Buckling Optimization of Radially-Graded, Thin-Walled, Long Cylinders under External Pressure

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***Abstract*—**This paper presents a generalized formulation for the problem of buckling optimization of anisotropic, radially graded, thin-walled, long cylinders subject to external hydrostatic pressure. The main structure to be analyzed is built of multi-angle fibrous laminated composite lay-ups having different volume fractions of the constituent materials within the individual plies. This yield to a piecewise grading of the material in the radial direction; that is the physical and mechanical properties of the composite material are allowed to vary radially. The objective function is measured by maximizing the critical buckling pressure while preserving the total structural mass at a constant value equals to that of a baseline reference design. In the selection of the significant optimization variables, the fiber volume fractions adjoin the standard design variables including fiber orientation angles and ply thicknesses. The mathematical formulation employs the classical lamination theory, where an analytical solution that accounts for the effective axial and flexural stiffness separately as well as the inclusion of the coupling stiffness terms is presented. The proposed model deals with dimensionless quantities in order to be valid for thin shells having arbitrary thickness-to-radius ratios. The critical buckling pressure level curves augmented with the mass equality constraint are given for several types of cylinders showing the functional dependence of the constrained objective function on the selected design variables. It was shown that material grading can have significant contribution to the whole optimization process in achieving the required structural designs with enhanced stability limits.

***Keywords*—**Buckling instability, structural optimization, functionally graded material, laminated cylindrical shells, external hydrostatic pressure.

I. Introduction

STRUCTURAL applications of composite materials are increasing in several areas where high stiffness/weight ratio and long fatigue life are most beneficial [1]. A common application is the design of composite cylindrical shells under the action of external hydrostatic pressure, which might cause collapse by buckling instability [2], [3]. Examples are the underground and underwater pipelines, rocket motor casing, boiler tubes subjected to external steam pressure, and reinforced submarine structures. Previous numerical and experimental studies have shown that failure due to structural buckling is a major risk factor for thin laminated cylindrical shells. Anastasiadis and Simitses [2] studied the buckling of long laminated cylindrical shells under external radial pressure using higher order deformation theory. Their formulation, however, was restricted to symmetric lay-ups with respect to the mid-surface, to eliminate the coupling terms, as well as constant-directional pressure. More conservative results for a true fluid pressure were given by Rasheed and Yousif [3], [4] who applied standard energy formulation to derive the kinematics and equilibrium equations and the classical lamination theory to express the needed constitutive equations. Another refined treatment of the inplane buckling of rings was given by Hodges [5]. Formulation was based on a non-linear theory for stretching and bending of anisotropic beams having constant initial curvature in their plane of symmetry with the only restriction of small strain in the prebuckling state.

Recently, the incorporation of material grading in which the physical and mechanical properties vary spatially can play an important role in the design optimization of a variety of structural systems [6]. *FGM*s have been comprehensively researched, and are almost a commercial reality offering great promise in several applications. The basic knowledge on the use of *FGMs* and their wide applications can be found in [7]. Considering their applications in composite structures, Chen and Gibson [8] performed experimental and theoretical analyses to determine the in-plane fiber distribution in unidirectional reinforced composites. They considered distributions represented by polynomial functions, and applied Galerkin's method to calculate the required coefficients from the resulting algebraic equations. Chi and Chung [9] studied the mechanical behavior of *FGM* plates under transverse loading, where a constant Poisson's ratio and variable moduli of elasticity throughout the plate thickness was assumed. The volume fraction of the constituents materials were defined by simple power-laws, and closed form solutions using Fourier series were given for the case of simply-supported plates. Another work [10] considered buckling of simply supported three-layer circular cylindrical shell under axial compressive load. Classical shell theory was implemented under the assumption of very small thickness/radius and very large length/radius ratios. A recent paper by Batra and Iaccarino [11] dealt with radial deformation of *FGM* cylinders that loaded by hydrostatic pressures from the inner and outer surfaces. Closed-form solutions were given for axisymmetric plane strain of isotropic and incompressible second-order elastic material with moduli varying only in the radial direction. In the field of structural optimization, several papers appeared on the topic of buckling and stability optimization. Maalawi [12] presented a model for buckling optimization of elastic columns under mass equality constraint. He showed that the use of piecewise models in structural optimization gives excellent results and can be

