Additional Considerations on a Sequential Life Testing Approach using a Weibull Model

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***Abstract*—**In this paper we will develop further the sequential life test approach presented in a previous article by [1] using an underlying two parameter Weibull sampling distribution. The minimum life will be considered equal to zero. We will again provide rules for making one of the three possible decisions as each observation becomes available; that is: accept the null hypothesis H_0; reject the null hypothesis H_0; or obtain additional information by making another observation. The product being analyzed is a new type of a low alloy-high strength steel product. To estimate the shape and the scale parameters of the underlying Weibull model we will use a maximum likelihood approach for censored failure data. A new example will further develop the proposed sequential life testing approach.

***Keywords*—**Sequential Life Testing, Underlying Weibull Model, Maximum Likelihood Approach, Hypothesis Testing.

I. Introduction

THE two-parameter Weibull distribution is widely used as a failure model, particularly for mechanical and metallurgical components. It has a shape parameter β, which specifies the shape of the distribution, and a scale parameter θ, which represents the characteristic life of the distribution. Both parameters are positive.The Weibull density function *f(t)*is given by.

$$f(t) = \frac{\beta}{\theta^{\beta}} (t_i)^{\beta-1} e^{-(t/\theta)^{\beta}}; \quad t \geq 0 \qquad (1)$$

Here, *t*representsthe time to failure of a component or part. The scale parameter θ (the characteristic life) is positive and is the 63.21 percent point of the distribution of T. The shape parameter β which is also positive, specifies the shape of the distribution. As β increases the mode of the distribution approaches the scale parameter θ. The hypothesis testing situations will be given by:

1. For the scale parameter θ: H_0: $\theta \geq \theta_0$; H_1: $\theta < \theta_0$

The probability of accepting the null hypothesis H_0 will be set at $(1-\alpha)$ if $\theta = \theta_0$. Now, if $\theta = \theta_1$ where $\theta_1 < \theta_0$, then the probability of accepting H_0 will be set at a low level γ. H_1 represents the alternative hypothesis.

2. For the shape parameter β: H_0: $\beta \geq \beta_0$; H_1: $\beta < \beta_0$

The probability of accepting H_0will be set again at $(1-\alpha)$in the case of $\beta = \beta_0$. Now, if $\beta = \beta_1$, where $\beta_1 < \beta_0$, then the probability of accepting H_0 will also be set at a low level γ.

II. Sequential Testing

The development of a sequential test uses the likelihood ratio (*LR*) given by the following relationship proposed by [2] and [3]:

$$LR = L_{1;n}/L_{0;n}$$

The sequential probability ratio (*SPR*) will be given by:

$$SPR = L_{1;n}/L_{0;n}$$

Based on the paper from [4], for the Weibull case the (*SPR*) will be given by:

$$\text{SPR} = \left(\frac{\beta_1}{\theta_1^{\beta_1}} \times \frac{\theta_0^{\beta_0}}{\beta_0} \right)^n \prod_{i=1}^{n} \left(t_i \right)^{\beta_1 - \beta_0} \times$$

$$\times \exp\left[-\sum_{i=1}^{n} \left(\frac{t_i^{\beta_1}}{\theta_1^{\beta_1}} - \frac{t_i^{\beta_0}}{\theta_0^{\beta_0}} \right) \right]$$

So, the continue region becomes $A < SPR < B$, where $A = \gamma/(1-\alpha)$ and $B = (1-\gamma)/\alpha$. We will accept the null hypothesis H_0 if $SPR \geq B$ and we will reject H_0 if $SPR \leq A$. Now, if $A < SPR < B$, we will take one more observation. Then, we get:

$$n \ln\left(\frac{\beta_1}{\theta_1^{\beta_1}} \times \frac{\theta_0^{\beta_0}}{\beta_0} \right) - \ln\left[\frac{(1-\gamma)}{\alpha} \right] < \text{W} <$$

$$< n \ln\left(\frac{\beta_1}{\theta_1^{\beta_1}} \times \frac{\theta_0^{\beta_0}}{\beta_0} \right) + \ln\left[\frac{(1-\alpha)}{\gamma} \right] \qquad (2)$$

$$\text{W} = \sum_{i=1}^{n} \left(\frac{t_i^{\beta_1}}{\theta_1^{\beta_1}} - \frac{t_i^{\beta_0}}{\theta_0^{\beta_0}} \right) + (\beta_0 - \beta_1) \sum_{i=1}^{n} \ln(t_i) \qquad (3)$$

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III. The Maximum Likelihood Approach

