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Finite-time stability analysis of fractional-order with multi-state time delay

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Abstract—In this paper, the finite-time stabilization of a class of multi-state time delay of fractional-order system is proposed.First, we define finite-time stability with the fractional-order system.Second,by using Generalized Gronwall's approach and the methods of the inequality,we get some conditions of finite-time stability for the fractional system with multi-state delay.Finally,a numerical example is given to illustrate the result.

Keywords—Finite-time stabilization, Fractional-order system, Gronwall inequality.

I. INTRODUCTION

IN recent years, many studies focus on a lot in fractional order systems.They study aspects of fractional order systems.For instance, in [1],the author study the existence of solutions for fractional differential equations,and in[2], the author study existence and uniqueness of solutions for the linear time-delay differential equations of fractional order systerms.It comes to time-delay systems, time-delays are often present invarious engineering systems such as biological,economical systems,chemical processes. Time-delays are described by differential-difference equations which belong to a class of functional differential equations [3]. Stability analysis is one of the most important issues for control systems, although this problem has been investigated for time-delay systems over many years in [4]. Recently, for the first time, finite-time stability analysis of fractional time-delay systems is presented and reported on paper [5].And in [6], a stability test procedure is proposed for linear nonhomogeneous fractional order systems with a pure time delay using a recently obtained generalized Gronwall's inequality.Here, the finite-time stabilization of a class of multi-state time delay of fractional-order system using Gronwall's approach is proposed.The main contribution of this paper is to introduce multi-state time delay of fractional-order system,and when$\tau_i = 0$ [6]is the special circumstances of this paper.

II. FUNDAMENTALS OF FRACTIONAL DERIVATIVE

There are many ways to define the fractional integral and derivative, and three definitions are generally used in recent studies, they are Riemann-Liouville definition, Grünwald-Letnikov definition and Caputo definition. Given , Riemann-Liouville definition of q-th order fractional derivative operator $0 < q < 1$ is given by [7]

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-\tau)^{-q} f(\tau)\, d\tau \tag{1}$$

Where $\Gamma(\cdot)$ is the Gamma function generalizing factorial for non-integer arguments

$$\Gamma(q) = \int_0^{+\infty} e^{-t} t^{q-1} dt \tag{2}$$

The Grünwald-Letnikov fractional derivative definition is given by [8]

$$aD_t^q f(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{j=0}^{[\frac{-}{\;}]} (-1)^j \begin{pmatrix} q \\ j \end{pmatrix} f(t - jh) \tag{3}$$

where$[\cdot]$ is a flooring operator.
And Caputo definition:

$${}_0D_t^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(\;)}(\tau)}{(t-\tau)^{\;+1-}} d\tau, m-1 < q < m \\ \frac{d}{dt} f(t), \qquad q = m \end{cases} \tag{4}$$

A linear time-invariant function-order system can be represented in the following state-space form:

$$D^q x(t) = Ax(t) + Bu(t) \tag{5}$$

Where$D^q x(t)$ denotes the Riemann-Liouville fractional derivative of order $q \in R$, $x(t) \in R^n$, $u(t) \in R^m$, $A \in R^{n \times n}$, $B \in R^{n \times m}$, $q = (q_1, q_2, \cdots, q_n) \in R^n$

III. MULTI-STATE TIME DELAY OF FRACTIONAL-ORDER SYSTEM WITH INPUT DELAY

Consider the following fractional order systerm:

$$\begin{cases} D^q x(t) = A_0 x(t) + \sum_{i=1}^{n} A_i x(t - \tau_i) + B_0 u,(t)\, t \geq 0 \\ x(t) = \Psi_x(t), t \in [-\tau, 0] \end{cases} \tag{6}$$

where D^q denotes Riemann-Liouville derivative of order q, $0 < q < 1$, $\Psi_x(\cdot)$ is a given continuous function on $[-\tau, 0]$, $\tau = max(\tau_1, \tau_2, \cdots, \tau_n)$,and τ_i is a constant with $\tau_i > 0$.In Eq.(6),$x(t) \in R^n$is a state vector,$u(t) \in R^m$is a input control vector,A_0, A_i, B_0 are constant system matrices of appropriate dimensions,and the system is defined over time interval $J = [0, T]$,where T is a positive number ,$u(t)$ is a given continuous function on $[0, T]$.