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promising for similar applications. Another work by Maalawi and El-Chazly [13] dealt with both stability and dynamic optimization of multi-element beam type-structures. They formulated the associated optimization problems in a standard mathematical programming solved by the interior penalty function technique. In the field of aeroelasticity, Librescu and Maalawi [6] introduced the underlying concepts of using material grading in optimizing subsonic wings against torsional instability. They developed exact mathematical models allowing the material physical and mechanical properties to change in the wing spanwise direction, where both continuous and piecewise structural models were successfully implemented. For fibrous laminated composite structures, the optimization of ply angles and thicknesses could allow the properties of the laminate to be tailored to a specific application. Chattopadhyay and Ferreira [14] performed a study to investigate the maximum buckling load of a cylinder subject to ply stress constraints using material and geometric design variables. A closed form shell equation was utilized for the buckling load calculation.

Little may be found in the literature that deals with buckling optimization of *FGM* cylinders under external hydrostatic pressure. The aim of the present study is, therefore, to incorporate the effect of changing the fiber volume fraction in each lamina aiming at the achievement of enhanced stability limits of such shell-type structures. Based on the mathematical concepts developed in a recent paper by the author [15], a useful optimization tool has been built for designing efficient configurations with improved buckling stability. This allows the search for optimal volume fractions that maximize the buckling pressure without violating the imposed mass constraint and manufacturing restrictions as well. Actually, substantial improvement in the overall stability level has been attained showing the usefulness of the proposed optimization model in arriving at the needed optimum designs for a variety of thin-walled anisotropic long cylinders having arbitrary thickness/radius ratio.

II. Constitutive Relationships

The structural model used to represent the composite laminated shell-type structures under study is schematically shown in Fig. 1. The 1, 2 and 3 are the principal directions of an orthotropic lamina, defined as follows:

- Direction (1): Principal fiber direction, also called fiber longitudinal direction.
- Direction (2): In-plane direction perpendicular to fibers, transversal direction.
- Direction (3): Out-of-plane direction perpendicular to fibers; normal direction.

Table I gives the mathematical formulas for determining the different elastic moduli for known type, properties and volume fractions of the fiber and matrix materials. The factor ξ is called the reinforcing efficiency and can be determined experimentally for specified types of fiber and matrix materials. Experimental results fall within a band of $1<\xi<2$. Usually, ξ is taken as 100% for theoretical analysis procedures, especially in case of glass and carbon composites.

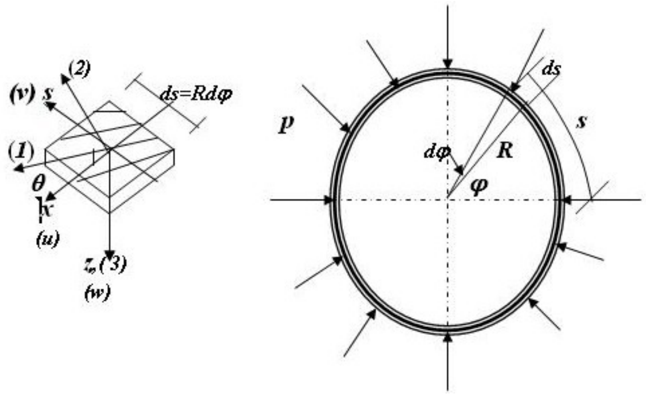


Fig. 1 Laminated composite ring/cylindrical shell under external pressure

(***u*** displacement in the axial direction ***x***, ***v*** in the tangential direction ***s***, ***w*** in the radial direction ***z***).

TABLE I
Composite Properties Formulas [16]

Elastic Propert y	Mathematical Formula*
E_{11}	$E_m V_m + E_{1f} V_f$
E_{22}	$E_m (1+\xi\eta V_f)/(1-\eta V_f)$; $\eta=(E_{2f}-E_m)/(E_{2f}+\xi E_m)$
G_{12}	$G_m (1+\xi\eta V_f)/(1-\eta V_f)$; $\eta=(G_{12f}-G_m)/(G_{12f}+\xi G_m)$
ν_{12}	$\nu_m V_m + \nu_{12f} V_f$

*Subscripts *"m"* and *"f"* refer to properties of matrix and fiber materials, respectively. Assuming no voids are present, then $V_m+V_f=1$.

