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Probabilities and the Persistence of Memory in a BINGO-like Carnival Game

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Abstract—Seemingly simple probabilities in the m-player game BINGO have never been calculated. These probabilities include expected game length and the expected number of winners on a given turn. The difficulty in probabilistic analysis lies in the subtle interdependence among the m-many BINGO game cards in play. In this paper, the game I GOT IT!, a BINGO variant, is considered. This variation provides enough weakening of the inter-player dependence to allow probabilistic analysis not possible for traditional BINGO. The probability of winning in exactly k turns is calculated for a one-player game. Given a game of m-many players, the expected game length and tie probability are calculated. With these calculations, the game's interesting payout scheme is considered.

Keywords—Conditional probability, games of chance, n-person games, probability theory.

I. Introduction

BINGO is strictly regulated in the U.S., Europe and Australia. Complicated legal requirements often restrict the offering of BINGO to religious, service and charity groups. Perhaps inadvertently, these regulations have given rise to BINGO-like games, which although technically legal, share much in common with their more regulated cousin. I GOT IT! is one of the more widely legal variants of 75-ball BINGO. From the perspective of the statistician, I GOT IT! represents not only a circumvention of local gambling laws, but also a mathematical opportunity. I GOT IT! provides a level of probabilistic analysis not yet attained for traditional BINGO.

Analysis of BINGO probabilities is tricky. No one knows, for example, the expected number of turns it takes to complete an m-player game of BINGO. In the best work to date on the subject, Agard and Shackleford [1] rigorously calculated the probabilities behind a single-player game, and showed how subtle dependence among m-many game boards makes a precise calculation of expected game length a difficult, and still open, question. In this paper, we adapt some of the techniques in [1], and consider new probabilistic questions under the modest weakening of BINGO hypotheses that comes with I GOT IT!. Knowing probabilities for the single-player I GOT IT! game does in fact allow calculations of expected game length and tie probabilities in the m-player game.

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A. Background

In I GOT IT!, players sit in front of a row of hoppers, or plexiglass containers, each of which is sloped down to a five-by-five square grate at the bottom. When instructed to do so, each player throws a small rubber ball into his hopper; the ball bounces inside until it loses speed and settles into one of the twenty-five squares in the grate, or *board*.[1] See Fig. 1.

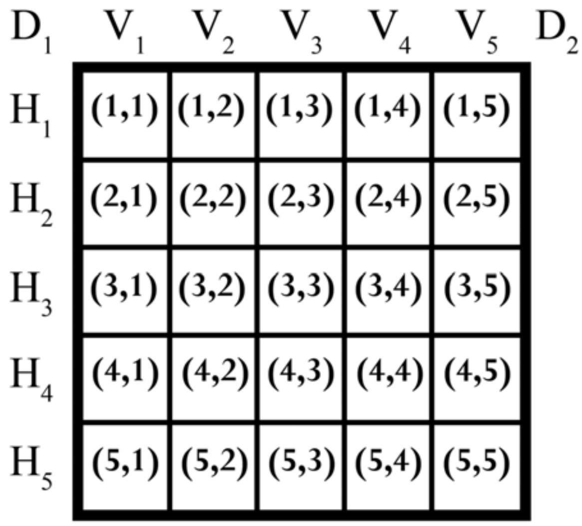

Fig. 1. The 5x5 board with twelve strike possibilities and reference coordinates.

Play proceeds with the throwing of a second, third, ...kth, ball, and ends when one or more players have achieved a *strike*. A strike is attained when five balls fill any of the five rows, $\{H_1, \ldots, H_5\}$, five columns, $\{V_1, \ldots, V_5\}$, or two diagonals $\{D_1, D_2\}$, of the board.[2] See Fig. 2.

Payouts in I GOT IT! represent a substantial deviation from typical BINGO: if after the kth turn exactly one player achieves a strike, then that player is awarded a prize of his choice. Unlike in traditional BINGO, there is no 'pot'

[1] In contrast to the traditional 75-ball BINGO card, there is no "free space" in the center square in I GOT IT!. In this way, the game considered here more closely approximates the BINGO version popular in Sweden.

[2] Not surprisingly, winners must call out "I got it!" to announce their strike.

