Q_k(t)$ can be expressed successively in terms of $Q_0(t)$ as follows

$$Q_1(t) = \left(\frac{D}{\mu} - 1 \right) Q_0(t) \qquad (18)$$

$$Q_2(t) = \left[\left(\frac{D}{\mu} \right)^2 - \left(\frac{D}{\mu} \right) \right] Q_0(t) \qquad (19)$$

$$= \left[\left(\frac{\alpha}{\mu} \right)^2 \left(\frac{D}{\alpha} \right)^2 - \left(\frac{\alpha}{\mu} \right) \left(\frac{D}{\alpha} \right) \right] Q_0(t)$$

After some algebra, the general expression for $Q_k(t)$ for k>0 becomes

$$Q_k(t) = \left[\left(\frac{\alpha}{\mu} \right)^k T_k^{(B)} \left(\frac{D}{\alpha} \right) - \left(\frac{\alpha}{\mu} \right)^{k-1} T_{k-1}^{(B)} \left(\frac{D}{\alpha} \right) \right] Q_0(t) \qquad (20)$$

From (20) it is clear that the general expression for $Q_k(t)$ is a weighted sum of the derivatives of $Q_0(t)$. Once $Q_0(t)$ is known the required results can be found. These can then be combined with the steady-state results to obtain the time-varying occupancy probabilities $P_k(t)$. Previously [6,7] a set of functions are introduced, and defined according to

$$V_{-k}^{(B)}(x) = \sum_{l=0}^{\infty} \frac{x^{l(B+1)+k}}{l!(lB+k)!} \qquad (21a)$$

$$V_k^{(B)}(x) = \sum_{l=\sigma_k}^{\infty} \frac{x^{l(B+1)-k}}{l!(lB-k)!} \qquad (21b)$$

where $\sigma_k = \lceil k/B \rceil$, the smallest integer not less than k/B. To make the expression more manageable, two other quantities $q_k(t)$ and $h_k(t)$ are defined

$$h_k(t) = V_k^{(B)}(\alpha t) - \frac{\alpha}{\mu} V_{k+1}^{(B)}(\alpha t) \qquad (22a)$$

$$q_k(t) = \left(\frac{\lambda}{\mu} \right)^{\frac{-k}{B+1}} V_{-k}^{(B)}(\alpha t) \qquad (22b)$$

Here $h_k(t)$ is the convolution kernel that appears in the expression for $Q_k(t)$, as given by

$$Q_k(t) = q_{i-k}(t) \exp[-(\lambda+\mu)t] + \left(\frac{\alpha}{\mu} \right)^k \int_0^t \mu Q_0(t-\tau) \exp[-(\lambda+\mu)\tau] h_k(\tau) d\tau \qquad (23)$$

Setting k=0 in (23) gives the integral equation whose solution would give the result for $Q_0(t)$,

$$Q_0(t) = q_i(t) \exp[-(\lambda+\mu)t] + \int_0^t \mu Q_0(t-\tau) \exp[-(\lambda+\mu)\tau] h_0(\tau) d\tau \qquad (24)$$

In [7] it is argued convincingly that rather than attempting to solve (24) directly, which is a challenge, it is better to exploit the fact that the systems $\left(B=1, I_k() \right)$ and $\left(B>1, V_k^{(B)}() \right)$ are isomorphic, and noting that by appropriately adapting the known solution for $Q_0(t)$ in $\left(B=1, I_k() \right)$, the corresponding results for the function $Q_0(t)$ here can be expressed as

$$Q_0(t) = \sum_{p=i}^{\infty} \left(\frac{\mu}{\alpha} \right)^p \left[V_{-p}^{(B)}(\alpha t) - B V_{-(p+B+1)}^{(B)}(\alpha t) \right] \qquad (25)$$

Thus far $Q_0(t)$ has been determined. The expression for $Q_k(t)$ in *(20)* requires other properties of the functions used in (25). In [7] it is shown that

$$\left(\frac{D}{\alpha} \right)^m V_{-p}^{(B)}(\alpha t) = \sum_{q=0}^{m} \binom{m}{q} \left[V_{-[p+(m-q)B]+q}^{(B)}(\alpha t) \right] \qquad (26)$$

It now remains to combine this with (20) and the definition of the polynomials $T_k^{(B)}(x)$ to obtain a closed form expression for $Q_k(t)$.