For a generally orthotropic material, the stress-strain relations are:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{ss}\\ \tau_{xs}\end{bmatrix}=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{ss}\\ \gamma_{xs}\end{bmatrix} \tag{1}$$

The elements of the *K-th* lamina stiffness matrix, $[\overline{Q}]$, depend on the elastic properties and fiber orientation angle. Their general mathematical expressions are given in appendix (A). Using Kirchoff plate theory [18], the displacements of a material point distance z from the middle surface are:

$$\begin{aligned} u(x,s,z) &= u_o(x,s) - z\,\frac{\partial w_o}{\partial x} \\ v(x,s,z) &= v_o(x,s) - z\,\left(\frac{\partial w_o}{\partial s} - \frac{v_o}{R}\right), \quad (s \cong R\varphi,\ \frac{z}{R} << 1) \\ w(x,s,z) &= w_o(x,s) \end{aligned} \tag{2}$$

where $u_o(x,s)$, $v_o(x,s)$ and $w_o(x,s)$ are the displacements of a generic point *(x, s)* on the shell middle surface *(z=0)* in *x, s* and z directions, respectively. The strain-displacement relations in terms of the middle surface strains and shell curvatures are given in the following [18]:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{ss} \\ \gamma_{xs} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^o_{xx} \\ \varepsilon^o_{ss} \\ \gamma^o_{xs} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx} \\ \kappa_{ss} \\ \kappa_{xs} \end{Bmatrix} \quad (3)$$

The middle surface strains and curvatures are:

$$\begin{Bmatrix} \varepsilon^o_{xx} \\ \varepsilon^o_{ss} \\ \gamma^o_{xs} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial s} + \frac{w_o}{R} + \frac{1}{2}(\frac{\partial w_o}{\partial s} - \frac{v_o}{R})^2 \\ \frac{\partial u_o}{\partial s} + \frac{\partial v_o}{\partial x} \end{Bmatrix} \quad (4a)$$

$$\begin{Bmatrix} \kappa_{xx} \\ \kappa_{ss} \\ \kappa_{xs} \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial^2 w_o}{\partial x^2} \\ \frac{\partial}{\partial s}(\frac{\partial w_o}{\partial s} - \frac{v_o}{R}) \\ 2\frac{\partial}{\partial x}(\frac{\partial w_o}{\partial s} - \frac{v_o}{R}) \end{Bmatrix} \quad (4b)$$

The resultant forces and moments per unit length applied at the middle surface are defined by the integrals:

<u>Forces</u>: $$\begin{Bmatrix} N_x \\ N_s \\ N_{xs} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{ss} \\ \tau_{xs} \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{ss} \\ \tau_{xs} \end{Bmatrix} dz \quad (5a)$$

<u>Moments</u>: $$\begin{Bmatrix} M_x \\ M_s \\ M_{xs} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{ss} \\ \tau_{xs} \end{Bmatrix} z dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{ss} \\ \tau_{xs} \end{Bmatrix} z dz \quad (5b)$$

where h is the total shell thickness and z_{k-1} and z_k , $k=1,2,\ldots n$ are the coordinates of the *k-th* lamina boundaries measured from the middle surface. Substituting for the stresses in terms of strains as defined in (1) and (3), we get:

$$\begin{Bmatrix} N_x \\ N_s \\ N_{xs} \\ M_x \\ M_s \\ M_{xs} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^o_{xx} \\ \varepsilon^o_{ss} \\ \gamma^o_{xs} \\ \kappa_{xx} \\ \kappa_{ss} \\ \kappa_{xs} \end{Bmatrix} \quad (6)$$

where the extensional stiffness's are: $A_{ij} = h \sum_{k=1}^{n} (\overline{Q}_{ij})_k (\hat{z}_k - \hat{z}_{k-1})$,

the bending-extensional stiffness's: $B_{ij} = \frac{h^2}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (\hat{z}^2_k - \hat{z}^2_{k-1})$

and the bending stiffness's: $D_{ij} = \frac{h^3}{3} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (\hat{z}^3_k - \hat{z}^3_{k-1})$

$\hat{z}_k = z_k / h$ is a dimensionless coordinate, and $\hat{h}_k = \hat{z}_k - \hat{z}_{k-1}$ is the dimensionless thickness of the *kth* lamina.

