A note on potentially power-positive sign patterns

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Abstract—In this note, some properties of potentially power-positive sign patterns are established, and all the potentially power-positive sign patterns of order ≤ 3 are classified completely.

Keywords—Sign pattern; Potentially eventually positive sign pattern; Potentially power-positive sign pattern.

I. Introduction

In qualitative and combinatorial matrix theory, a methodology based on the signs of the elements of a matrix is very often useful in the study of some properties of matrices. A sign pattern is a matrix with entries in $\{+,-,0\}$. For a real matrix A, sgn(A) is the sign pattern whose entries are the signs of the corresponding entries of A. If $\mathcal{A}$ is an n-by-n sign pattern, the qualitative class of $\mathcal{A}$, denoted by $Q(\mathcal{A})$, is the set of all n-by-n real matrices A with sgn(A)= $\mathcal{A}$, and we call A is a realization of $\mathcal{A}$. A subpattern of $\mathcal{A}$ is an $n \times n$ sign pattern obtained from $\mathcal{A}$ by replacing some (possibly, none) nonzero entries of $\mathcal{A}$ with zeros. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is a superpattern of $\mathcal{B}$. A permutation pattern is simply a sign pattern matrix with exactly one entry in each row and column equal to $+$, and the remaining entries equal to 0. A product of the form $\mathcal{S}^T\mathcal{A}\mathcal{S}$, where $\mathcal{S}$ is a permutation pattern and $\mathcal{A}$ is a sign pattern matrix of the same order as $\mathcal{S}$, is called a permutation similarity. Two sign patterns $\mathcal{A}$ and $\mathcal{B}$ are equivalent if $\mathcal{A} = \mathcal{P}^T\mathcal{B}\mathcal{P}$, or $\mathcal{A} = \mathcal{P}^T\mathcal{B}^T\mathcal{P}$, where $\mathcal{P}$ is a permutation pattern. A pattern $\mathcal{A}$ is reducible if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T\mathcal{A}\mathcal{P} = \left(\begin{array}{cc} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{array} \right),$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is irreducible if it is not reducible.

For a sign pattern $\mathcal{A}$, we define the positive part of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha^+_{ij}]$ and the negative part of $\mathcal{A}$ to be $\mathcal{A}^- = [\alpha^-_{ij}]$, where $\alpha^+_{ij} = +$ if $\alpha_{ij} = +$, $\alpha^+_{ij} = 0$ if $\alpha_{ij} = 0$ or $-$, and $\alpha^-_{ij} = -$ if $\alpha_{ij} = -$, $\alpha^-_{ij} = 0$ if $\alpha_{ij} = 0$ or $+$.

Since graph theoretical methods are often useful in the study of sign patterns, we now introduce some graph theoretical concepts (see, for example, [1, 2]).

An n-by-n sign pattern $\mathcal{A}$ has signed digraph $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \cdots, n\}$ and a positive (negative) arc from i to j if and only if α_{ij} is positive (negative). A (directed) simple cycle of length k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_1)$ such that the vertices $i_1, \cdots, i_k$ are distinct. A digraph D is primitive if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. A sign pattern $\mathcal{A}$ is primitive if its signed digraph $\Gamma(\mathcal{A})$ is primitive. For a nonnegative sign pattern $\mathcal{A}$, the Frobenius Test for primitivity tells us that the signed digraph $\Gamma(\mathcal{A})$ is primitive if and only if for some (and hence for all) $A \in Q(\mathcal{A})$, there exists a nonnegative integer m such that $A^k > 0$ for all $k \geq m$.

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In order to state our results clearly, we need the following definitions.

Definition 1.1. [3] *An n-by-n real matrix A is eventually positive if there exists a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$.*

Definition 1.2. [4] *An n-by-n real matrix A is power-positive if there exists a positive integer k such that $A^k > 0$.*

Definition 1.3. [5, 6] *An n-by-n sign pattern $\mathcal{A}$ is potentially eventually positive (PEP) if there exists some real matrix $A \in Q(\mathcal{A})$ that is eventually positive.*

Definition 1.4. [7] *An n-by-n sign pattern $\mathcal{A}$ is potentially power-positive (PPP) if there exists some $A \in Q(\mathcal{A})$ that is power-positive.*

In [5], Ellison, Hogben, Tsatsomeros studied the sign patterns that require eventual positivity or require eventual nonnegativity. Sign patterns that allow eventual positivity have been studied in [6]. A characterization of potentially power-positive sign patterns was given in [7] by Catral, Hogben, Olesky and Driessche. It was shown that the sign pattern $\mathcal{A}$ is potentially power-positive if and only if $\mathcal{A}$ or $-\mathcal{A}$ is potentially eventually positive. However, for the characterization of potentially eventually positive sign patterns is still open, the characterization of potentially power-positive sign patterns is open and the classification of potentially power-positive sign patterns is also open.