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Let us denote by $C([a,b])$ the space of all continuous real functions defined on $[a,b]$ and by $C([a,b],R^n)$ the Banach space of continuous functions mapping the interval $[a,b]$ into R^n with the topology of uniform convergence.Let $C = C([-\tau,0],R^n)$,if $[a,b]=[-\tau,0]$,and designate the norm of an element $\|\Psi_x\|_C$ in C by

$$\|\Psi_x\|_C = \sup_{-\tau\le\theta\le 0}\|\Psi(\theta)\| \tag{7}$$

Before proceeding further,we will introduce the following some definition and lemmas which will be used in the next section.

Definition 3.1 The system given by homogeneous state equation (6) $(u(t)\equiv 0,\forall t)$, satisfying initial condition $x(t) = \Psi_x(t)$, $-\tau\le t\le 0$ is finite-time stable w.r.t.$\{\delta,\varepsilon,J\}$, if and only if:

$$\|\Psi_x\|_C < \delta \tag{8}$$

imply:

$$\|x(t)\| < \varepsilon, \forall t\in J \tag{9}$$

Where J denotes time interval $J=[0,T]$.

Definition 3.2 The system given by (6) satisfying initial condition $x(t) = \Psi_x(t)$, $-\tau\le t\le 0$ is finite-time stable w.r.t.$\{\delta,\varepsilon,q_u,J\}$, if and only if:

$$\|\Psi_x\|_C < \delta \tag{10}$$

and

$$\|u(t)\| < q_u, \forall t\in J \tag{11}$$

imply:

$$\|x(t)\| < \varepsilon, \forall t\in J \tag{12}$$

Where J denotes time interval $J=[0,T]$.
Let

$$f(t) = \int_0^t (t-s)^p\|x(l)\|dl, \forall t\in J, p>0 \tag{13}$$

we have the following definition.

Lemma 3.1 ([9] Generalized Gronwall Inequality) Suppose $x(t)$, $a(t)$ are nonnegative and local integrable on $0\le t<T$,some $T\le+\infty$,and $g(t)$is a nonnegative,nondecreasing continuous function defined on $0\le t<T$,$g(t)\le M$ =const,$q>0$,with

$$x(t)\le a(t) + g(t)\int_0^t (t-s)^{q-1}x(s)\,ds \tag{14}$$

on this interval.Then

$$\begin{aligned} x(t) &\le a(t) \\ &+\int_0^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(q))}{\Gamma(nq)}(t-s)^{nq-1}a(s)\right]ds \end{aligned} \tag{15}$$

where $0\le t<T$.

Lemma 3.2([9]) Under the hypothesis of Lemma 3.1, let $a(t)$ be a nondecreasing function on $[0,T)$ .Then holds:

$$x(t)\le a(t)E_q\left(g(t)\cdot\Gamma(q)\cdot t^q\right) \tag{16}$$

Where E_q is the Mittag-Leffler function defined by

$$E_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kq+1)} \tag{17}$$

IV. MAIN RESULTS

Theorem 4.1 The system given by (6) satisfying initial condition $x(t) = \Psi_x(t)$, $-\tau\le t\le 0$ is finite-time stable w.r.t.$\{\delta,\varepsilon,q_u,J\}$,if the following condition is satisfied:

$$\left[1+\frac{(n+1)\sigma t^q}{\Gamma(q+1)}+\frac{q_u\cdot b_0\cdot t^q}{\delta\Gamma(q+1)}\right]E_q\left((n+1)\sigma t^q\right) < \frac{\varepsilon}{\delta} \tag{18}$$

where $\sigma_{max}(\cdot)$ being the largest singular value of matrix $(\cdot)$ and

$$\begin{aligned} \sigma_1 &= \max_{1\le i\le n}\{\sigma_{\max}(A_i)\} \\ \sigma &= \max\{\sigma_{\max}(A_0)\,,\sigma_1\} \\ b_o &= \sigma_{\max}(B_0) \end{aligned} \tag{19}$$