III. Analytical Buckling Model

In order to restrict the time of calculation to acceptable values for the developed optimization tool, the analytical formulation shall be based on the derivations given in [4] and [15], which are based on the assumption of small hoop strain and rotation of circumferential elements. Such an approach provides good sensitivity to lamination parameters, and allowing the search for the needed optimal stacking sequences and volume fractions, which maximizes the buckling pressure in a reasonable computational time. The governing differential equations of anisotropic long cylinders subjected to external pressure are [18]:

$$\begin{aligned} &M'_{ss} + R(N'_{ss} - \beta N_{ss}) = \beta \, p R^2 \\ &M''_{ss} - R[N_{ss} + (\beta N_{ss})' + p(w_o + v'_o)] = p R^2 \end{aligned} \quad (7)$$

where the prime denotes differentiation with respect to angular position φ, and $\beta = (v_o - w'_o)/R$. Two possible solutions for (7) can be obtained; one for the pre-buckled state and the other termed as the bifurcation solution obtained by perturbing the displacements about the pre-buckling solution. For laminated composite long cylindrical shells the only significant strain components are the hoop strain (ε^o_{ss}) and the circumferential curvature (κ_{ss}) of the mid-surface. The out-of-plane displacements are restrained (that is: $\varepsilon^o_{xx} = \gamma^o_{xs} = \kappa_{xs} = 0$). Therefore, (6) can be reduced to:

$$\begin{Bmatrix} N_{ss} \\ M_{ss} \end{Bmatrix} = \begin{bmatrix} A_{22} & B_{22} \\ B_{22} & D_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon^o_{ss} \\ \kappa_{ss} \end{Bmatrix} \quad (8)$$

The final closed form solution for the critical buckling pressure is given by the following mathematical expression [4], [15]:

$$p_{cr} = 3\left[\frac{D_{ani}}{R^3}\right]\left[\frac{1-(\psi^2/\alpha)}{1+\alpha+2\psi}\right], \ \psi = (\frac{1}{R})(\frac{B_{ani}}{A_{ani}}), \alpha = (\frac{1}{R^2})(\frac{D_{ani}}{A_{ani}}) \quad (9)$$

It is to be noticed here that the formula given in (9) is only valid for thin cylinders with thickness-to-radius ratio $(h/R) \leq 0.1$. In case of thin orthotropic cylinders with fibers parallel to x-axis and $\psi=0$ and $\alpha<<1$, (9) reduces to:

$$p_{cr} = (h/R)^3 E_{22} / 4(1 - \nu_{12}\nu_{21}) \quad (10)$$

where E_{22} is the hoop modulus, ν_{12} Poisson's ratio for axial load and $\nu_{21}=\nu_{12}E_{22}/E_{11}$. In cases with fibers perpendicular to the shell axis, E_{22} should be replaced by E_{11}.

IV. Optimization Problem Statement

The associated optimization problem shall seek maximization of the critical buckling pressure p_{cr} (i.e. minimization of $-p_{cr}$) while maintaining the total structural mass constant at a value equals to that of a reference baseline design. Optimization variables include the fiber volume fraction (V_{fk}), thickness (h_k) and fiber orientation angle (θ_k) of the individual *k-th* ply, $k=1, 2,\ldots..n$ (total number of plies). Side constraints are always imposed on the design variables for geometrical, manufacturing or logical reasons to avoid having unrealistic odd shaped optimum designs.

A. Definition of the Baseline Design

An essential phase in formulating an optimization problem is to appropriately define a baseline design to which the resulting optimal designs can be compared. The baseline design has been selected to be a unidirectional, orthotropic, single layer cylinder with the fibers parallel to the shell axis *x and* with equal volume fractions of fiber and matrix materials, i.e. $V_f=V_m= 50\%$. Optimized shell designs shall have the same total structural mass, properties of the matrix and fiber materials, mean radius R and total shell thickness h of the baseline design. Therefore, the preassigned parameters, which are not subject to change in the optimization process, ought to be the type of material of construction, mean radius and total thickness of the shell.

B. Optimization Model

Coupling the analytical buckling shell model to a standard nonlinear mathematical programming procedure can perform the search for the required optimized lamination. The design variable vector, $\vec{x}_d$, which is subject to change in the optimization process, is defined as:

$$\vec{x}_d = (V_{fk}, \hat{h}_k, \theta_k)_{k=1,2,\ldots n} \qquad (11)$$

where the dimensionless thickness of the *k-th* lamina is defined by $\hat{h}_k = h_k / h$. The total structural mass M is kept equal to the baseline design mass M_0, so the dimensionless mass $\hat{M} = M / M_0$ equals 1. Since the fiber volume fraction of the baseline design V_{f0} equals 50%, a feasible design must satisfy the constraint equation $\sum_{k=1}^{n} V_{fk}\hat{h}_k = 0.5$, where V_{fk} is the fiber volume fraction of the k-th lamina. Therefore, the buckling optimization problem considered herein might be cast in the following standard mathematical programming form:

