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A systematic approach for finding Hamiltonian cycles with a prescribed edge in crossed cubes

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Abstract—The crossed cube is one of the most notable variations of hypercube, but some properties of the former are superior to those of the latter. For example, the diameter of the crossed cube is almost the half of that of the hypercube. In this paper, we focus on the problem embedding a Hamiltonian cycle through an arbitrary given edge in the crossed cube. We give necessary and sufficient condition for determining whether a given permutation with n elements over Z_n generates a Hamiltonian cycle pattern of the crossed cube. Moreover, we obtain a lower bound for the number of different Hamiltonian cycles passing through a given edge in an n-dimensional crossed cube. Our work extends some recently obtained results.

Keywords—Interconnection network, Hamiltonian, Crossed cubes, Prescribed edge.

I. Introduction

THe ring structure is a fundamental network for multiprocessor systems and suitable for developing simple algorithms with low communication cost. Many efficient algorithms were designed with respect to rings for solving a variety of algebraic problems, graph problems, and some parallel applications, such as those in image and signal processing [2], [9]. To carry out a ring-structure algorithm on a multiprocessor computer or a distributed system, the processes of the parallel algorithm need to be mapped to the nodes of the interconnection network in the system such that any two adjacent processes in the cycle are mapped to two adjacent node of the network. Due to execute a parallel program efficiently, the targeted interconnection network posses a Hamiltonian cycle, i.e., a cycle that passes every node of the network exactly once if the number of processes in the ring-structure parallel algorithm equals the number of nodes of the interconnection network. On the other hand, each link in a parallel distributed system may be assigned with distinct bandwidth, thus, it is meaningful to study the problem of how to embed a Hamiltonian cycle into a network such that these cycles pass through a special edge.

Hypercubes are the most well known of all interconnection networks for parallel computing, given their basic simplicity, their generally desirable topological and algorithmic properties. Thus, many practical parallel computer systems, such as Intel iPSC, the nCUBE family [6], the SGI's Origin 2000 [10], and the Connection Machine [11], employ the hypercubes as the interconnection network. The crossed cube proposed by Efe [3] is one of the most notable variations of hypercube, but some properties of the former are superior to those of the latter. For example, the diameter of the crossed cube is almost the half of that of the hypercube. With regard to cycles embedding of crossed cubes, many interesting results have received considerable attention [1], [5], [7], [8], [12], [13], [14]. In particular, Zheng and Latifi [14] introduced to the notion of *reflected link label sequences* and proposed a kind of codeword, called *Generalized Gray Code*. Applying these concepts, they showed that CQ_n can embed cycles of arbitrary length from 4 to 2^n. In this paper, we consider the problem of embedding a Hamiltonian cycle passing through a prescribed edge in the crossed cube. We introduce a new concept, the *cycle pattern*, and use it to propose a systematic approach for embedding a desired Hamiltonian cycle in the crossed cube. In particular, we give necessary and sufficient condition for determining whether or not a given permutation with n elements over Z_n generates a Hamiltonian cycle pattern of the crossed cube. Our work extends some recently obtained results in [12], [14].

The rest of this paper is organized as follows. Section II introduces definitions and reflected edge label sequence that will be used throughout this paper. In Section III, we propose cycle pattern concept and give necessary and sufficient condition for determining whether or not a given permutation with n elements over Z_n generates a Hamiltonian cycle pattern of the crossed cube. Based on this concept, how many distinct Hamiltonian cycles pass through a given edge in CQ_n is calculated in Section IV. Conclusions are given in the final section.

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II. Preliminaries

A topology of an interconnection network is conveniently represented by an undirected simple graph $G = (V, E)$, where $V(G)$ and $E(G)$ is the vertex set and the edge set of G, respectively. Throughout this paper, vertex and node, edge and link, graph and network are used interchangeably. For graph terminology and notation not defined here we refer the reader to [9]. A walk in a graph is a finite sequence $\omega : \lambda_0, e_1, \lambda_1, e_2, \lambda_2, \ldots, \lambda_{k-1}, e_k, \lambda_k$ whose terms are alternately vertices and edges such that, for $1 \le i \le k$, the edge e_i has ends λ_{i-1} and λ_i, thus each edge e_i is immediately preceded and succeeded by the two vertices with which it is incident. In particular, a walk ω is called a path if all internal vertices, λ_i for $1 \le i \le k-1$, of the walk ω are distinct. The first vertex λ_0 of ω is called its start vertex, and the vertex λ_k is called a last vertex. Both of them are called end-vertices of the path ω. For simplicity, the path ω is also denoted by $\lambda_0, \lambda_1, \ldots, \lambda_k$. If $\lambda_0 = \lambda_k$, then ω is called a cycle. A cycle of