In this paper, we address on the potentially power-positive sign patterns of order ≤ 3. This work is organized as follows: Some definitions and notations are given in Section 1. In Section 2, some properties of potentially power-positive sign patterns are discussed. In Section 3, the potentially power-positive sign patterns of order ≤ 3 are classified. Conclusions and open questions are given in Section 4.

II. Some properties of PPP sign patterns

We begin this section by stating some properties of power-positive matrices without any proof.

Lemma 2.1. *Let A be an n-by-n real matrix. Then the following statements are equivalent:*

(1) A is power-positive.

(2) $-A$ is power-positive.

(3) A^T is power-positive.

(4) P^TAP is power-positive, where P is a permutation matrix of the same order.

Lemma 2.2. (see [6, Theorem 2.1]) *Let $\mathcal{A}$ be an n-by-n sign pattern. If its signed digraph $\Gamma(\mathcal{A}^+)$ is primitive, then $\mathcal{A}$ is PEP.*

Next, we discuss some properties of PPP sign patters. Following [6], we use $[+]$ (respectively, $[-]$) to denote a sign pattern consisting entirely of positive (respectively, negative) entries.

Proposition 2.3. *If $\mathcal{A}$ is the checkerboard block sign pattern*

$$\begin{pmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with square diagonal blocks and there are at least two diagonal blocks, then $\mathcal{A}$ is not PPP.

Proof: Proposition 2.3 follows from the fact that the checkerboard block sign pattern $\mathcal{A}$ and its negation $-\mathcal{A}$ are not PEP.

Proposition 2.4. *If $\mathcal{A}$ is the block sign pattern,*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix},$$

with $\mathcal{A}_{12} = \mathcal{A}_{12}^{-}$, $\mathcal{A}_{21} = \mathcal{A}_{21}^{+}$, and $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ square, then $\mathcal{A}$ is not PPP. Moreover, if $\mathcal{A}_{12} = \mathcal{A}_{12}^{+}$, $\mathcal{A}_{21} = \mathcal{A}_{21}^{-}$ and $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ square, then $\mathcal{A}$ is also not PPP.

Proof: We first claim that the block pattern

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{pmatrix},$$

with $\mathcal{B}_{12} = \mathcal{B}_{12}^{+}$, $\mathcal{B}_{21} = \mathcal{B}_{21}^{-}$ and diagonal block $\mathcal{B}_{11}$ and $\mathcal{B}_{22}$ square, is not PEP. By a way of contradiction suppose $\mathcal{B}$ is PEP. Then there exists some real matrix $B \in Q(\mathcal{B})$ such that B is eventually positive. By Theorem 2.2 in [3], B^T is eventually positive. It follows that sign pattern $\mathcal{B}^T =$sgn(B^T) is PEP. Then Theorem 5.2 in [6] is contradicted. Hence $\mathcal{B}$ is not PEP and $-\mathcal{A}$ is also not PEP. Therefore, $\mathcal{A}$ is not PPP.

Theorem 2.5. *Let $\mathcal{A}$ be an n-by-n sign pattern. If $\mathcal{A}$ is PPP, then every superpattern of $\mathcal{A}$ is PPP. If $\mathcal{A}$ is not PPP, then every subpattern is not PPP.*

Proof: If $\mathcal{A}$ is PPP, then eithor $\mathcal{A}$ or $-\mathcal{A}$ is PEP. If $\mathcal{A}$ is PEP, then every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ is PEP and PPP. If $-\mathcal{A}$ is PEP, then every superpattern $(\widehat{-\mathcal{A}})$ of $(-\mathcal{A})$ is PEP and $-\hat{\mathcal{A}}$, the negation of every superpattern of $\mathcal{A}$, is PEP. Hence, every superpattern $\hat{\mathcal{A}}$ is PPP. For the second statement, if $\mathcal{A}$ is not PPP but has a subpattern that is PPP, then it is a contradiction to the first statement.

Proposition 2.6. *Let $\mathcal{A}$ be an n-by-n sign pattern. Then the following statements are equivalent:*

(1) $\mathcal{A}$ is PPP.

(2) $-\mathcal{A}$ is PPP.

(3) $\mathcal{A}^T$ is PPP.

(4) $\mathcal{P}^T\mathcal{A}\mathcal{P}$ is PPP, where $\mathcal{P}$ is an n-by-n permutation pattern.

Proof: Theorem 2.7 follows directly from Lemma 2.1.

The following conditions are necessary for an n-by-n sign pattern to be potentially power-positive. We state them without proof.

Proposition 2.8. *Let $\mathcal{A}$ be an n-by-n sign pattern. If $\mathcal{A}$ is PPP, then the following hold:*

(1) $\mathcal{A}$ is irreducible.

(2) Every row of $\mathcal{A}$ has at least one nonzero.

(3) Every column of $\mathcal{A}$ has at least one nonzero.