$$\begin{aligned}
&\text{Minimize} && -\hat{p}_{cr} \\
&\text{subject to} && \sum_{k=1}^{N} V_{fk}\hat{h}_k = 0.5 \\
& && \sum_{k=1}^{n} \hat{h}_k = 1 \\
& && V_L \le V_{fk} \le V_U \\
& && h_L \le \hat{h}_k \le h_U, \\
& && \theta_L \le \theta_k \le \theta_U \quad k=1, 2, \ldots.n
\end{aligned} \qquad (12)$$

where $\hat{p}_{cr} = p_{cr} / p_{cro}$ is the dimensionless critical buckling pressure and the subscripts "L" an "U" denote the lower and upper bounds imposed on the various design variables. In a real-world manufacturing process, the filament-winding angles θ_k must be chosen from a limited range of allowable lower (θ_L) and upper (θ_U) values according to technology references. This optimization problem may be thought as a search in a $3n$-dimensional space for a point corresponding to the minimum value of the objective function and such that it lie within the region bounded by subspaces representing the constraint functions [19]. The *MATLAB Optimization Toolbox* [20] offers routines named *"fmincon"* and *"fminsearch"* implementing both constrained and unconstrained formulations. The *MATLAB* facilities are invoked for interacting to the routines, which calculates the required numerical values of the original objective function and constraints. The two equality constraints in (12) can be used to discard any two variables of the whole set of design variables defined in (11), reducing the dimensionality of the optimization problem to $(3n-2)$.

V. Results and Discussions

The given approach outlined before shall be applied to several cases of study of thin-walled, anisotropic, radially graded, long cylinders subjected to external hydrostatic pressure. Table II gives the different properties of the chosen composite materials, which have favorable characteristics and are desirable for the manufacturing of cylindrical shell-type structures [4]. Appropriate values for the orthotropic properties and critical buckling pressure (p_{cro}) of the baseline design are given in Table III.

TABLE II
Composite Material Properties [1]

Property*	Fiber			Matrix	
	E-glass	S-glass	Carbon (AS-4)	Epoxy (3501-6)	Vinyl-ester
Young's moduli (GPa)	E_{1f} : 73	86	235	E_m : 4.30	3.50
	E_{2f} : 73	86	15		
Shear moduli (GPa)	G_{12f} : 30	35	27	G_m : 1.60	1.30
Poisson's ratio	ν_{12f} : 0.23	0.23	0.20	ν_m : 0.35	0.35
Mass density (g/cm^3)	ρ_f : 2.54	2.49	1.81	ρ_m : 1.27	1.15

The first case study to be examined herein is a long thin-walled cylindrical shell fabricated from E-glass/epoxy composites with the lay-up made of only two plies $(n=2)$ having fibers parallel to the x-axis (i.e. $\theta_1=\theta_2=0$). Considering the case with no side inequality constraints imposed on the design variables, Fig. 2 shows the developed $\hat{p}_{cr}$-level curves, augmented with the mass equality constraint, in $(V_{f1}$-$\hat{h}_1)$ design space.

TABLE III
BUCKLING PRESSURE OF THE BASELINE DESIGN

Material Type	Orthotropic mechanical properties $(GPa)^*$				p_{cro} x $(h/R)^3$ "GPa"
	E_{11}	E_{22}	G_{12}	ν_{12}	
E-Glass/Epoxy	38.65	11.18	4.21	0.29	2.865
S-Glass /Epoxy	45.15	11.404	4.285	0.29	2.913
Carbon/Epoxy	119.65	7.60	4.16	0.275	1.909
E-Glass/VinylEster	38.25	9.325	3.50	0.29	2.379

* Volume fractions: $V_f = V_m = 50\%$. Reinforcing efficiency factor $\xi = 100\%$.