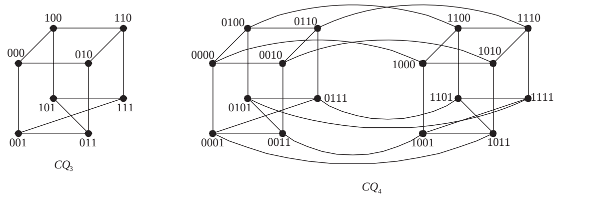

Fig. 1. Crossed cubes CQ_3 and CQ_4.

length l is called a l-cycle. A path (respectively, cycle) which traverses each vertex of G exactly once is *Hamiltonian path* (respectively, *Hamiltonian cycle*).

An n-dimensional crossed cube, denoted as CQ_n, was first proposed by Efe [3]. It is derived by "crossing" some edges in Q_n. With exactly same hardware cost as hypercube, it has been shown that such a simple variation gains important benefits such as greatly reduced diameter. To define crossed cubes, the notion so called "*pair related*" relation is introduced. Let R = $\{(00, 00), (10, 10), (01, 11), (11, 01)\}$. Two binary strings u_1u_0 and v_1v_0 are pair related, denoted as $u \sim v$, if and only if $(u, v) \in R$. Subsequently, a crossed cube of dimension n is an undirected graph consisting of 2^n vertices labeled from 0 to $2^n - 1$ and defined recursively as following:

Definition 1: [3] The crossed cube CQ_1 is a complete graph with two vertices labeled by 0 and 1, respectively. For $n \geq 2$, an n-dimensional crossed cube CQ_n consists of two $(n-1)$-dimensional sub-crossed cubes, CQ^0_{n-1} and CQ^1_{n-1}, and a perfect matching between the vertices of CQ^0_{n-1} and CQ^1_{n-1} according to the following rule:

Let $V(CQ^0_{n-1}) = \{0u_{n-2}u_{n-3}\cdots u_0 : u_i = 0 \text{ or } 1\}$ and $V(CQ^1_{n-1}) = \{1v_{n-2}v_{n-3}\cdots v_0 : v_i = 0 \text{ or } 1\}$. The vertex $u = 0u_{n-2}u_{n-3}\cdots u_0 \in V(CQ^0_{n-1})$ and the vertex $v = 1v_{n-2}v_{n-3}\cdots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if

(1) $u_{n-2} = v_{n-2}$ if n is even, and
(2) $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$, for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

An edge $(u, v) \in E(CQ_n)$ is labeled by j if $u_j \neq v_j$ and $u_i = v_i$ for $j + 1 \leq i \leq n - 1$, i.e, v is the j-th dimensional neighbor (abbreviated as j-neighbor) of u, denoted by $u[j]v$ or $v[j]u$. It is observed that each vertex u in CQ_n has n neighbors in CQ_n; u has exactly one j-neighbor for $0 \leq j \leq n-1$. As a consequence, there are 2^{n-1} edges labeled by j, $0 \leq j \leq n-1$, in CQ_n. For example, the graphs shown in Figure 1 are CQ_3 and CQ_4.

A path in CQ_n might be specified by the source vertex and a sequence of labels detailing the edges to be traversed, for example, the path in CQ_3 detailed as having the source vertex 000 and then following the edges labeled 1,2,1 (also denoted [1,2,1]) is actually the path $000, 010, 110, 100$, also denoted 000[1,2,1]100. Besides, 000[1,2]110,100, 000,010[2,1]100, and 000[1,2,1]100 are represented the identical path $000, 010, 110, 100$. Therefore, the sequence $L = [d_1, d_2, \ldots, d_m]$ is called an *Edge Label Sequence* in CQ_n if two adjacent labels are not identical where $d_i \in Z_n$, $Z_n = \{0, 1, \ldots, n-1\}$, for $1 \leq i \leq m$.

