World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:7, 2011

On fractional (k, m)-deleted graphs with constrains conditions

Sizhong Zhou, Hongxia Liu

Abstract—Let G be a graph of order n, and let $k \geq 2$ and $m \geq 0$ be two integers. Let $h : E(G) \rightarrow [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is called a fractional (k, m)-deleted graph if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. In this paper, it is proved that G is a fractional (k, m)-deleted graph if $\delta(G) \geq k + m + \frac{m}{k+1}$, $n \geq 4k^2 + 2k - 6 + \frac{(4k^2+6k-2)m-2}{k-1}$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x, y) = 2$. Furthermore, it is shown that the result in this paper is best possible in some sense.

Keywords—graph, degree condition, fractional k-factor, fractional (k, m)-deleted graph.

I. INTRODUCTION

IN this paper, we consider finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G-S = G[V(G) \setminus S]$. Let S and T be two disjoint vertex subsets of G, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other end in T. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. We define the distance $d_G(x, y)$ between two vertices x and y as the minimum of the lengths of the (x, y) paths of G.

Let $k \geq 1$ be an integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \rightarrow [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. Zhou [1] introduced firstly the definition of a fractional (k, m)-deleted graph, that is, a graph G is called a fractional (k, m)-deleted graph if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. A fractional (k, m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$.

Sizhong Zhou is with the School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, People's Republic of China, e-mail: zsz_cumt@163.com.

Hongxia Liu is with the School of Mathematics and Informational Science, Yantai University, Yantai, Shandong 264005, People's Republic of China, e-mail: mqy7174@sina.com

Manuscript received January 22, 2010; revised September 16, 2010.

Some people studied graph factors [2–6]. Yu and Liu [7] obtained a Fan-type condition for a graph to have a fractional k-factor. Liu and Zhang [8] gave a toughness condition for a graph to have a fractional k-factor. Cai and Liu [9] showed a stability number condition for graphs to have fractional k-factors. Zhou [10] obtained some sufficient conditions for graphs to have fractional k-factors. Zhou [1,11] obtained two sufficient conditions for graphs to be fractional (k, m)-deleted graphs.

The following results on k-factors, fractional k-factors and fractional (k, m)-deleted graphs are known.

Theorem 1. ([12]). Let G be a connected graph of order n with $\delta(G) \geq k$, where k is a positive integer, kn is even and $n \geq 8k^2 + 12k + 6$. If G satisfies

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x, y) = 2$, then G has a k-factor.

Theorem 2. ([7]). Let G be a connected graph of order n with $\delta(G) \geq k$, where k is a positive integer and $n \geq 8k^2 + 12k + 6$. If G satisfies

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x, y) = 2$, then G has a fractional k-factor.

Theorem 3. ([1]). Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph of order n with $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2} + 2(2k+1)m$, $\delta(G) \geq k + m + \frac{(m+1)^2-1}{4k}$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

Theorem 4. ([11]). Let $k \geq 1$ and $m \geq 1$ be two integers. Let G be a graph of order n with $n \geq 4k - 5 + 2(2k+1)m$. If

$$\delta(G) \geq \frac{n}{2},$$

then G is a fractional (k, m)-deleted graph.

In this paper, we proceed to study the fractional (k, m)-deleted graphs and obtain a Fan-type condition for a graph to be a fractional (k, m)-deleted graph. Our result is the following theorem which is an extension of Theorems 1 and 2.

Theorem 5. Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph of order n with $n \geq 4k^2+2k-6+\frac{(4k^2+6k-2)m-2}{k-1}$. If $\delta(G) \geq k+m+\frac{m}{k+1}$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x,y)=2$, then G is a fractional (k,m)-deleted graph.

Our result is stronger than Theorem 4 if $k \geq 2$ and the order n is sufficiently large. Set $m=0$ in Theorem 5. Then we get the following corollary.

Corollary 1. Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k^2+2k-6$. If $\delta(G) \geq k$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x,y)=2$, then G has a fractional k-factor.

Obviously, the result of Corollary 1 is stronger than Theorem 2 if $k \geq 2$.

II. The Proof of Theorem 5

The proof of Theorem 5 relies heavily on the following lemma.

Lemma 2.1. ([1]). Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k,m)-deleted graph if and only if for any subset S of $V(G)$

$$k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \geq 0,$$

where $T = \{x : x \in V(G)\setminus S, d_{G-S}(x) - d_H(x) + e_H(x,S) \leq k-1\}$.

Proof of Theorem 5. Suppose that G satisfies the assumption of Theorem 5, but is not a fractional (k,m)-deleted graph. Then from Lemma 2.1, there exists some subset S of $V(G)$ such that

$$k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \leq -1, \quad (1)$$

where $T = \{x : x \in V(G)\setminus S, d_{G-S}(x) - d_H(x) + e_H(x,S) \leq k-1\}$ and H is some subgraph of G with m edges. It is easy to see that $d_{G-S}(x) - d_H(x) + e_H(x,S) \geq 0$ for any $x \in V(G)$. Since $|E(H)| = m$, we have $\sum_{x\in T} d_H(x) - e_H(S,T) \leq 2m$.