C. Overflow / Blocking Probability

Given that a system can hold no more than N packets (there is room for only N customers), the probability that there will be an overflow (blocking) at time t is then the probability that this quantity is exceeded.

$$P_{ovfl}^{(B)}(N) = \sum_{k=N+1}^{\infty} \left\{ P_k + Q_k(t) \exp[-(\lambda+\mu)t] \right\} \qquad (27)$$

It has to be assumed here that the system capacity N is

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larger than the batch size (N>B). This condition is necessary since the system should be able hold at least one batch upon its arraival. In that case the sum can be broken into two parts to give

$$P_{ovfl}^{(B)}(N) = -\left(\frac{\rho}{B}\right)^{N/(B+1)} T_N^{(B)}\left(\left(1+\frac{\rho}{B}\right)\left(\frac{B}{\rho}\right)^{1/(B+1)}\right) P_0 \qquad (28)$$

$$- \exp[-(\lambda + \mu)t]\left(\frac{\alpha}{\mu}\right)^N T_N^{(B)}\left(\frac{D}{\alpha}\right) Q_0(t)$$

The fundamental role played by the polynomials $T_k^{(B)}(x)$ is evident in (28).

D. Transient Proportion in Overflow/Blocking Probability

The proportion of the overflow probability taken by the transient term can be used to give a sense of the significance of the transients in the overflow assessment. To this end the ratio η_{trans} is defined as follows

$$\eta_{trans} = \frac{1}{P_{ovfl}^{(B)}(N)} \exp[-(\lambda + \mu)t] \sum_{k=N+1}^{\infty} Q_k(t) \qquad (29)$$

Since the results presented below are functions of μt, the normalized time, it is necessary to express (29) so as to reflect this. Combining the definitions of α and ρ, together with (28) and (29), it follows that

$$\eta_{trans} = \frac{\exp\left(-\left(1+\frac{\rho}{B}\right)\mu t\right) T_N^{(B)}\left(\frac{D}{\alpha}\right) Q_0(t)}{T_N^{(B)}\left(\left(1+\frac{\rho}{B}\right)\frac{\mu}{\lambda}\right) P_0 + \exp\left(-\left(1+\frac{\rho}{B}\right)\mu t\right) T_N^{(B)}\left(\frac{D}{\alpha}\right) Q_0(t)} \qquad (30)$$

It will be interesting to observe how long it takes for this proportion to decay to zero (as it should) as the load is varied.

V. RESULTS AND DISCUSSION

The results given are in two sets, the first one being the plain overflow/blocking probabilities as a function of time. The second set are the proportion of the overflow that is in the transient component. All the results are shown for a capacity of 10 packets (N=10) when the arrival batch size is 3. The first four depict the results starting from the empty state (i=0), while the last three depict the results for a non-empty initial state (i>0). The results presented are only for B=3 and N=10. Other values of B and N may be considered if required.

Fig.6 shows the results of the overflow probability (blocking probability) for low loads (ρ = 0.05). Initially the probability of overflow is relatively low (5×10⁻⁷), then rising to a peak of 6×10⁻⁷ at μt = 6., and then settles finally at 5×10⁻⁷, for μt > 20.