A walk, $\omega(L, u) = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m$, in CQ_n can be generated with respect to a given edge label sequence L and a given vertex u as follows: $\lambda_0 = u$, and λ_j is the d_j-neighbor of λ_{j-1} in CQ_n where $1 \leq j \leq m$, i.e, $\lambda_{j-1}[d_{j-1}]\lambda_j$. Thus, this walk $\omega(L, u)$ is also represented as $\lambda_0[L]\lambda_m$. In particular, the edge label sequence L is interesting in this paper when it generates a loop-free path $\omega(L, u)$ starting from any vertex u in CQ_n.

Hereafter, we are interesting a special edge label sequence called reflected link label sequence generated by a systematic method. A reflected edge label sequence of length 2^k is generated from a permutation with k elements over Z_n. Let $\pi_k = \langle d_1, d_2, \ldots, d_k \rangle$, $1 \leq k \leq n$, be a permutation over Z_n with k elements, and let $\pi_k(i) = \langle d_1, d_2, \ldots, d_i \rangle$. The *Reflected Edge Label Sequence*, RL_{π_k} defined by π_k, be generated recursively as follows:

$$\begin{aligned} RL_{\pi_k(1)} &= d_1 \\ RL_{\pi_k(i)} &= RL_{\pi_k(i-1)}, d_i, RL_{\pi_k(i-1)}, 1 \leq i \leq k; \text{ and} \\ RL_{\pi_k} &= RL_{\pi_k(k)} \end{aligned}$$

As a result, the RL_{π_k} defined by arbitrary permutation π_k over Z_n is a symmetry edge label sequence in CQ_n. Zheng and Latifi [14] observe the following lemma.

Lemma 1: [14] For any vertex u in CQ_n and any π_n permutation over Z_n with n elements, the walk $\omega(RL_{\pi_n}, u)$ corresponds to a Hamiltonian path of CQ_n that start from u.

Lemma 2: Assume that π_n is a permutation, $\langle d_1, d_2, \ldots, d_n \rangle$, with n elements over on Z_n. Then, the total number of d_k, $1 \leq k \leq n$, in RL_{π_n} equals to 2^{n-k}.

Lemma 3: Assume that π^0_n and π^1_n are two distinct permutations with n elements over on Z_n. Then, $\omega(RL_{\pi^0_n}, u)$ and $\omega(RL_{\pi^1_n}, v)$ correspond two distinct Hamiltonian paths of CQ_n for any two vertices u and v of CQ_n.

Proof. Let $\pi^0_n = \langle d_1, d_2, \ldots, d_n \rangle$ and $\pi^1_n = \langle d'_1, d'_2, \ldots, d'_n \rangle$ be two distinct permutations over on Z_n. Since $\pi^0_n \neq \pi^1_n$, there exist k and h such that $k \neq h$ and $d_k = d'_h$. By Lemma 2 and for any two vertices $u, v \in V(CQ_n)$, the path $\omega(RL_{\pi^0_n}, u)$ and $\omega(RL_{\pi^1_n}, v)$ passes through 2^{n-k} and 2^{n-h} edges in dimension d_k and d'_h of CQ_n, respectively. As a conclusion, $\omega(RL_{\pi^0_n}, u)$ and $\omega(RL_{\pi^1_n}, v)$ correspond to two distinct Hamiltonian paths of CQ_n. □

III. Cycle patterns

Let L be an edge label sequence of CQ_n with l elements. We called L an *l-cycle pattern*, *l-CP* for short, of CQ_n if the walk $\omega(L, u)$ forms an l-cycle for every vertex u. Particularly, appending d_k to the end of the sequence RL_{π_k}, we obtain an edge label sequence $[RL_{\pi_k}, d_k]$ where $\pi_k = \langle d_1, d_2, \ldots, d_k \rangle$. For convenience, we use C_{π_k} to denote the special sequence $[RL_{\pi_k}, d_k]$ in the following; thus, C_{π_k} contains 2^k edge labels for any permutation, π_k, with k elements. In other word, we are interested the special permutations, π_k for $1 \leq k \leq n$, with k elements over Z_n satisfy that C_{π_k} is a 2^k-CP of CQ_n. Such permutation π_k is called a 2^k-*CP generator* of CQ_n. In particular, π_n is called a *Hamiltonian cycle pattern generator* if C_{π_n} is a 2^n-CP.