(4) The minimum number of nonzero entries of $\mathcal{A}$ is $n + 1$.

III. PPP SIGN PATTERS OF ORDER ≤ 3

In this section, we classify the n-by-n PPP sign patterns of order ≤ 3. To state clearly, we use the notation ? to denote one of $0, +, -$, $\ominus$ to denote one of $0, -$, $\oplus$ to denote one of $0, +$.

Proposition 3.1. *The 1-by-1 sign pattern $\mathcal{A}$ is PPP if and only if $\mathcal{A}$ is either $[+]$ or $[-]$, and the 2-by-2 sign pattern $\mathcal{A}$ is PPP if and only if $\mathcal{A}$ is equivalent to either $\begin{pmatrix} + & + \\ + & ? \end{pmatrix}$ or its negation, i.e., $\begin{pmatrix} - & - \\ - & ? \end{pmatrix}$.*

Proof: The first statement is obvious. The second statement follows from the fact that the 2-by-2 sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{A}$ is equivalent to $\begin{pmatrix} + & + \\ + & ? \end{pmatrix}$.

Theorem 3.2. *Let $\mathcal{A}$ be a 3-by-3 sign pattern such that $\Gamma(\mathcal{A}^+)$ and $\Gamma(\mathcal{A}^-)$ are not primitive. Then $\mathcal{A}$ is PPP if and only if $\mathcal{A}$ is equivalent to*

$$\mathcal{B} = \begin{pmatrix} + & - & 0 \\ + & \oplus & - \\ - & + & + \end{pmatrix},$$

or its negation

$$\begin{pmatrix} - & + & 0 \\ - & \ominus & + \\ + & - & - \end{pmatrix}.$$

Proof: Sufficiency: if $\mathcal{A}$ is equivalent to $\mathcal{B}$ (respectively, $-\mathcal{B}$), then $\mathcal{A}$ (respectively, $-\mathcal{A}$) is a superpattern of sign pattern sgn(B) (respectively, sgn$(-B)$), where

$$B = \begin{pmatrix} 1.3 & -0.3 & 0 \\ 1.3 & 0 & -0.3 \\ -0.31 & 0.3 & 1.01 \end{pmatrix}.$$

It follows that $\mathcal{A}$ (respectively, $-\mathcal{A}$) is PEP by Example 2.2 in [6]. Therefore, $\mathcal{A}$ is PPP.

Necessity: if $\mathcal{A}$ is PPP, then either $\mathcal{A}$ or $-\mathcal{A}$ is PEP. If $\mathcal{A}$ is PEP, then $\Gamma(\mathcal{A}^+)$ is primitive or $\mathcal{A}$ is equivalent to $\mathcal{B}^*$ by Theorem 6.4 in [6], where

$$\mathcal{B}^* = \begin{pmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{pmatrix}.$$

As $\Gamma(\mathcal{A}^-)$ is not primitive, the (1, 3)-th entry of $\mathcal{B}^*$ must be 0, and the (2, 2)-th entry of $\mathcal{B}^*$ must be $-$ or 0. It shows that $\mathcal{A}$ is equivalent to $\mathcal{B}$. If $-\mathcal{A}$ is PEP, then $\Gamma((-\mathcal{A})^+)$ is primitive or $\mathcal{A}$ is equivalent to $-\mathcal{B}^*$. As $\Gamma((-\mathcal{A})^+)$ is primitive if and only if $\Gamma(\mathcal{A}^-)$ is primitive, it follows that $\mathcal{A}$ is equivalent to$-\mathcal{B}^*$. By a similar discussion, we have $\mathcal{A}$ is equivalent to

$$\begin{pmatrix} - & + & 0 \\ - & \ominus & + \\ + & - & - \end{pmatrix}.$$

The following result follows readily from Theorem 3.2 and characterizes the potentially power-positive 3-by-3 sign patterns completely.

Corollary 3.3. *Let $\mathcal{A}$ be a 3-by-3 sign pattern. Then $\mathcal{A}$ is potentially power-positive if and only if one of the following conditions hold: (1) $\Gamma(\mathcal{A}^+)$ is primitive.*

(2) $\Gamma(\mathcal{A}^-)$ is primitive.

(3) $\mathcal{A}$ is equivalent to $\mathcal{B} = \begin{pmatrix} + & - & 0 \\ + & \oplus & - \\ - & + & + \end{pmatrix}$.

(4) $\mathcal{A}$ is equivalent to $\begin{pmatrix} - & + & 0 \\ - & \ominus & + \\ + & - & - \end{pmatrix}$.

IV. Concluding remarks

We have extended some sufficient conditions for PEP sign pattern to PPP sign pattern. Some properties of PPP sign patterns are established. Finally, we classified the PPP sign patterns of order ≤ 3. However, identification of the sufficient and necessary conditions for an n-by-n sign pattern ($n \geq 4$) to be PEP or PPP remain still open.

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