According to [1], the maximum likelihood estimator for the shape and scale parameters of a two parameter Weibull sampling distribution is given by:

$$\frac{dL}{d\theta} = -\frac{r\beta}{\theta}\ \frac{\beta \times \sum_{i=1}^{r}(t_i)^{\beta}}{\theta^{\beta+1}} + \frac{\beta(n-r)(t_r)^{\beta}}{\theta^{\beta+1}} = 0 \qquad (4)$$

$$\frac{dL}{d\beta} = \frac{r}{\beta} - r\ln(\theta) + \sum_{i=1}^{r}\ln(t_i) - \sum_{i=1}^{r}\left(\frac{t_i}{\theta}\right)^{\beta} \times \ln\left(\frac{t_i}{\theta}\right) - (n-r)\left(\frac{t_r}{\theta}\right)^{\beta}\ln\left(\frac{t_r}{\theta}\right) = 0 \qquad (5)$$

From (4) we obtain:

$$\theta = \left(\left[\sum_{i=1}^{r}(t_1)^{\beta} + (n-r)(t_r)^{\beta}\right]\Big/ r\right)^{1/\beta} \qquad (6)$$

Using (6) for θ in (5) and after some mathematical manipulation, (5) reduces to:

$$\frac{r}{\beta} + \sum_{i=1}^{r}\ln(t_i) - \frac{r \times \left[\sum_{i=1}^{r}(t_i)^{\beta}\ln(t_i) + (n-r)(t_r)^{\beta}\ln(t_r)\right]}{\sum_{i=1}^{r}(t_i)^{\beta} + (n-r)(t_r)^{\beta}} = 0 \qquad (7)$$

Equation (7) must be solved iteratively.

In a previous article [1] an example was presented to illustrate the proposed approach. We will now analyze four new different situations for the hypothesis testing considered in this paper.

IV. Example

A new low alloy-high strength steel product will be life tested. Since this is a new product, there is little information available about the possible values that the parameters of the corresponding Weibull underlying sampling distribution could have. To estimate the shape and the scale parameters of this sampling model we will use a maximum likelihood approach for censored failure data. Some preliminarily life testing was performed in order to determine an estimated value for the two Weibull parameters. Using the maximum likelihood estimator approach we obtained the following values for these parameters:

$$\theta = 2{,}500{,}000 cycles;\quad \beta = 2.5$$

It was decided that $\alpha = 0.05$ and $\gamma = 0.10$. Initially, we elect the null hypothesis parameters to be θ_0 = 2,500,000 cycles; with $\beta_0 = 2.5$; $\alpha = 0.05$ and $\gamma = 0.10$ and choose some possible values for the alternative parameters θ_1 and β_1, and see how this choice will alter the results of the test. After that, we will change the values of the null hypothesis parameters and verify how the test results will behave. So, we choose θ_1 = 2,000,000 cycles and $\beta_1 = 1.5$. Then, using (2) and (3), we have:

$$n ln\left(\frac{1.5}{2{,}000{,}000^{\,1.5}} \times \frac{2{,}500{,}000^{\,2.5}}{2.5}\right) - ln\left[\frac{(1-0.10)}{0.05}\right] =$$

$$n \times 14.55569 - 2.89037$$

$$n ln\left(\frac{1.5}{2{,}000{,}000^{\,1.5}} \times \frac{2{,}500{,}000^{\,2.5}}{2.5}\right) + ln\left[\frac{(1-0.05)}{0.10}\right] =$$

$$n \times 14.55569 + 2.25129$$

Then, we get:

$$n \times 14.55569 - 2.89037 < W < n \times 14.55569 + 2.25129$$

$$W = \sum_{i=1}^{n}\left(\frac{t_i^{\,1.5}}{2{,}000{,}000^{\,1.5}} - \frac{t_i^{\,2.5}}{2{,}500{,}000^{\,2.5}}\right) + 1.0 \times \sum_{i=1}^{n} ln(t_i)$$

After a sequential test graph has been developed for this life-testing situation, a random sample is taken.

The procedure is then defined by the following rules:

1. If $W \geq n \times 14.5557 + 2.2513$, we will accept H_0.
2. If $W \leq n \times 14.5557 - 2.8904$, we will reject H_0.
3. If $n \times 14.5557 - 2.8904 < W < n \times 14.5557 + 2.2513$, we will take one more observation.

In this first case, 7 units were tested to allow the decision of accepting the null hypothesis H_0. These values for the corresponding number of cycles (time to failure) of these 7 units were the following: *2,467,263.2; 1,574,362.1; 2,010,281.3; 2,361,826.1; 1,016,274.8; 2,605,312.2; 2,663,115.0* cycles. Table I and Fig. 1 show the results of this test.