Herein, fundamental properties of CQ_n are proposed in order to help for constructing 2^k-CP generator of CQ_n. Let $B_2 = \{00, 01, 10, 11\}$ and $\gamma : B_2 \rightarrow B_2$ be a bijection mapping, which is defined as follows: $\gamma(u_1u_0) = v_1v_0$ if and only if u_1u_0 and v_1v_0 are pair related, i.e., $u_1u_0 \sim v_1v_0$. Let $\mathcal{C}_b : B_2 \rightarrow B_2$ and $\mathcal{C}_f : B_2 \rightarrow B_2$ be two bijection mappings, which are defined as follows $\mathcal{C}_b(u_1u_0) = u_1\overline{u_0}$ and $\mathcal{C}_f(u_1u_0) = \overline{u_1}u_0$, respectively. Thus, a compose function $g \circ f : B_2 \rightarrow B_2$ defined by $(g \circ f)(u) = g(f(u))$ for all $u \in B_2$ is obtained where $g, f \in \{\gamma, \mathcal{C}_f, \mathcal{C}_b\}$. The following lemma is useful in the proof of Lemma 5. It is not difficult to verify by straightforward; hence, the detail proof is omitted.

Lemma 4: For $k \geq 1$, let u be any 2-bit binary string. Then,

(1) $f^{2k}(u) = u$ where $f \in \{\gamma, \mathcal{C}_f, \mathcal{C}_b\}$,
(2) $\mathcal{C}_f \circ \gamma(u) = \gamma \circ \mathcal{C}_f(u)$, and
(3) $\mathcal{C}_b \circ \gamma(u) = \mathcal{C}_f \circ \gamma \circ \mathcal{C}_b(u)$.

Consequently, Lemma 5 gives necessary and sufficient conditions for determining 2^k-CP generator of CQ_n for $2 \leq k \geq n$.

Lemma 5: For $n \geq 3$ and $2 \leq k \leq n$, let $\pi_k = \langle d_1, \ldots, d_{k-1}, d_k \rangle$ be a permutation over Z_n with k elements where $2 \leq k \leq n$. Then, π_k is not a 2^k-CP generator if and only if $\min\{d_{k-1}, d_k\}$ is even and $|d_k - d_{k-1}| \geq 2$. Moreover, for any vertex u in CQ_n, two end-vertices of the walk $\omega(C_{\pi_k}, u)$ are only different from the g-th bit position where $g = \min\{d_{k-1}, d_k\} + 1$ if π_k is not a 2^k-CP generator in CQ_n.

Proof. The lemma is proved by induction on k. We first claim the base case for $k = 2$. Without loss of generality, we may assume that $d_1 < d_2$. Also let u be a vertex of CQ_n.

($\Rightarrow$) Suppose that $d_1 = d_2 - 1$ if d_1 is even; otherwise, d_1 is an odd integer. It is claimed that $u[d_1, d_2, d_1, d_2]v$ forms a 4-cycle in CQ_n, i.e., $u = v$. It is obvious that $u = v$ if $d_1 = d_2 - 1$. So, we just consider only d_1 is an odd integer. Since $d_1 < d_2$ and d_1 is odd, $u_j = v_j$ for $j > d_1$ or $j < d_1 - 1$. We check only whether $u_{d_1}u_{d_1-1} = v_{d_1}v_{d_1-1}$ or not. Note that $v_{d_1}v_{d_1-1} = (\gamma \circ \mathcal{C}_f)^2(u_{d_1}u_{d_1-1})$. By Lemma 4, $(\gamma \circ \mathcal{C}_f)^2(u_{d_1}u_{d_1-1}) = u_{d_1}u_{d_1-1}$. Therefore, $u = v$.