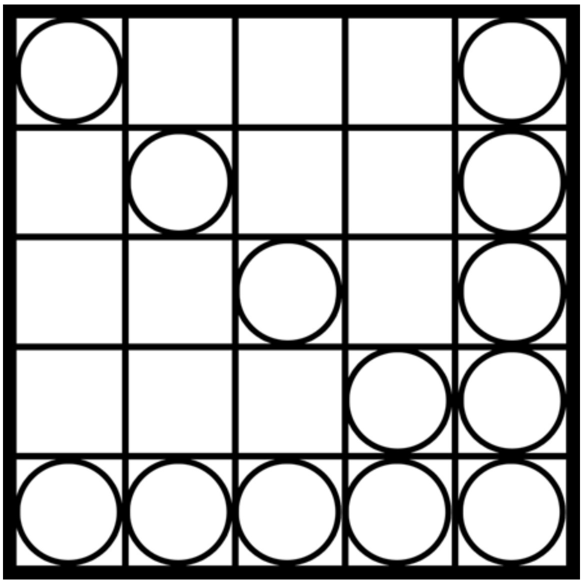

Fig. 2. Board with 3 strikes: H_5, V_5, D_1.

to split among multiple winners. In I GOT IT!, if on the kth turn, exactly l-many players, $l \geq 2$, simultaneously achieve a strike, then each of the l-winners is awarded a free game, and no payout of cash value is made.

II. Analysis

A. Persistence of memory

It is necessary to determine the probability of completing a strike in k or fewer turns, $1 \leq k \leq 25$, on a single board. In this paper, it is assumed there is an equal chance of the ball coming to rest in any unfilled square of the grid, an assumption consistent with the authors' observation of twenty hours of game play. In the 5x5 grid, given k squares covered, there are exactly $\binom{25}{k}$ possible configurations of the board. For $0 \leq k \leq 5$, all $\binom{25}{k}$ combinations have an identical likelihood. However, analysis of BINGO and I GOT IT! is substantially complicated by the subtle fact that for $k > 5$, configurations of k-many covered squares do *not* carry the same probability. To illustrate this, consider an arbitrary k-ball configuration $\hat{B}_k$. Define a *parent state* of $\hat{B}_k$ as any configuration $\hat{B}_{k-1}$ formed by the removal from the board $\hat{B}_k$ of a single ball. Denote parent states, and parent states of parent states, etc. as *antecedents*, and working in the opposite direction, denote as *descendant* states any offspring of parent states.

Were it the case that any k-ball board occur with equal probability, then all k-ball boards must share an equal number of parent states. This reasonable notion is dispelled, however, since I GOT IT! games must end with the first strike. This requirement partitions into three disjoint sets all possible configurations of the 25-square board: some configurations are *legal*, some are only potentially legal, or *critical*, whereas others are inherently *illegal* boards. Fig. 3 illustrates this trichotomy.

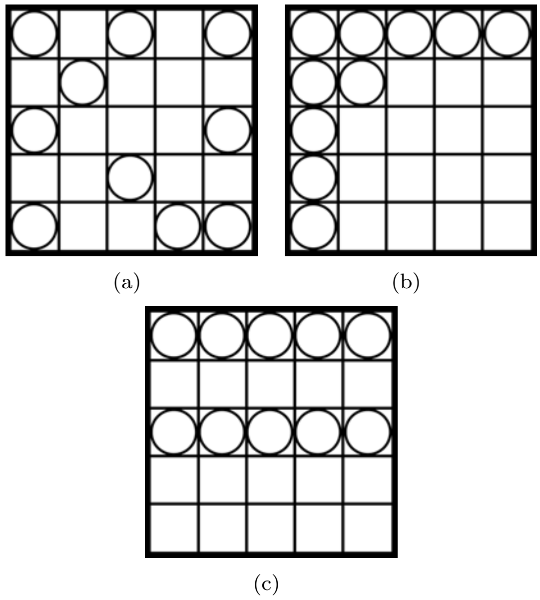

Fig. 3. Three combinations of $k = 10$ balls: (a) is a legal board, (b) is a critical board, and (c) is an illegal board.

Fig. 3(a) is a legal board with no completed strikes. The board has ten distinct parent states, in the sense that removing any ball would result in a nine-ball nonwinning board. Yet in I GOT IT!, there is a certain *persistence of memory* which can be seen in Figs. 3(b) and 3(c). Fig. 3(b) can be attained legally from only one parent state; any nine-ball parent of Fig. 3(b) with position (1,1) covered is actually a winning board itself. Thus the configuration in Fig. 3(b) is attainable only if $(1, 1)$ is the tenth ball thrown. Fig. 3(c) shows a board with two completed strikes, as in Fig. 3(b), but is in fact an illegal board. Any ball removed from the configuration would still represent a winning nine-ball parent state. Fig. 3(c) is a descendant of no nonwinning parents, an impossibility in game play.