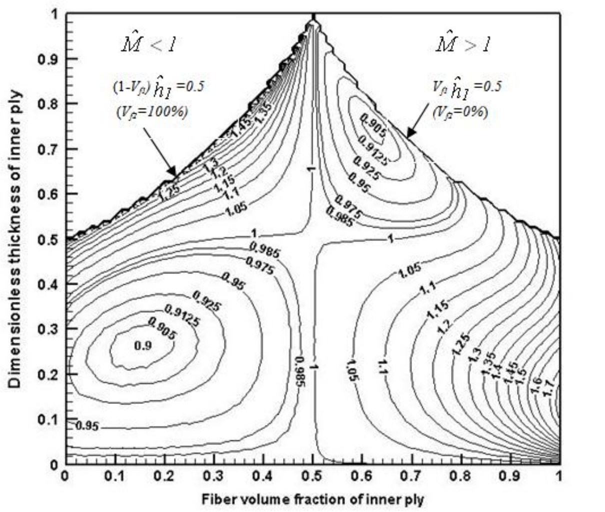


Fig. 2 The optimum tent-like design space containing $\hat{p}_{cr}$-isomerits augmented with the mass equality constraint $\hat{M} = 1.0$. Case of two-layer, E-glass/epoxy cylinder with fibers parallel to cylinder axis ($\theta_1=\theta_2=0$).

It is seen that such a constrained objective function is well behaved in the selected design space having the shape of a tent with its ceiling formed by two curved lines, above which the mass equality constraint is violated. Their zigzagged pattern is due to the obliged turning of many contours, which are not allowed to penetrate the tent's ceiling and violate the mass equality constraint. The curve to the left represent a 100% fiber volume fraction of the outer ply, V_{f2}, while the other curve to the right represents zero volume fraction, that is $V_{f2}=0\%$. Two local minima with $\hat{p}_{cr}$ near a value of 0.90 can be observed: one to the lower left zone near the design point $(V_{fk}, \hat{h}_k)_{k=1,2}$ = (0.15, 0.25), (0.6165, 0.75) while the other lies at the upper right zone close to the point (0.625, 0.745), (0.135, 0.255). This represents degradation in the stability level by about 10.6% below the baseline value. On the other hand, the unconstrained absolute optimum value of the dimensionless critical buckling pressure was found to be 1.7874 at the design point (1.0, 0.145), (0.415, 0.855). A more realistic optimum design has been obtained by imposing the side constraints: $0.25 \leq V_{fk} \leq 0.75,\ k=1,\ 2$. The attained solution is $(\hat{p}_{cr})_{max}$ =1.2105 at the design point (0.75, 0.215), (0.4315, 0.785), showing that good shell designs with higher stability level ought to have a thinner inner layer with higher fiber volume fraction and a thicker outer layer with less volume fraction.

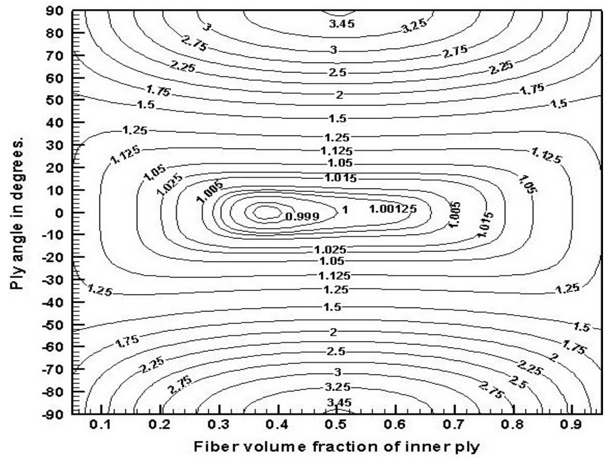


Fig. 3 $\hat{p}_{cr}$ - Isomerits in $(V_{f1}$-$\theta)$ design space under mass equality constraint. Case of long cylinder constructed from two balanced, E-glass/epoxy layers.