($\Leftarrow$) Suppose that d_1 is even and $d_1 < d_2 - 1$. It is observed that $u_j = v_j$ for $j > d_1 + 1$ or $j < d_1$, and $v_{d_1+1}v_{d_1} = (\gamma \circ \mathcal{C}_b)^2(u_{d_1+1}u_{d_1})$. By Lemma 4, $(\gamma \circ \mathcal{C}_b)^2(u_{d_1+1}u_{d_1}) = \mathcal{C}_f(u_{d_1+1}u_{d_1}) = \overline{u_{d_1+1}}u_{d_1}$. Therefore, $u[d_1, d_2, d_1, d_2]v$ is not a 4-cycle and u, v are only different from the d_1-th bit position.

As a subsequence, suppose that the lemma is true for $k \leq m - 1$. Let π_m be a permutation, $\langle d_1, \ldots, d_{m-1}, d_m \rangle$, with m elements over Z_n. The C_{π_m} will be represented by $[RL_{\pi_m(m-1)}, d_m, RL_{\pi_m(m-1)}, d_m]$ where $RL_{\pi_m(m-1)}$ is a reflected edge label sequence generated by the permutation $\langle d_1, d_2, \ldots, d_{m-2}, d_{m-1} \rangle$. Let $u = u_{n-1}u_{n-2} \ldots u_0$ be an arbitrary vertex of CQ_n. The walk $\omega(C_{\pi_m}, u)$ is written as $u[RL_{\pi_m(m-1)}]x[d_m]y[RL_{\pi_m(m-1)}]z[d_m]\mathbf{v}$, where $u[RL_{\pi_m(m-1)}]x$ is the path $\omega(RL_{\pi_m(m-1)}, u)$ and $y[RL_{\pi_m(m-1)}]z$ is the path $\omega(RL_{\pi_m(m-1)}, y)$. According to whether or not $\pi_m(m-1)$ is a 2^{m-1}-CP generator, the proof is divided into two parts as follows.

Case 1: $\pi_m(m-1)$ is a 2^{m-1}-CP generator.

Hence u and y is a d_{m-1}-neighbor of x and z, respectively, i.e., $u[d_{m-1}]x$ and $y[d_{m-1}]z$. Therefore, $u[d_{m-1}]x[d_m]y[d_{m-1}]z[d_m]v$ is a path having the source vertex u and then following the edge label sequence $[d_{m-1}, d_m, d_{m-1}, d_m]$.

($\Rightarrow$) Suppose that $\langle d_{m-1}, d_m \rangle$ is a 4-CP generator. Thus, $u = v$. Therefore, $\omega(C_{\pi_k}, u)$ is a 2^k-cycle in CQ_n.

($\Leftarrow$) Suppose that $\langle d_{m-1}, d_m \rangle$ is not a 4-CP generator. By induction hypothesis, u and v are only different from the g-th bit position where $g = \min\{d_{m-1}, d_m\} + 1$. Hence $\omega(C_{\pi_m}, u)$ is not a 2^m-cycle of CQ_n.

Case 2: $\pi_m(m-1)$ is not a 2^{m-1}-CP generator.

By induction hypothesis, $u[RL_{\pi_m(m-1)}]x[d_{m-1}]w$ does not form a cycle, i.e, $u \neq w$. Moreover, u and w are only different from the h-th bit position where $h = \min\{d_{m-2}, d_{m-1}\} + 1$. Let $g = \min\{d_{m-1}, d_m\} + 1$. In this case, there are six situations with respect to the relation of d_{m-2}, d_{m-1}, and d_m. The proof of each situation is similar. Thus, we discuss only the case of $d_{m-2} > d_{m-1} > d_m$ in the following, that is, $h = d_{m-1} + 1$ and $g = d_m + 1$. By induction hypothesis, $\langle d_{m-2}, d_{m-1} \rangle$ is not a 4-CP generator. By induction hypothesis, d_{m-1} is an even integer. Obviously, the vertex $x = x_{n-1}x_{n-2} \ldots x_0$ satisfies that

$$\begin{aligned} x_i &= u_i \text{ for } i > h, \\ x_hx_{h-1} &= \overline{u_hu_{h-1}}, \text{ and} \\ x_{2j+1}x_{2j} &= \gamma(u_{2j+1}u_{2j}) \text{ for } \tfrac{h-3}{2} \geq j \geq 0. \end{aligned}$$