Because of the complication introduced by legal, critical, and illegal game boards, a probabilistic examination of merely $C = \sum_{i=0}^{25} \binom{25}{i} \approx 3.4 \times 10^7$ configurations does not suffice. Eliminating all boards with winning antecedents requires respect to the *order* of construction within the C-many board configurations. Even after exploiting the D_4 symmetry in the set of winning boards, we are still left with $25!/8 > 1.9 \times 10^{24}$ possible permutations to consider, a daunting task for even the fastest of computers.

B. Probabilities behind the one-player game

To eliminate winning antecedents from consideration, one can enumerate all possible subsets of the twelve-element set B of all strikes:

$$B = \{H_1, \ldots, H_5, V_1, \ldots, V_5, D_1, D_2\} .$$

Let B_i be the set of all of all i-element subsets of B. Denote as $b_{i,j}$ the elements of each B_i, where j runs from 1 to

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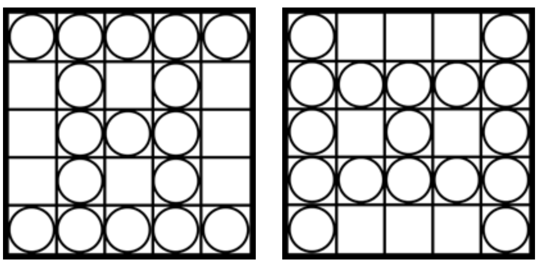
TABLE I
COUNT OF WINNING CONFIGURATIONS BY NUMBER OF SIMULTANEOUS STRIKES AND MINIMUM NUMBER OF SQUARES REQUIRED

Squares	Number of Simultaneous Strikes (i)											
Covered (k)	1	2	3	4	5	6	7	8	9	10	11	12
5	12											
6												
7												
8												
9		46										
10		20										
11												
12			48									
13			152									
14				8								
15			20	148								
16				188	8							
17				141	120	2						
18					232	8						
19					360	136	4					
20				10		304	24					
21					70	272	182	10				
22						188	264	56				
23							276	228	36			
24								129	96	16		
25					2	14	42	72	88	50	12	1
Total	12	66	220	495	792	924	792	495	220	66	12	1

$\binom{12}{i}$. Since each $b_{i,j}$ is a collection of strikes, it is possible to calculate for all 4,905 possible nonempty subsets the minimum number of balls which would be required to simultaneously produce these $i = 1, \ldots, 12$-many strikes. Table I is a count of all subsets of B of size i (top row), requiring exactly k-many balls (first column) to cover. As an example, Table I illustrates that there are exactly two distinct configurations of six strikes which require exactly seventeen squares covered. See fig. 4.



(a) (b)

Fig. 4. The two unique board configurations with $k = 17$, $i = 6$.

The entries in Table I can be used to remove winning antecedents from consideration in calculating winning probabilities in the one-player game. First, recall several formulas from basic probability theory.[2]

Denote the completion of any set of n of 25 squares as A_n. The probability of achieving this set in at most k turns is then

$$P(A_n \leq k) = \binom{25-n}{k-n}\binom{25}{k}^{-1}. \quad (1)$$

To calculate the probability of a union of events

$$E = \bigcup_{i=1}^{n} E_i,$$

with E_i not necessarily disjoint, recall the general addition formula:

$$P(E) = P(\bigcup_{i=1}^{n} E_i) = \sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \cdots \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cdots \cap E_n). \quad (2)$$

Let S_i be the discrete random variable which assumes the value k when the ith strike is achieved after filling exactly k squares. Let $S = \min\{S_1, S_2, \ldots, S_{12}\}$. The chance, then, of achieving the first strike in k or fewer balls is given by:

$$P(S \leq k) = P\left(\bigcup_{i=1}^{12} S_i \leq k\right).$$

Combining (1) and (2) and the entries from Table I above gives the single-game probability of a strike occurring in k or fewer turns:

$$P(S \leq k) = P\left(\bigcup_{i=1}^{12} S_i \leq k\right) = \sum_{i=1}^{12} (-1)^{i+1} \sum_{n=5}^{k} a_{n,i} \binom{25-n}{k-n}\binom{25}{k}^{-1}, \quad (3)$$

where the $a_{n,i}$ are precisely those entries appearing in the nth row and ith column of Table I. Evaluating (3) for $k =$

$5, 6, \ldots, 25$ gives the cumulative probability of achieving a strike in k or fewer turns, as shown in Table II.