Fig.7 shows the results of the overflow probability (blocking probability) for moderate loads (ρ = 0.50). The trend is similar to the one in the preceding figure. Initially the probability of overflow is once again relatively low (2.17×10⁻²), then rising to a peak of 2.25×10⁻² at μt = 9., and then settles finally at 2.17×10⁻², for μt > 70. It takes longer here to reach steady state than for lower loads
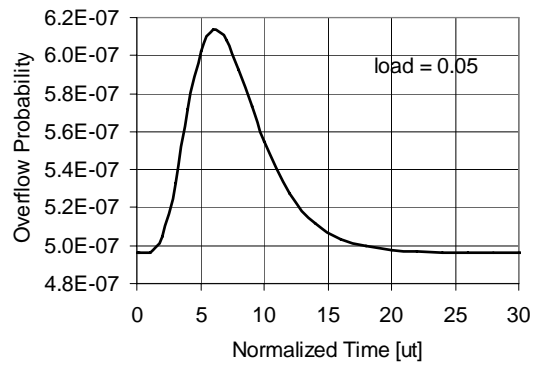

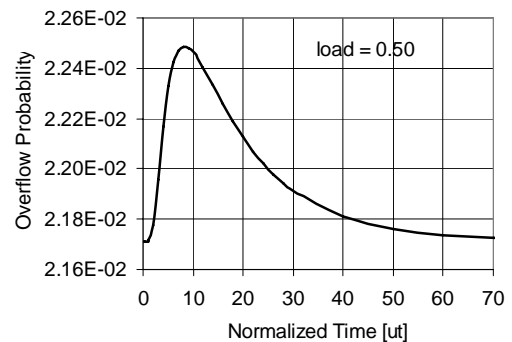Fig. 6 Overflow probability for low loads (ρ = 0.05) starting from the empty state (i=0) with B=3 and N=10

Fig.7 Overflow probability for medium load (ρ = 0.50) starting from the empty state (i=0) with B=3 and N=10

Fig.8 shows the results of the overflow probability for heavy loads (ρ = 0.95). Once again the trend is similar to those in the preceding figures. Initially the probability is low (0.742), then rising to a peak of 0.748 at μt = 10., and then settles finally at 0.743, for μt > 100. Here too it takes longer to reach steady state than for lower loads.

Fig.9 shows the results for the proportion of transients in the overflow probability for different loads, and starting from the empty state (i=0). Whereas the transient component eventually goes to zero, the relative magnitude is less than 1% after μt = 30, with the heavier loads experiencing the least variation. For smaller loads, the transient component is more pronounced and decays faster.
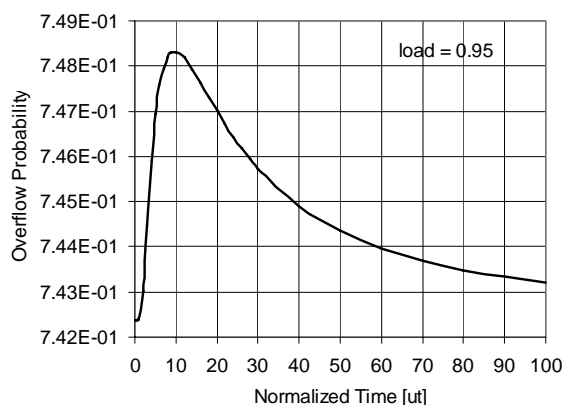
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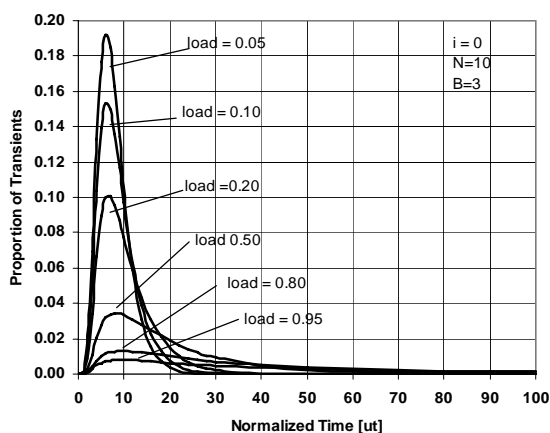Fig. 8 Overflow probability for heavy load (ρ = 0.95) starting from the empty state (i=0) with B=3 and N=10

Fig. 9 Proportion of transients in the overflow probability for different loads starting from the empty system (i=0) with B=3 and N=10

μt = 100 (i.e. 70 units of time later than for the empty initial state).