($\Rightarrow$) Suppose that $C_{\langle d_{m-1}, d_m \rangle}$ is a 4-CSK. We will claim that C_{π_m} is a 2^m-CSK in CQ_n, i.e., $\omega(C_{\pi_m}, u)$ is a 2^m-cycle. Since $d_{m-1} > d_m$ and by induction hypothesis, d_m is an odd integer. Since y is the d_m-neighbor of x, the vertex $y = y_{n-1}y_{n-2} \ldots y_0$ satisfies that

$$\begin{aligned} y_i &= u_i \text{ for } i > h, \\ y_hy_{h-1} &= \overline{u_hu_{h-1}}, \\ y_{2l+1}y_{2l} &= \gamma(u_{2l+1}u_{2l}) \text{ for } \tfrac{h-3}{2} \geq j \geq \tfrac{d_m+1}{2}, \\ y_{d_m}y_{d_m-1} &= \mathcal{C}_f \circ \gamma(u_{d_m}u_{d_m-1}), \text{ and} \\ y_j &= u_j \text{ for } (d_m - 2) \geq j \geq 0. \end{aligned}$$

Note that $y[RL_{\pi_m(m-1)}]z$. Thus, the vertex $z = z_{n-1}z_{n-2} \ldots z_0$ satisfies that

$$\begin{aligned} z_i &= u_i \text{ for } i > h, \\ z_hz_{h-1} &= u_hu_{h-1}, \\ z_{2l+1}z_{2l} &= (\gamma)^2(u_{2l+1}u_{2l}) \text{ for } \tfrac{h-3}{2} \geq j \geq \tfrac{d_m+1}{2}, \\ z_{d_m}z_{d_m-1} &= \gamma \circ \mathcal{C}_f \circ \gamma(u_{d_m}u_{d_m-1}), \text{ and} \\ z_{2j+1}z_{2j} &= \gamma(u_{2j+1}u_{2j}) \text{ for } \tfrac{d_m-3}{2} \geq j \geq 0. \end{aligned}$$

By Lemma 4, we have that $(\gamma)^2(u_{2l+1}u_{2l}) = u_{2l+1}u_{2l}$. Hence $z_{2l+1}z_{2l} = u_{2l+1}u_{2l}$ for $\frac{h-3}{2} \geq j \geq \frac{d_m+1}{2}$. Since $\mathcal{C}_f \circ \gamma = \gamma \circ \mathcal{C}_f$ (by Lemma 4), $\gamma \circ \mathcal{C}_f \circ \gamma(u_{d_m}u_{d_m-1}) = \mathcal{C}_f(u_{d_m}u_{d_m-1})$. Consequently, the vertex z can be represented by

$$\begin{aligned} z_i &= u_i \text{ for } i > d_m, \\ z_{d_m}z_{d_m-1} &= \overline{u_{d_m}}u_{d_m-1}, \text{ and} \\ z_{2j+1}z_{2j} &= \gamma(u_{2j+1}u_{2j}) \text{ for } \tfrac{d_m-3}{2} \geq j \geq 0. \end{aligned}$$

Thus, it is observed that vertex z is the d_m-neighbor of u, i.e., $z[d_m]u$. Since $z[d_m]v$, $u = v$ and hence $\omega(C_{\pi_m}, u)$ is a 2^m-cycle in CQ_n.

($\Leftarrow$) Suppose that $\langle d_{m-1}, d_m \rangle$ is not a 4-CP generator. It is recalled that $\omega(C_{\pi_m}, u) = u[RL_{\pi_m(m-1)}]x[d_m]y[RL_{\pi_m(m-1)}]z[d_m]\mathbf{v}$. We will claim that $\omega(C_{\pi_m}, u)$ is not a 2^m-cycle of CQ_n, and $u_g = \overline{v_g}$ and $u_i = v_i$ for all $0 \leq i \neq g \leq n - 1$ where $g = \mathbf{min}\{d_{m-1}, d_m\} + 1$.