To investigate the effect of the ply angle, another case of study has been considered for a cylinder constructed from two balanced plies $(\pm\,\theta)$ with equal thicknesses and same material properties of E-glass/epoxy composites. This type of stacking sequence is widely used in filament wound circular shells since such a manufacturing process inherently dictates adjacent $(\pm\,\theta)$ layers. Fig. 3 shows the developed isomerits in the $(V_{f1}$-$\theta)$ design space which are well behaved, monotonic and symmetric about the horizontal line of zero ply angles. A local minimum can be observed near the design point (V_{f1},θ) = $(0.375,\ 0.0)$ with $\hat{p}_{cr} = 0.9985$, indicating a degradation in the stability level below the baseline design. It is also seen that the absolute maximum occurs at the design points $(V_{f1},\theta)=(0.5,\pm 90^o)$ *with* $(\hat{p}_{cr})_{max}=3.45766$, which means that the dimensional critical pressure, $p_{cr}=3.45766x2.865=9.906\ x(h/R)^3\ GPa$. Fig. 4 depicts the final global optimum designs of cylinders constructed from adjacent $(+\ \theta)$ and $(-\ \theta)$ plies for the different types of the selected composite materials. All shall have the same optimal solution $(V_{fk}, \hat{h}_k)_{k=1,2}$ = (0.75, 0.215), (0.4315, 0.785), independent upon the shell thickness-to radius ratio (h/R), a major contribution of the given formulation. Two distinct ranges can be observed; $0^o<\theta<30^o$ and $30^o<\theta<90^o$. In the former, the glass fibrous composites excels the carbon ones in resisting buckling, while in the second range the buckling pressure of carbon composite is much exceeding that of the glass types, reaching a remarkable value of 32.686 $(h/R)^3$ *GPa* for hoop wound cylinders.

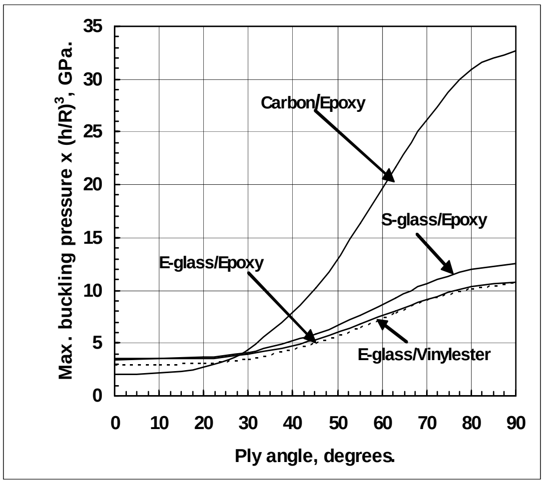
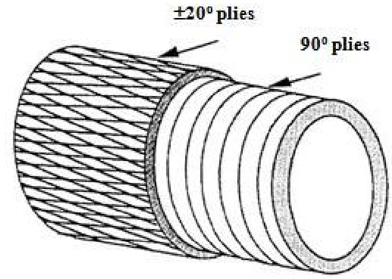


Fig. 4 Variation of the absolute maximum buckling pressure with ply angle for balanced *(± θ°)* cylinders with structural mass preserved constant.

Other cases of study include optimization of multi-layered cylinders made of *AS-4* carbon/epoxy composites. Two different constructions proposed by Rasheed and yousif [3] will be considered herein. The first one is called a lumped-layup construction with the inner half of its wall composed of 90^o hoop layers and the outer half made of $\pm 20^o$ helically wound layers, as shown in Table IV. The second type, given in Table V, has different stacking sequence where the $\pm 20^o$ layers are sandwiched in between outer and inner 90^o hoop layers. The attained Optimum solutions in both cases indicate substantial increase in the critical buckling pressure as compared with the non-optimal solutions presented in [3], [4]. It is also seen that good designs shall have thicker hoop wound layers with higher volume fraction of the fibers near the upper limiting values imposed by the manufacturers. On the other hand, the sandwiched helically wound layers are seen to be thinner and have less fiber volume fractions.

TABLE IV
BUCKLING OPTIMIZATION OF $[90^o/\pm 20^o]$ LAYUP, AS-4 COMPOSITE CYLYNDER

(h/R)	$P_{cr,max} = 9.37x\ (10h/R)^3$ MPa		
	Reference [3]	Present Optimum	% Gain
1/50	0.064	0.075	17.19%
1/25	0.516	0.596	15.50%
1/20	1.013	1.171	15.60%
1/15	2.418	2.776	14.81%



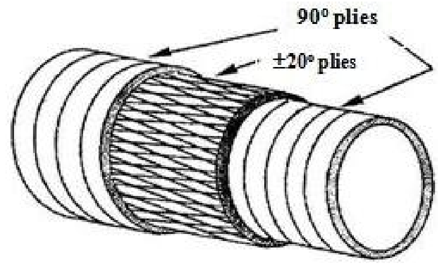
Optimum solution

Two Helical layers: $(V_f, \hat{h}, \theta) = (0.250, 0.225, \pm 20^o)$

Two Hoop layers: $(V_f, \hat{h}, \theta) = (0.705, 0.275, 90^o)$

TABLE V
BUCKLING OPTIMIZATION OF $[90^o/\pm 20^o/90^o]$ LAYUP, AS-4 COMPOSITE CYLYNDER

(h/R)	$P_{cr,max} = 36.634x\ (10h/R)^3$ MPa		
	Reference [3]	Present Optimum	% Gain
1/50	0.231	0.293	26.84%
1/25	1.848	2.344	26.84%
1/20	3.608	4.579	26.91%
1/15	8.552	10.854	26.92%