Fig. 10 Proportion of transients in the overflow probability for different loads starting from a non-empty state with 5 packets with B=3 and N=10

Fig. 11 Proportion of transients in the overflow probability for different loads starting from a non-empty state with 10 packets with B=3 and N=10
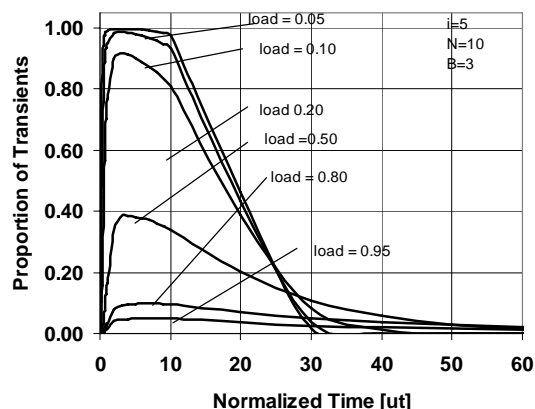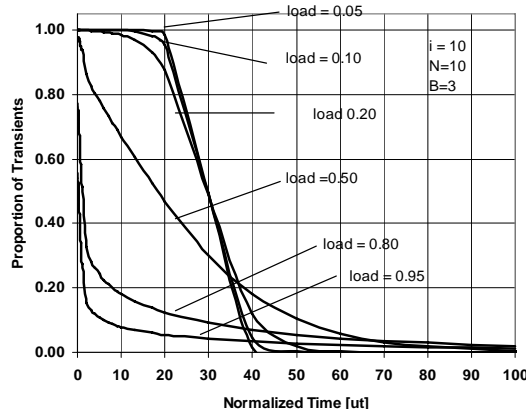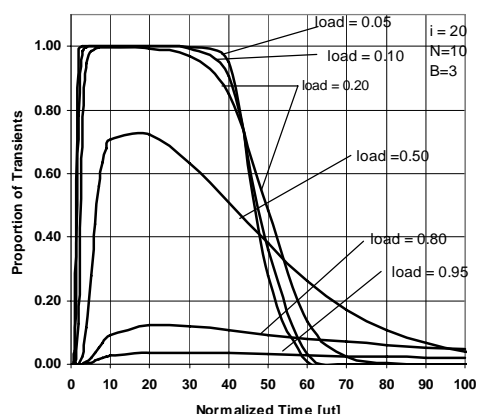

Fig. 12. Proportion of transients in the overflow probability for different loads starting from a non-empty state with 20 packets with B=3 and N=10

Fig.10 shows the results for the proportion of transients for different loads, and starting from a non-empty system (i=5). The similarities with the empty initial state are that the relative magnitude is less than 1% after μt = 30, with the heavier loads experiencing the least variation. The difference here is that the relative magnitude of the transient component is high (approaching unity) for low loads.

Fig.11 shows the results for the proportion of transients in the overflow probability for different loads, and starting from a non-empty system (i=10). The similarities with the preceding case are that the heavier loads experience the least variation, and the relative magnitude of the transient component is high (approaching unity) for low loads. The major difference here is that the 1% point is reached after μt = 50 (i.e. 20 units of time later).

Finally Fig.12 shows the results for the proportion of transients in the overflow probability for different loads, starting from a non-empty system (i=20). The similarities with the preceding case are that the heavier loads experience the least variation, and the relative magnitude of the transient component is high (approaching unity) for low loads. The major difference here is that the 1% point is reached after

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VI. CONCLUSION

In conclusion, the paper has addressed the problem of analyzing the transient behaviour of a queuing system with Poisson arrivals of fixed-size batches, and exponential service times. In the process, there emerged a family of polynomials $T_k^{(B)}(x)$ which are seen to generalize Chebyshev polynomials of the second kind $U_k(x)$. The similarities of these polynomials are evident in the descriptions, in the appearance of their graphs, and also in the generating functions.