Since $d_{m-1} > d_m$ and by induction hypothesis, d_m is an even integer. Hence the vertex $y = y_{n-1}y_{n-2}\ldots y_0$ satisfies that

$$\begin{aligned} y_i &= u_i \text{ for } i > h, \\ y_h y_{h-1} &= \overline{u_h u_{h-1}}, \\ y_{2l+1}y_{2l} &= \gamma(u_{2l+1}u_{2l}) \text{ for } \tfrac{h-3}{2} \geq j \geq \tfrac{d_m+2}{2}, \\ y_{d_m+1}y_{d_m} &= \mathcal{C}_b \circ \gamma(u_{d_m+1}u_{d_m}), \text{ and} \\ y_j &= u_j \text{ for } (d_m-1) \geq j \geq 0. \end{aligned}$$

Note that $y[RL_{\pi_{m}(m-1)}]z$. Thus, the vertex $z = z_{n-1}z_{n-2}\ldots z_0$ satisfies that

$$\begin{aligned} z_i &= u_i \text{ for } i > h, \\ z_h z_{h-1} &= u_h u_{h-1}, \\ z_{2l+1}z_{2l} &= (\gamma)^2(u_{2l+1}u_{2l}) \text{ for } \tfrac{h-3}{2} \geq j \geq \tfrac{d_m+2}{2}, \\ z_{d_m+1}z_{d_m} &= \gamma \circ \mathcal{C}_b \circ \gamma(u_{d_m+1}u_{d_m}), \text{ and} \\ z_{2j+1}z_{2j} &= \gamma(u_{2j+1}u_{2j}) \text{ for } \tfrac{d_m-2}{2} \geq j \geq 0. \end{aligned}$$

Since $\mathcal{C}_b \circ \gamma = \mathcal{C}_f \circ \gamma \circ \mathcal{C}_b$ and $\gamma \circ \mathcal{C}_f = \mathcal{C}_f \circ \gamma$ (by Lemma 4), $\gamma \circ \mathcal{C}_b \circ \gamma(u_{d_m}u_{d_m-1}) = \mathcal{C}_f\mathcal{C}_b(u_{d_m}u_{d_m-1}) = \overline{u_{d_m}u_{d_m-1}}$. Consequently, the vertex z can be represented by

$$\begin{aligned} z_i &= u_i \text{ for } i > d_m, \\ z_{d_m+1}z_{d_m} &= \overline{u_{d_m+1}u_{d_m}}, \text{ and} \\ z_{2j+1}z_{2j} &= \gamma(u_{2j+1}u_{2j}) \text{ for } \tfrac{d_m-2}{2} \geq j \geq 0. \end{aligned}$$

Thus, it is observed that vertex z and u are not adjacent in CQ_n. Since $z[d_m]v$, the vertex $v = v_{n-1}v_{n-2}\ldots v_0$ satisfies that

$$\begin{aligned} v_i &= u_i \text{ for } i \neq d_m, \text{ and} \\ v_{d_m+1} &= \overline{u_{d_m+1}}. \end{aligned}$$

Therefore, vertex u and v are only different from the g-th bit position where $g = d_m + 1$, that is, $g = \min\{d_{m-1}, d_m\} + 1$. □

Given a permutation π_n over on Z_n with n elements, one can determine whether or not the permutation π_n can generate a Hamiltonian cycle pattern, C_{π_n}, of CQ_n by only inspecting the last two numbers d_{n-1} and d_n of π_n. Thus, we have the following corollary.

Corollary 1: For $n \geq 3$, let $\pi_n = \langle d_1, \ldots, d_{n-1}, d_n\rangle$ be a permutation over Z_n with n elements. Then, C_{π_n} is a Hamiltonian cycle pattern of CQ_n if and only if

(1) $\min\{d_{n-1}, d_n\}$ is odd, or
(2) $|d_n - d_{n-1}| = 1$.

IV. Distinct Hamiltonian cycles passing a given edge

Given a Hamiltonian cycle pattern generator π_n and any vertex u in CQ_n, the walk $\omega(C_{\pi_n}, u)$ corresponds to a Hamiltonian cycle in CQ_n. In this section, we will construct several distinct Hamiltonian cycles with respect to Hamiltonian cycle pattern in use such that they pass through the same prescribed edge.