TABLE I
SEQUENTIAL TEST RESULTS FOR THE WEIBULL MODEL
$\beta_1 = 1.5$; $\theta_1 = 2{,}000{,}000$ $\beta_0 = 2.5$; $\theta_0 = 2{,}500{,}000$

Unit Nº	Lower Limit	Upper Limit	Value of W
1	11.66532	16.80698	15.12122
2	26.22101	31.36267	29.77428

3	40.77670	45.91837	44.71597
4	55.33239	60.47406	59.80671
5	69.88808	75.02975	73.89523
6	84.44377	89.58543	89.04640
7	98.99946	104.14110	104.2067

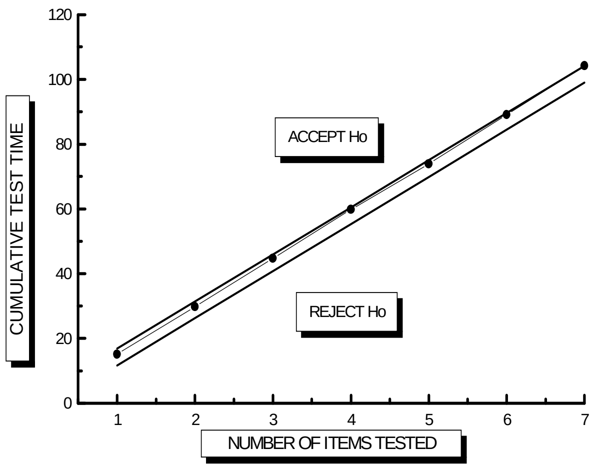

Fig. 1Sequential test graph for the two-parameter Weibull model

Next, we modify the value of β_1, the shape parameter of the alternative hypothesis, making it closer to the value of β_0, the shape parameter of the null hypothesis. So, we choose the value of 2.0 for β_1. Table II shows the results of this test.

TABLEII
SEQUENTIAL TEST RESULTS FOR THE WEIBULL MODEL
$\beta_1 = 2.0$; $\theta_1 = 2{,}000{,}000$ $\beta_0 = 2.5$; $\theta_0 = 2{,}500{,}000$

Unit Number	Lower Limit	Upper Limit	Value of W
1	4.698672	9.840336	7.913573
2	12.28772	17.42938	15.35320
3	19.87676	25.01842	23.04058
4	27.46581	32.60747	30.90511
5	35.05485	40.19651	37.97378
6	42.64389	47.78556	45.94856
7	50.23294	55.37460	53.94793
8	57.82198	62.96365	61.72543
9	65.41103	70.55269	69.74948
10	73.00007	78.14173	77.82352
11	80.58911	85.73077	85.65101
12	88.17815	93.31982	93.91177

The choice for the value of the alternative shape parameter (β_1 = 2.0), being closer to the value of the null hypothesis shape parameter (β_0 = 2.5) made it necessary to continue the test through 12 units, so a decision could be made to accept the null hypothesis.

Now, we decided to verify if a null shape parameter value relatively wrong will cause the null hypothesis to be rejected by this sequential life testing procedure. We choose the following values for the alternative and null shape parameters β_1 and β_0 (β_1 = 2.5; β_0 = 3.5). Table III shows the results of this test.

TABLE III
SEQUENTIAL TEST RESULTS FOR THE WEIBULL MODEL
$\beta_1 = 2.5$; $\theta_1 = 2{,}000{,}000$ $\beta_0 = 3.5$; $\theta_0 = 2{,}500{,}000$

Unit Number	Lower Limit	Upper Limit	Value of W
1	10.72396	15.86562	13.60100
2	24.33828	29.47994	27.75229
3	37.95261	43.09427	41.82779
4	51.56693	56.70860	55.58061
5	65.18126	70.32292	69.42410
6	78.79559	83.93725	82.78290
7	92.40991	97.55157	96.02476
8	106.0242	111.1659	109.9740
9	119.6386	124.7802	123.0851
10	133.2529	138.3946	135.8842
11	146.8672	152.0089	149.7309
12	160.4816	165.6232	160.3708

In this third case, 12 units had to be life-tested to allow the decision of rejecting the null hypothesis H_0. A relatively poor choice of the value for the null shape parameter (β_0 = 3.5), caused this rejection. So, we can verify that this sequential life testing procedure is shown to be sensitive to “wrong” choices for the null shape parameter values.

Finally, we decided to verify if a null scale parameter value relatively wrong will cause the null hypothesis to be rejected by this sequential life testing procedure. We choose the following values for the alternative and null scale parameters θ_1 and θ_0 (θ_1 = 2,500,000 cycles; θ_0 = 2,000,000 cycles).

Table IV shows the results of this test.