Now, we prove the following claims.

Claim 1. $1 \leq |S| < \frac{n}{2}$.

Proof. If $S = \emptyset$, then from (1), $\delta(G) \geq k+m+\frac{m}{k+1} \geq k+m$ and $d_H(x) \leq m$ for any $x \in V(G)$, we get $-1 \geq \sum_{x\in T}(d_G(x) - d_H(x) - k) \geq \sum_{x\in T}(\delta(G) - m - k) \geq 0$, a contradiction. Hence, $|S| \geq 1$.

On the other hand, according to (1), $|S|+|T| \leq n$ and $d_{G-S}(x) - d_H(x) + e_H(x,S) \geq 0$ for any $x \in V(G)$, we obtain

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&\geq k|S| - k|T| \geq k|S| - k(n-|S|) = 2k|S| - kn,
\end{aligned}$$

which implies $|S| < \frac{n}{2}$. This completes the proof of Claim 1.

Claim 2. $|T| > |S|$.

Proof. In terms of (1) and $d_{G-S}(x) - d_H(x) + e_H(x,S) \geq 0$ for any $x \in V(G)$, we have that $-1 \geq k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \geq k|S| - k|T|$, which implying $|T| > |S|$. The proof of Claim 2 is complete.

Claim 3. $|T| \geq k+1$.

Proof. Suppose that $|T| \leq k$. Then from (1), $\delta(G) \geq k+m+\frac{m}{k+1} \geq k+m$ and $d_H(x) \leq m$ for any $x \in V(G)$, we obtain

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&\geq |T||S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&= \sum_{x\in T}(|S| + d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&\geq \sum_{x\in T}(\delta(G) - m - k) \geq 0,
\end{aligned}$$

which is a contradiction. This completes the proof of Claim 3.

Claim 4. $|S| < \frac{n}{2} - (k+m-1)$.

Proof. Suppose that $|S| \geq \frac{n}{2} - (k+m-1)$. Then using (1), $|S|+|T| \leq n$ and $\sum_{x\in T} d_H(x) - e_H(S,T) \leq 2m$, we have

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&= k|S| + \sum_{x\in T} d_{G-S}(x) - (\sum_{x\in T} d_H(x) - e_H(S,T)) \\
&\quad -k|T| \\
&\geq k|S| + \sum_{x\in T} d_{G-S}(x) - 2m - k|T| \\
&\geq k|S| + \sum_{x\in T} d_{G-S}(x) - 2m - k(n-|S|) \\
&= 2k|S| + \sum_{x\in T} d_{G-S}(x) - 2m - kn \\
&\geq 2k(\frac{n}{2} - (k+m-1)) + \sum_{x\in T} d_{G-S}(x) - 2m - kn \\
&= -2k(k+m-1) + \sum_{x\in T} d_{G-S}(x) - 2m,
\end{aligned}$$

that is,

$$\sum_{x\in T} d_{G-S}(x) \leq 2k(k+m-1) + 2m - 1.$$

Consequently, it follows from Claim 2 and $n \geq 4k^2+2k-6+\frac{(4k^2+6k-2)m-2}{k-1}$ that

$$\begin{aligned}
\frac{\sum_{x\in T} d_{G-S}(x)}{|T|} &\leq \frac{2k(k+m-1)+2m-1}{|S|+1} \\
&\leq \frac{2k(k+m-1)+2m-1}{\frac{n}{2} - (k+m-1) + 1} \\
&\leq 1 - \frac{1}{k}.
\end{aligned}$$

Combining the inequalities above with Claim 3, we obtain

$$\sum_{x\in T} d_{G-S}(x) \le (1-\frac{1}{k})|T| = |T| - \frac{1}{k}|T| < |T|-1. \quad (2)$$

Set $T_0 = \{x : x \in T, d_{G-S}(x) = 0\}$. Note that $|T_0| \ge 2$ holds by (2). For each $x \in T_0$, $d_G(x) \le |S| < \frac{n}{2}$ by Claim 1. Since T_0 is an independent set of G and G satisfies the assumption of Theorem 5, the neighborhoods of the vertices in T_0 are disjoint. Therefore, we obtain

$$\begin{aligned} |S| &\ge |\bigcup_{x\in T_0} N_G(x)| \ge \delta(G)|T_0| \\ &\ge (k+m+\frac{m}{k+1})|T_0| \ge (k+m)|T_0|. \quad (3) \end{aligned}$$

Using (2) and the definition of T_0, we have

$$(1-\frac{1}{k})|T| \ge \sum_{x\in T} d_{G-S}(x) \ge |T| - |T_0|,$$

which implies

$$|T_0| \ge \frac{1}{k}|T|. \quad (4)$$

According to (3) and (4), we get

$$|S| \ge (k+m)|T_0| \ge (1+\frac{m}{k})|T| \ge |T|.$$

That contradicts Claim 2. This completes the proof of Claim 4.

Claim 5. $e_G(S,T) \le (k+m)|S|$.