Proof: In accordance with the property of the fractional order $0<q<1$, one can obtain a solution in the form of the equivalent Volterra integral equation:

$$\begin{aligned} x(t) &= x(0) + \tfrac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A_0x(s) \\ &+\sum_{i=1}^{n}A_ix(s-\tau_i) + B_0u(s)\,ds \end{aligned} \tag{20}$$

Applying the norm $\|\cdot\|$ on Eq.(20)and using appropriate property of the norm,it follows that

$$\begin{aligned} \|x(t)\| &\le \|x(0)\| + \tfrac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \\ &\quad\times\left\|A_0x(s)+\sum_{i=1}^{n}A_ix(s-\tau_i)+B_0u(s)\right\|ds \\ &\le \|\Psi_x\|_C + \tfrac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\|A_0\|\,\|x(s)\| \\ &\quad+\sum_{i=1}^{n}\|A_i\|\,\|x(s-\tau_i)\| + \|B_0\|\,\|u(s)\|)ds \\ &\le \|\Psi_x\|_C + \tfrac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\sigma(n+1) \\ &\quad\times\sup_{s-\tau\le t^*\le s}\|x(t^*)\| + \|\Psi_x\|_C + b_0q_u)ds \\ &\le \|\Psi_x\|_C \\ &\quad+\tfrac{\sigma(n+1)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\sup_{s-\tau\le t^*\le s}\|x(t^*)\|\,ds \\ &\quad+\tfrac{1}{\Gamma(q)}\left(\sigma(n+1)+\|\Psi_x\|_C + b_0q_u\right) \\ &\quad\times\int_0^t (t-s)^{q-1}ds \\ &= \left(1+\tfrac{\sigma(n+1)t}{\Gamma(q+1)}\right)\|\Psi_x\|_C + \tfrac{b_0q\ \ t}{\Gamma(q+1)} \\ &\quad+\tfrac{\sigma(n+1)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\sup_{s-\tau\le t^*\le s}\|x(t^*)\|ds \end{aligned} \tag{21}$$

let

$$\begin{aligned} a(t) &= \|\Psi_x\|_C\left[1+\tfrac{(n+1)\sigma\cdot t}{\Gamma(q+1)}\right] + \tfrac{q\ \ \cdot b_0\cdot t}{\Gamma(q+1)} \\ g(t) &= \tfrac{(n+1)\sigma}{\Gamma(q)} \end{aligned} \tag{22}$$

by(21),we have

$$\|x(t)\|\le a(t) + g(t)\int_0^t (t-s)^{q-1}\sup_{s-\tau\le t^*\le s}\|x(t^*)\|\,ds \tag{23}$$

Obviously,the right of the Eq.(23)is the nondecreasing continuous functions defined on $[0.T]$.We have

$$\begin{aligned}\sup_{t-\tau\leq t^*\leq t}\|x(t^*)\| &\leq a(t)\\ &+g(t)\int_0^t (t-s)^{q-1}\sup_{t-\tau\leq t^*\leq t}\|x(t^*)\|ds\end{aligned}\tag{24}$$

Now, one may apply generalized Gronwall inequality, here, obviously, it is easy to show:

$$\begin{aligned}\|x(t)\| &\leq a(t)\cdot E_q\left(g(t)\cdot\Gamma(q)\cdot t^q\right)\\ &\leq a(t)\cdot E_q\left((n+1)\sigma\cdot\Gamma(q)\cdot t^q\right)\end{aligned}\tag{25}$$

and

$$\begin{aligned}\|x(t)\| \leq &\left[\delta\left(1+\tfrac{(n+1)\sigma\cdot t}{\Gamma(q+1)}\right)+\tfrac{q\ \cdot b_0\cdot t}{\Gamma(q+1)}\right]\cdot\\ &\times E_q\left((n+1)\sigma\cdot\Gamma(q)\cdot t^q\right)\end{aligned}\tag{26}$$

Hence, using the basic condition of Theorem4.1, relation (18) yields:

$$\|x(t)\|<\varepsilon,\forall t\in J_0\tag{27}$$

This is a proof of the theorem.
When $u(t)=0$,we can get Theorem 4.2.