TABLEIV
SEQUENTIAL TEST RESULTS FOR THE WEIBULL MODEL
$\beta_1 = 2.0$; $\theta_1 = 2{,}500{,}000$ $\beta_0 = 2.5$; $\theta_0 = 2{,}000{,}000$

Unit Number	Lower Limit	Upper Limit	Value of W
1	3.694526	8.836189	6.642992
2	10.27942	15.42109	13.62447
3	16.86432	22.00599	20.51506
4	23.44922	28.59088	27.22959
5	30.03412	35.17579	34.12661
6	36.61901	41.76068	40.66241
7	43.20391	48.34558	47.14869
8	49.78810	54.93048	53.96407
9	56.37371	61.51538	60.39636
10	62.95861	68.10027	66.70446
11	69.54350	74.68517	73.46568
12	76.12840	81.27007	79.01801
13	82.71330	87.85497	84.12077
14	89.29819	94.43986	91.06505
15	95.88309	101.0248	97.96452
16	102.4680	107.6097	104.3059
17	109.0529	114.1946	111.2875
18	115.6378	120.7795	117.6909
19	122.2227	127.3643	124.5001
20	128.8076	133.9493	131.3394

In this last case, even after 20 units have been life tested, it is not possible to make the decision of accepting the null hypothesis H_0 or rejecting the null hypothesis H_0.

A relatively poor choice of the value for the null scale parameter (θ_0 = 2,000,000 cycles), has caused this impasse. In cases like this, instead of obtaining additional information by making another observation, it seems more logical to apply the stopping-rule mechanism (truncation) presented in [1].

V. CONCLUSIONS

The sequential life testing approach developed in this work provides rules for working with the null hypothesis H_0 in situations where the underlying sampling distribution is the Weibull model. After each observation one of three possible decisions is made:

1. Accept the null hypothesis H_0.
2. Reject the null hypothesis H_0.
3. Take one more observation.

In the example presented, we analyzed 4 different situations for the hypothesis testing considered in this paper. Table I shows the sequential test results for the Weibull distribution, where β_1 = 1.5; θ_1 = 2,000,000 cycles; β_0 =2.5; θ_0 = 2,500,000 cycles.

In this first case it was necessary to use only 7 units of the product under analysis to reach the decision to accept the null hypothesis H_0.

In the second case, the test had to be continued through 12 units before a decision could be made to accept the null hypothesis. Table II shows the results of the test when β_1 = 2.0; θ_1 = 2,000,000 cycles;β_0 = 2.5; θ_0 = 2,500,000 cycles. We used the value of the alternative shape parameter (β_1 = 2.0) because it is closer to the value of the null hypothesis shape parameter β_0 = 2.5 (the value we believe to be "true" for this parameter).

Table III shows the results of the test when we have β_1 = 2.5; θ_1 = 2,000,000 cycles; β_0 = 3.5; θ_0 = 2,500,000 cycles. In this third case, 12 units had to be life-tested to allow the decision of rejecting the null hypothesis H_0. A relatively poor choice of the value for the null shape parameter (β_0 = 3.5), caused this rejection. So, we can verify that this sequential life testing procedure is shown to be sensitive to "wrong" choices for the null shape parameter values.

Finally, Table IV shows the results of the test when β_1 = 2.0; θ_1 = 2,500,000 cycles;β_0 = 2.5; θ_0 = 2,000,000 cycles. In this last case, even after 20 units have been life tested, it is not possible to make the decision of accepting the null hypothesis H_0 or rejecting the null hypothesis H_0. A relatively poor choice of the value for the null scale parameter (θ_0 = 2,000,000 cycles), has caused this impasse. In cases like this, instead of obtaining additional information by making another observation, it seems more logical to apply the stopping-rule mechanism (truncation) presented in [1]. This fact shows the advantage of such a truncation mechanism to be used in a sequential life test approach.

REFERENCES

[1] D. I. De Souza, D. R. Fonseca and D. Kipper, "Appling Eyring's Accelerated Life Test Model to Times to Breakdown of Insulating Fluid: A Combined Approach of an Accelerated and a Sequential Life Testing,", *World Academy of Science, Engineering and Tecknology:* Year 6, Issue 7, pp. 77-82, September 2010.

[2] K. Kapur andL. R. Lamberson,*Reliability in Engineering Design.* New York:John Willey & Sons, Inc., 1977.

[3] Daniel I. De Souza,"Sequential Life Testing with a Truncation Mechanism for an Underlying Weibull Model,"*Towards a Safer World, ESREL 2001 Conference*, Torino, Italy, vol. 3, pp. 1539 –1546, 16–20 September, 2001.

[4] Daniel I. De Souza, "Sequential Life-Testing with Truncation Mechanisms for Underlying Three-Parameter Weibull and Inverse Weibull Models," *COMADEM 2004 Conference,*Cambridge, U.K.,pp. 260-271, August 2004.