TABLE II
SINGLE-PLAYER CUMULATIVE PROBABILITY DISTRIBUTION FOR THE NUMBER OF TURNS k TO COMPLETE A STRIKE

k	$P(S \leq k)$	k	$P(S \leq k)$	k	$P(S \leq k)$
5	0.000225861	12	0.173533835	19	0.987713156
6	0.001355167	13	0.272983482	20	0.999096556
7	0.004743083	14	0.402219231	21	1
8	0.012648221	15	0.554396163	22	1
9	0.028435982	16	0.712216253	23	1
10	0.056685716	17	0.849883734	24	1
11	0.103042132	18	0.943648845	25	1

The expected number of balls thrown until the first strike is achieved is 14.8972, and there is approximately a 55% chance that a player will have completed a strike in fifteen or fewer turns. By the time the player has thrown his nineteenth ball, there is effectively a 99% probability that he will have completed a strike. And since any 22-ball board is illegal, striking in 21 or fewer turns has a probability of exactly one.

C. Probabilities behind the m-player game

Unlike the case in BINGO, in I GOT IT!, each of the players' boards are independent until the first strike is achieved, ending the game. It is possible to use the binomial distribution to extend single-player winning probabilities to the m-player game. Given m players, r winners on a single turn, and the probability p of *no* single-game winner on a specific turn, the probability of exactly r winners is given by:

$$P(X = r) = \binom{m}{r} p^r (1 - p)^{m-r}$$

Here, the p-values—the probability of *no* single-game winner on a specific turn—are found by subtracting the $P(S \leq k)$ entries of Table II from unity. As an example, consider the case of eight players. By use of the binomial distribution, we obtain Table III, which gives the probability of zero, one, or multiple winners on a particular turn.

Table III shows, for example, that an eight-player game which has reached the seventh turn will end with a single winner, in that turn, in just over 3.5% of cases. However, the entries in Table III do not take into account the fact that any game may never reach the seventh turn. Therefore, it is necessary to multiply all of the entries in Table III by the probability that there were zero winners on each prior turn. For example, the probability that there is one winner on exactly the seventh turn in Table III is 0.0367. In order for the game even to have reached the seventh turn, there must have been no winners on the fifth turn, and no winners on the sixth turn. This gives the following:

Probability of *exactly* one winner for *exactly* $k = 7$
$$= 0.0367 \times 0.9982 \times 0.9892 = 0.0362.$$

TABLE III
BINOMIAL PROBABILITY OF ZERO, ONE, OR MULTIPLE WINNERS ON THE kTH TURN IN AN EIGHT-PLAYER GAME

k	0	1	Multiple Winners
4	1	0	0
5	0.9982	0.0018	≈ 0
6	0.9892	0.0107	0.0001
7	0.9627	0.0367	0.0006
8	0.9032	0.0926	0.0043
9	0.7939	0.1859	0.0202
10	0.6270	0.3014	0.0716
11	0.4190	0.3850	0.1960
12	0.2177	0.3656	0.4167
13	0.0780	0.2344	0.6875
14	0.0163	0.0878	0.8959
15	0.0016	0.0155	0.9830
16	4.7047E-05	0.0009	0.9990
17	2.5788E-07	1.1680E-05	≈ 1
18	1.0168E-10	1.3621E-08	≈ 1
19	5.1942E-16	3.3404E-13	≈ 1
20	4.4382E-25	3.9265E-21	≈ 1
21	0	0	1

Each of the Table III probabilities were re-calculated using the same recursive method. These conditional probabilities are listed in Table IV.

TABLE IV
CONDITIONAL BINOMIAL PROBABILITY OF ZERO, ONE, OR MULTIPLE WINNERS ON THE kTH TURN IN AN EIGHT-PLAYER GAME

k	0	1	Multiple Winners
4	1	0	0
5	0.9982	0.0018	1.4271E-06
6	0.9874	0.0107	5.1051E-05
7	0.9506	0.0362	0.0006
8	0.8585	0.0880	0.0040
9	0.6816	0.1596	0.0173
10	0.4273	0.2054	0.0488
11	0.1790	0.1645	0.0838
12	0.0390	0.0655	0.0746
13	0.0030	0.0091	0.0268
14	4.9596E-05	0.0003	0.0027
15	7.7098E-08	7.6737E-07	4.8752E-05
16	3.6272E-12	7.1814E-11	7.7022E-08
17	9.3539E-19	4.2366E-17	3.6272E-12
18	9.5108E-29	1.2741E-26	9.3539E-19
19	4.9401E-44	3.1770E-41	9.5108E-29
20	2.1926E-68	1.9398E-64	4.9401E-44
21	0	0	2.1926E-68