Theorem 4.2 The liner autonomous system given by (6) satisfying initial condition $x(t)=\Psi_x(t)$, $-\tau\leq t\leq 0$ is finite-time stable w.r.t.$\{\delta,\varepsilon,J\}$,$\forall t\in J$ if the following condition is satisfied:

$$\left(1+\frac{(n+1)\cdot\sigma\cdot t^q}{\Gamma(q+1)}\right)E_q\left((n+1)\sigma\cdot t^q\right)\leq\frac{\varepsilon}{\delta}\tag{28}$$

Proof: The proof immediately follows from the proof of Theorem 4.1 applying the same procedure taking into account Eqs.(8)and (28).

V. AN ILLUSTRATIVE EXAMPLE

Using a time-delay PD^q compensator on a linear system of equations with respect to the small perturbation$z(t)=y(t)-y_d(t)$,one can obtain:

$$\begin{aligned}\overset{\bullet}{z}(t)+\omega z(t) &= K_{p_1}z(t-\tau_1)+K_{D_1}\cdot\tfrac{dz^{(\ )}(t-\tau_1)}{dt}\\ &+K_{p_2}z(t-\tau_2)+K_{D_2}\cdot\tfrac{dz^{(\ )}(t-\tau_2)}{dt}+u(t)\end{aligned}\tag{29}$$

Where$q=\frac{1}{2}$,$\omega=2$,$K_{P_1}=3$,$K_{D_1}=4$,$K_{P_2}=0.1$,$K_{D_2}=0.2$, and $u(t)$is feed forward control,K_P ,K_D are gain matrix. Also,all initial values are zeros.introducing:

$$\begin{aligned}x_1(t) &= z_1(t)\\ x_2(t) &= \tfrac{d^{1\ 2}z(t)}{dt^{1\ 2}}\end{aligned}\tag{30}$$

and

$$D_t^q x_1(t)=D_t^{1/2}z_1(t)=x_2(t)\tag{31}$$

$$\begin{aligned}D_t^q x_2(t) &= D_t^{1/2}\left(D_t^{1/2}z(t)\right)=\dot{z}(t)\\ &= -2x_1(t)+3x_1(t-\tau_1)+4x_2(t-\tau_1)\\ &\quad+0.1x_1(t-\tau_2)+0.2x_2(t-\tau_2)+u(t)\end{aligned}\tag{32}$$

Or ,in condensed form,where $x(t)=(x_1,x_2)^T$,we can obtain this as:

$$\begin{aligned}D_t^{1/2}x(t) &= \begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ &+\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}x_1(t-\tau_1)\\ x_2(t-\tau_1)\end{bmatrix}\\ &+\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}0.1 & 0.2\end{bmatrix}\begin{bmatrix}x_1(t-\tau_2)\\ x_2(t-\tau_2)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)\end{aligned}\tag{33}$$

or

$$D_t^{1/2}x(t)=A_0x(t)+A_1(t-\tau_1)+A_2(t-\tau_2)+B_0u(t)\tag{34}$$

with the initial state of the function:

$$x(t)=\psi_x(t)=0,-\tau\leq t\leq 0\tag{35}$$

And now, we check the finite-time stability w.r.t

$$\{t_0=0,J=[0\ ,\ 10],\delta=0.1,\varepsilon=100,\tau_1=0.1,\tau_2=0.01,\ q_u=1\}\tag{36}$$

where $\Psi_x(t)=0$,$\forall t\in[-0.1,0]$.
From the initial data and Eqs.(33)and (6)one can obtain: $\|\psi_x(t)\|_C<0.1$, $\sigma_{max}(A_0)=2$, $\sigma_{max}(A_1)=5$, $\sigma_{max}(A_2)=\sqrt[2]{0.05}$,$b_0=1$
Then,we can obtain: $\sigma=5$.
Applying the condition of Theorem (4.1) we can get:

$$\begin{aligned}&\left[1+\tfrac{(2+1)\cdot 5\cdot T^{\ 0\ 5}}{\Gamma(0.5+1)}+\tfrac{1\cdot 1\cdot T^{\ 0\ 5}}{0.1\cdot\Gamma(0.5+1)}\right]\\ &\quad\cdot E_{0.5}\left((2+1)\cdot 5\cdot {T_e}^{0.5}\right)<\tfrac{100}{0.1}\end{aligned}\tag{37}$$

and then $T\approx 0.15$.
T_e being"estimated time" of finite time stability.

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