Optimum solution

Two Helical layers: $(V_f, \hat{h}, \theta) = (0.2925, 0.235, \pm 20^o)$

Two Hoop layers: $(V_f, \hat{h}, \theta) = (0.6835, 0.265, 90^o)$

VI. CONCLUSION

An efficient mathematical approach for enhancing the buckling stability limits of thin-walled anisotropic long cylinders with radial material grading has been developed. The formulation of an optimal lamination design against buckling has been thoroughly investigated, where useful design charts are given for several types of cylinders showing the functional dependence of the critical buckling pressure on the fiber volume fractions, ply thickness and stacking sequence as well. Side constraints are imposed on the design variables in order to avoid having odd-shaped optimized configurations with unrealistic values of the volume fractions and ply angles. An analytical buckling model has been implemented, which provides good sensitivity to lamination parameters, and allowing the search for the needed optimal design in an acceptable computational time. The proposed model deals with dimensionless quantities in order to be applicable for handling thin shells having arbitrary thickness-to-radius ratios, which is a major contribution of this work. Results have indicated that the optimized laminations induce significant increases, always exceeding several tens of percent, of the buckling pressures with respect to the reference or baseline design. Types of composites considered included E-glass/epoxy, S-glass/epoxy, carbon/epoxy and E-glass/vinyl-ester. It has been shown that the overall stability level of the laminated, radially graded composite shell structures under considerations can be substantially improved by finding the optimal ply thicknesses and fiber volume fractions without violating both the mass equality constraint as well as any of the imposed side constraints. The stability limits of the optimized shells have been substantially enhanced as compared with those of the reference or baseline designs. Future aspects shall consider buckling optimization of cylindrical shells with material grading in both the circumferential and axial directions.

APPENDIX (A)

The reduced form of Hooke's law for an orthotropic homogeneous lamina in a plane stress state is [1]:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} \qquad \text{(A-1)}$$

where the square matrix *[Q]* is defined in terms of material properties as follows:

$$Q_{11}=\frac{E_{11}}{1-\nu_{12}^2 E_{22}/E_{11}},\quad Q_{22}=\frac{E_{22}}{1-\nu_{12}^2 E_{22}/E_{11}},\quad Q_{12}=\frac{\nu_{12}E_{22}}{1-\nu_{12}^2 E_{22}/E_{11}},\ Q_{66}=G_{12} \qquad \text{(A-2)}$$

E_{11} and E_{22} are the young's moduli in the 1 and 2 directions, G_{12}, the shear modulus, and ν_{12} the major Poisson's ratio. The elements of the *K-th* lamina stiffness matrix, $[\overline{Q}]$, which is now referred to the reference axes of the cylindrical shell *(x, s, z)*, are given by:

$$\begin{aligned}
\overline{Q}_{11}&=U_1+U_2\cos 2\theta+U_3\cos 4\theta\\
\overline{Q}_{22}&=U_1-U_2\cos 2\theta+U_3\cos 4\theta\\
\overline{Q}_{12}&=U_4-U_3\cos 4\theta\\
\overline{Q}_{16}&=0.5U_2\sin 2\theta+U_3\sin 4\theta\\
\overline{Q}_{26}&=0.5U_2\sin 2\theta-U_3\sin 4\theta\\
\overline{Q}_{66}&=0.5(U_1-U_4)-U_3\cos 4\theta
\end{aligned} \qquad \text{(A-3)}$$

The terms U_i are solely function of the material properties and, hence the volume fractions. They are no longer termed as invariant as has been cited by several investigators, and are defined by the following expressions [17]:

$$\begin{aligned}
U_1&=(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{66})/8\\
U_2&=(Q_{11}-Q_{22})/2\\
U_3&=(Q_{11}+Q_{22}-2Q_{12}-4Q_{66})/8\\
U_4&=(Q_{11}+Q_{22}+6Q_{12}-4Q_{66})/8
\end{aligned} \qquad \text{(A-4)}$$

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