The polynomials are used to obtain results for the occupancy probabilities and subsequently, blocking (or overflow) probabilities for the system. The transient probability of overflow (equivalently the blocking probability) is used to characterize the transient behaviour. It is found that for all loads the blocking probability has the same shape as a function of time, starting low, rising to a peak and decaying to the steady state level. For small loads the actual levels are low (e.g. 5×10^{-7} for a load of 0.05), and for heavy loads the actual level are high (e.g. 0.745 for a load of 0.95).

The pattern in the results is to be expected since a heavily loaded system is more likely to experience overflow than a lightly loaded one. What is new here is that use is made of the polynomials in showing the significance of the transient component in the assessment of the overflow or blocking.

The proportion of the transient component in this quantity is found to be high for low loads, and lower for heavy loads. The time for the transients to decay to below 1% is longer when the system starts with more packets than when it starts with fewer or no packets.

Finally, it is interesting that there is a relationship between the Bessel functions and Chebyshev polynomials, and it is being found here that the same system can be described by functions that generalize Bessel functions, and also in terms of polynomials that generalize Chebyshev polynomials. It remains to establish the parallel relationships between the functions and the polynomials.

REFERENCES

[1] H. P. Schwefel, L. Lipsky, M. Jobmann, "On the Necessity of Transient Performance Analysis in Telecommunication Networks," *17th International Teletraffic Congress* (ITC17), Salvador da Bahia, Brazil, September 24-28 2001

[2] B. van Holt, C. Blondia, "Approximated Transient Queue Length and Waiting Time Distribution via Steady State Analysis", *Stochastic Models* **21**, pp.725-744, 2005

[3] T. Hofkens, K. Spacy, C. Blondia, "Transient Analysis of the D-BMAP/G/1 Queue with an Applications to the Dimensioning of Video Playout Buffer for VBR Traffic", *Proceedings of Networking*, Athens Greece, 2004

[4] D. M. Lucantoni, G. L. Choudhury, W. Witt, "The Transient BMAP/PH/1 Queue", *Stochastic Models* **10**, pp.461-478, 1994

[5] W. Böhm. S. G. Mohanty, "Transient Analysis of Queues with Heterogeneous Arrivals", *Queuing Systems,* Vol.18, pp.27-45, 1994

[6] V. K. Oduol, "Transient Analysis of a Single-Server Queue with Batch Arrivals Using Modeling and Functions Akin to the Modified Bessel Functions" *International Journal of Applied Science, Engineering and Technology, Vol. 5, No.1, pp.34-39, April 2009*

[7] V. K. Oduol, C. Ardil, *"Transient Analysis of a Single-Server Queue with Fixed-Size Batch Arrivals", International Journal of Electronics, Communications and Computer Engineering, Vol. 1, No.1, pp.55-50, May 2009*

[8] G. L. Choudhury, D. M. Lucantoni, W. Witt, "Multidimensional Transform Inversion with Application to the Transient M/G/1 Queue", *Annals of Applied Probability*, **4**, 1994, pp.719-740.

[9] J. Abate, G. L. Choudhury, W. Whitt, "An Introduction to Numerical Transform Inversion and its Application to Probability Models" In: W. Grassman, (ed.) *Computational Probability*, pp. 257–323. Kluwer, Boston , 1999.

[10] G. N. Higginbottom, Performance Evaluation of Communication Networks, Artech House 1998

[11] I. S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products,* Alan Jeffrey and Daniel Zwillinger (eds.) Seventh edition, Academic Press, Feb. 2007

[12] M. Abramowitz, I. A. Stegun (eds.), "Chapter 22", *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York, Dover, 1965.

[13] P. K. Suetin,. "Chebyshev polynomials", in Hazewinkel, Michiel, *Encyclopaedia of Mathematics*, Kluwer Academic Publishers, (2001)

[14] *Wikipedia* "Chebyshev polynomials" Available at http://en.wikipedia.org/wiki/Chebyshev_polynomials

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