Lemma 6: For $n \geq 3$, let π_n be arbitrary permutation, $\langle d_1, d_2, \ldots, d_n\rangle$, with n elements over on Z_n and (u, v) be any edge of CQ_n. Then, there exists a vertex z such that the Hamiltonian path $\omega(RL_{\pi_n}, z)$ of CQ_n passes through the edge (u, v).

Proof. Let π_n be a permutation, $\langle d_1, \ldots, d_{k-1}, d_k, \ldots, d_n\rangle$, with n elements over on Z_n and (u, v) be an edge in dimension d. Without loss of generality, $d_k = d$. Let $RL_{\langle d_1,\ldots,d_{k-1}\rangle}$ be a reflected edge label sequence defined by the permutation $\langle d_1, \ldots, d_{k-1}\rangle$. Obviously, $RL_{\langle d_1,\ldots,d_{k-1}\rangle}$ is a substring of RL_{π_n}. The proof is trivial if $k = 1, 2$; thus, we consider only the case of $k \geq 3$.

Let z be the d_{k-1}-th neighbor of u if $\langle d_1, \ldots, d_{k-1}\rangle$ is a 2^{k-1}-CP generator; otherwise, z be a vertex satisfies that

$$\begin{aligned} z_i &= u_i \text{ for } i > g, \\ z_g z_{g-1} &= \overline{u_g u_{g-1}}, \text{ and} \\ z_{2j+1}z_{2j} &= \gamma(u_{2j+1}u_{2j}) \text{ for } \tfrac{g-3}{2} \geq j \geq 0. \end{aligned}$$

, where $g = \min\{d_{k-2}, d_{k-1}\} + 1$.

By Lemma 5, we obtain the path $z[RL_{\langle d_1,\ldots,d_{k-1}\rangle}]u$; besides, $z[RL_{\langle d_1,\ldots,d_{k-1}\rangle}]u$ lies on the Hamiltonian path $\omega(RL_{\pi_n}, z)$. Since $u[d_k]v$, the path $\omega(RL_{\pi_n}, z)$ passes through the edge (u, v). □

With respect to Lemma 6, the subsequent theorem is immediately clear.

Theorem 1: For $n \geq 3$, let π_n be arbitrary Hamiltonian cycle pattern generator and (u, v) be any edge in CQ_n. Then, there exists a vertex z such that the Hamiltonian cycle $\omega(C_{\pi_n}, z)$ of CQ_n passes through the edge (u, v).

Given any edge (u, v) in dimension d, $0 \leq d \leq n-1$, and any two distinct Hamiltonian cycle pattern generators π_n^0 and π_n^1, by Lemma 3 and Theorem 1, we can generate two distinct Hamiltonian cycles of CQ_n based on π_n^0 and π_n^1 such that each cycle passes through the edge (u, v). Indeed, we can obtain a lower bound for the number of different Hamiltonian cycles passing a given edge by calculating how many different Hamiltonian cycle pattern generators in CQ_n. Subsequently, the following theorem is easy to verify by fundamental calculation.

Theorem 2: For $n \geq 3$, let (u, v) be any given edge in CQ_n. Then, there are at least m different Hamiltonian cycles in CQ_n passing through the edge (u, v) where $m = \frac{n^2}{2} \times (n-2)!$ if n is even; otherwise, $m = \frac{n^2-1}{2} \times (n-2)!$.

V. Conclusion

The crossed cube is one of most prominent variants of hypercube. Because crossed cubes are neither edge- nor vertex-symmetric, producing Hamiltonian cycles to pass any prescribed edge in a crossed cube is more intricate of a process than in a regular hypercube. In this paper, we apply the characterization of Hamiltonian cycles pattern extended from [14] to build a simple systematic approach to generate a Hamiltonian cycle passing through arbitrary prescribed edge.

Numerous variants of hypercube, for example, Möbius cubes, Twisted cubes, and Locally Twisted cubes, have been proposed and proved that there exists a Hamiltonian cycle passing through any given edge in them. Finding an algorithm to generate a desired Hamiltonian cycle passing through arbitrary given edge in these variants of hypercube is still open. We conjecture that such approach may be constructed by applying the concept of Hamiltonian cycle pattern to these networks.

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