D. Payout scheme

In most carnival games, every play results in a unique winner, who is then awarded a prize—perhaps a stuffed animal or T-shirt. In a game of BINGO, the 'pot' is either claimed by a unique winner, or split among multiple winners after every game.[3] I GOT IT!, however, is a rather devious departure from this format. Recall that only a *unique* winner is awarded a prize of real monetary value; any game won simultaneously by multiple players does not result in a payout of cash value. Instead, the winners are each awarded one free game, and the next round starts,

[3]BINGO operations differ on policies concerning ties, i.e. cases when more than one winner achieves a BINGO on the same turn. In some establishments, the pot is split among the simultaneous winners. In others, it is only the first player to call BINGO! who receives the prize.

with another round of entry fees collected from all other players. For this reason, the respective relative frequencies of single- and multiple-winner games becomes important to the analysis of the payout scheme.

For the purposes of this paper, the probabilities behind zero winners on any turn k are important only in subsequent calculations for the probabilities of one or multiple winners. Let the sample space consist of only the single- and multiple-winner columns of Table IV. Such a sample space is reasonable, as the respective probabilities of these events do in fact sum to unity. The column totals of Table IV show that in an eight-player game, there is a 74.12% chance of a unique winner and 25.88% chance of multiple winners. In games for a selected number of players, the probabilities for single versus multiple winners are given in Table V.

TABLE V
PROBABILITY OF SINGLE VERSUS MULTIPLE WINNERS IN THE m-PLAYER GAME

Players	1 Winner	Multiple Winners
5	0.7724	0.2276
10	0.7280	0.2720
15	0.7054	0.2946
20	0.6902	0.3098
25	0.6785	0.3215
30	0.6691	0.3309
35	0.6611	0.3389
40	0.6542	0.3458
50	0.6426	0.3574
60	0.6330	0.3670
100	0.6056	0.3944
200	0.5664	0.4336
300	0.5429	0.4571
400	0.5245	0.4755
576	0.5000	0.5000
1000	0.4676	0.5324

When there are 576 players in the game (a number far greater than allowed by the sixty-player operations we observed), there is a 50% chance of either one or multiple winners. And as one might expect, the probability of multiple winners increases monotonically with the number of players. See Fig. 5.

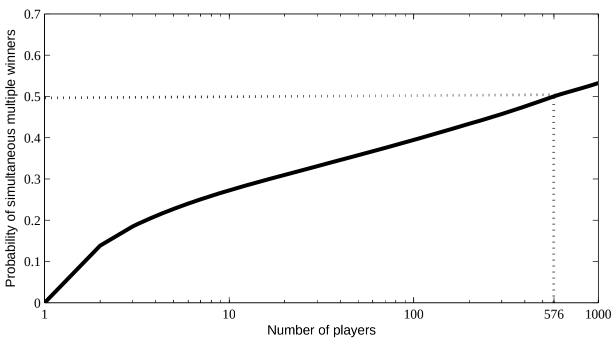


Fig. 5. Probability of multiple simultaneous winners as a function of the number of players.

E. Expected Game Length

Recall that in the single-player game, the expected number of turns required to achieve a strike was calculated to be 14.9872. In practice, this translates to an effective, or *real-world*, expectation of a fifteen-ball game. Cumulative binomial probabilities allows the extension of these expectations to a game of m-many players. One can expect a two-player game to be completed, on average, in twelve turns. Four players are the fewest required to reduce this expected game length to eleven turns; seven players are the fewest to lower expected game length to ten turns. Table VI lists the minimum number of players required to reduce effective expectation to the next lowest integer value of turns:

TABLE VI
LEAST NUMBER OF PLAYERS REQUIRED TO MANIFEST EFFECTIVE EXPECTED GAME LENGTH

Effective expectation of game length (turns)	Minimum players required
15	1
12	2
11	4
10	7
9	15
8	37
7	110
6	439
5	3,069

Note that whereas *unrounded* expected game length is a strictly decreasing function of the number of players, the *effective* expected game length attains its absolute minimum of five turns for $m = 3,069$ players and above.

III. CONCLUSION

In any carnival game, more players means more entry fees, and hence greater revenue for the game operator. Yet because of its unique payout scheme, I GOT IT! goes significantly further than that. In I GOT IT!, more players does mean greater revenue, as well as nonnegligibly shorter games (and thus more rounds of entry fees per unit time). But more players also translates to more ties, and more ties means *fewer winners* of prizes of cash value. And what of the free game coupons awarded to multiple winners? Assuming that the operation is not at capacity, these free games in fact only *increase* revenue, since more players lead to more ties, and more ties—got it?—lead to more money for the game operator.

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