Proof. Since $d_{G-S}(x) - d_H(x) + e_H(x,S) \le k-1$ for each $x \in T$ and $d_H(x) \le m$, we have $d_{G-S}(x) \le k+m-1$ for each $x \in T$. Combining this with Claim 4, we obtain

$$d_G(x) \le d_{G-S}(x) + |S| < \frac{n}{2} \quad (5)$$

for each $x \in T$. From (5) and the assumption of Theorem 5, $G[N_G(s) \cap T]$ is a complete induced subgraph of G for each $s \in S$. Note that $S \ne \emptyset$ by Claim 1. Thus, by $d_{G-S}(x) \le k+m-1$ for each $x \in T$, we have

$$e_G(s,T) \le \Delta(G[T]) + 1 \le k+m.$$

Hence, we obtain

$$e_G(S,T) \le (k+m)|S|.$$

The proof of Claim 5 is complete.

According to (1), $\sum_{x\in T} d_H(x) - e_H(S,T) \le 2m$, $\delta(G) \ge k+m+\frac{m}{k+1}$, Claim 2, Claim 3 and Claim 5, we have

$$\begin{aligned} -1 &\ge k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\ &\ge k|S| + \sum_{x\in T}(d_{G-S}(x) - k) - 2m \\ &= k|S| + \sum_{x\in T}(d_G(x) - k) - e_G(S,T) - 2m \\ &\ge k|S| + \sum_{x\in T}(\delta(G) - k) - (k+m)|S| - 2m \\ &\ge k|S| + (k+m+\frac{m}{k+1} - k)|T| \\ &\quad -(k+m)|S| - 2m \\ &= m(|T|-|S|) + \frac{m}{k+1}|T| - 2m \\ &\ge 0, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 5.

III. Remark

In Theorem 5, the bound in the assumption

$$\max\{d_G(x), d_G(y)\} \ge \frac{n}{2}$$

is best possible in the sense that we cannot replace $\frac{n}{2}$ by $\frac{n}{2}-1$. We can show this by constructing a graph $G = ktK_1 \vee (kt+1)K_1$, where $k \ge 2$ and $m \ge 0$ are two integers and t is enough large positive integer. Then it follows that $|V(G)| = n = 2kt+1$ and

$$\frac{n}{2} > \max\{d_G(x), d_G(y)\} = kt > \frac{n}{2} - 1$$

for any two vertices vertices x, y of $(kt+1)K_1 \subset G$ with $d_G(x,y) = 2$. Let $S = V(ktK_1) \subseteq V(G)$, $T = V((kt+1)K_1) \subseteq V(G)$ and H is any subgraph of G with m edges. Then $|S| = kt$, $|T| = kt+1$, $d_{G-S}(T) = 0$ and $\sum_{x\in T} d_H(x) - e_H(S,T) = 0$. Thus, we get

$$\begin{aligned} & k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\ = & k^2t - k(kt+1) = -k < 0. \end{aligned}$$

In terms of Lemma 2.1, G is not a fractional (k,m)-deleted graph.

Acknowledgment

This research was sponsored by Qing Lan Project of Jiangsu Province, and was supported by Jiangsu Provincial Educational Department (07KJD110048) and Shandong Province Higher Educational Science and Technology Program (J10LA14)

References

[1] S. Zhou, *A neighborhood condition for graphs to be fractional (k,m)-deleted graphs*, Glasgow Mathematical Journal 52(1)(2010), 33–40.
[2] S. Zhou, *Independence number, connectivity and (a,b,k)-critical graphs*, Discrete Mathematics 309(12)(2009), 4144–4148.
[3] S. Zhou and Q. Shen, *On fractional (f,n)-critical graphs*, Information Processing Letters 109(14)(2009), 811–815.
[4] H. Matsuda, *Fan-type results for the existence of $[a,b]$-factors*, Discrete Mathematics 306(2006), 688–693.
[5] J. R. Correa and M. Matamala, *Some remarks about factors of graphs*, Journal of Graph Theory 57(2008), 265–274.
[6] H. Liu and G. Liu, *Binding number and minimum degree for the existence of (g,f,n)-critical graphs*, Journal of Applied Mathematics and Computing 29(1-2)(2009), 207–216.
[7] J. Yu and G. Liu, *Fractional k-factors of graphs*, Chinese Journal of Engineering Mathematics 22(2)(2005), 377–380.
[8] G. Liu and L. Zhang, *Toughness and the existence of fractional k-factors of graphs*, Discrete Mathematics 308(2008), 1741–1748.
[9] J. Cai and G. Liu, *Stability number and fractional f-factors in graphs*, Ars Combinatoria 80(2006), 141–146.
[10] S. Zhou, *Some results on fractional k-factors*, Indian Journal of Pure and Applied Mathematics 40(2)(2009), 113–121.
[11] S. Zhou, *A minimum degree condition of fractional (k,m)-deleted graphs*, Comptes rendus Mathematique 347 (21-22)(2009), 1223–1226.
[12] T. Niessen, *A Fan-type result for regular factors*, Ars combinatoria, 46(1997), 277–285.