Torsional Statics of Circular Nanostructures: Numerical Approach

M.Z. Islam, C.W. Lim

Abstract**—**Based on the standard finite element method, a new finite element method which is known as nonlocal finite element method (NL-FEM) is numerically implemented in this article to study the nonlocal effects for solving 1D nonlocal elastic problem. An Eringen-type nonlocal elastic model is considered. In this model, the constitutive stress-strain law is expressed interms of integral equation which governs the nonlocal material behavior. The new NL-FEM is adopted in such a way that the postulated nonlocal elastic behavior of material is captured by a finite element endowed with a set of (cross-stiffness) element itself by the other elements in mesh. An example with their analytical solutions and the relevant numerical findings for various load and boundary conditions are presented and discussed in details. It is observed from the numerical solutions that the torsional deformation angle decreases with increasing nonlocal nanoscale parameter. It is also noted that the analytical solution fails to capture the nonlocal effect in some cases where numerical solutions handle those situation effectively which prove the reliability and effectiveness of numerical techniques.

Keywords**—**NL-FEM, nonlocal elasticity, nanoscale, torsion.

I. Introduction

MECHANICAL behavior of materials or structures and their understanding on a smaller and smaller length scale (i.e. micro- and nano-scale) are important in the design of micro-electro-mechanical-systems (MEMS) and nano-electro-mechanical-systems (NEMS). When the size of a body or a structure enters the micro- and nano-ranges, the material exhibits specific and interesting nonclassical mechanical, chemical, electrical properties etc. Due to the continuous reduction of device size into micro- and nano-scale in the present decade, the classical continuum theory is unable to predict the increasingly prominence of size effects [1]-[3] because it is a scale free theory. There are typically three approaches in the study of size effects in nanomechanics, i.e. experiment, numerical atomic-scale simulation and scale-dependent continuum mechanics method. Because control experiments in nanoscale are frequently very difficult and numerical atomic-scale simulations are highly computationally expensive, the scale-dependent continuum mechanics methods have been widely used not only due to its simplicity but also its possibility of deriving accurate analytical solutions. Based on the physical bases and constitutive aspects, the fundamental works on nonlocal elasticity which is a scale-dependent continuum mechanics model were done in a few papers [4]-[7]. Further improvements on nonlocal elasticity were presented in [8]-[13]. Nonlocal elasticity model of Eringen [11], [12] and his associate [13] is developed based on the assumption that the stress tensor at a point in an elastic continuum not only depends on the strain at that point but also depends on strains at all other points in the body. Consequently in the constitutive relation of a nonlocal theory Hooke's law (for local theory) is replaced by integration.

M.Z. Islam and C.W. Lim are with the Department of Civil and Architectural Engineering, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong, P.R. China (E-mail: jahurul_lec_ap@yahoo.com and bccwlim@cityu.edu.hk).

As the constitutive relation is of the integral form, the subsequent integro-partial differential equations interms of displacement field in the nonlocal elasticity are extremely difficult to solve. Under certain conditions using Green's function with a certain approximation errors, Eringen [14] has transformed the integral from into differential constitutive relation. Although this differential constitutive relation has been extensively used in [15]-[19] to study the mechanical properties of nanomaterial or nanostructure, the exact nonlocal boundary effects presented by the integration of kernel function in the integral formulations are not perfectly matched or transformed in the differential form which is a weak point of approximations. For this purpose, based on the standard finite element method, Polizzotto [20] has extended the classical principles of local theory for the nonlocal theory and this technique offers a fundamental base for nonlocal finite element method.

Although, research on transverse bending, buckling, vibration, wave propagation etc. is aplenty, few studies on torsional behaviors are found at present. Because torsional static is common for NEMS and some other nano-devices, their effects and behavior should not be discounted. In this paper, the torsional statics of circular nanostructures in the presence of combined distributed torque and fixed external end torque is investigated based on the nonlocal finite element method of Polizzotto [20]. In addition, the corresponding nonlocal equation of motion is also derived based on the differential constitutive relation [14]. To illustrate the accuracy of the present method, the obtained results are compared with those predicted by the analytical solutions.

II. Mathematical Formulations

A. Constitutive Relation

According to Eringen and co-worker [21], [22], the nonlocal theory differs from the local one only for the stress-strain constitutive relation. Based on this assumption, they developed a simplified elastic theory for linear homogeneous isotropic continua and the stress-strain relation for such a continuum is indeed assumed in the form:

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D} : \hat{\boldsymbol{\varepsilon}}(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \mathrm{V} \tag{1}$$

where V is the domain; $\mathbf{x}$ is a vector in this domain; $\mathbf{D}$ is the elastic moduli fourth-order tensor of isotropic local elasticity; $\boldsymbol{\sigma}(\mathbf{x})$ is the second-order tensor representing the stress field at $\mathbf{x}$; $\hat{\varepsilon}(\mathbf{x})$ is the second-order strain tensor representing the nonlocal strain field at $\mathbf{x}$. It is noted that the nonlocal strain at the field point $\mathbf{x}$ is the sum of the strain arising at $\mathbf{x}$ itself and the strain at $\mathbf{x}$ induced by strain arising at all $\mathbf{x}' \neq \mathbf{x}$ in V. In general, the second contribution can be expressed interms of integral which governs the nonlocal behavior of the material in the constitutive relation. According to Eringen [23] and Altan [24], the nonlocal strain, namely $\hat{\varepsilon}(\mathbf{x})$, is assumed in the form:

$$\hat{\boldsymbol{\varepsilon}}(\mathbf{x}) = \zeta_1 \boldsymbol{\varepsilon}(\mathbf{x}) + \zeta_2 \int_V \alpha(\mathbf{x}, \mathbf{x}') \boldsymbol{\varepsilon}(\mathbf{x}') dV' \tag{2}$$

In "(2)" the nonlocal elastic material can be physically interpreted as a two-phase elastic material; namely, phase 1 material (of volume fraction ζ_1) complying with local elasticity and phase 2 material (of volume fraction ζ_2) complying with nonlocal elasticity; ζ_1 and ζ_2 are the positive material constants with $\zeta_1 + \zeta_2 = 1$. It is also noted that the attenuation or kernel function $\alpha(\mathbf{x}, \mathbf{x}')$ is defined by the ratio $|\mathbf{x} - \mathbf{x}'|/\tau$ with nonlocal parameter $\tau = \frac{e_0 a}{L}$ where e_0 is a material constant, a is an internal characteristic length such as lattice parameter, granular distance while L is an external characteristic length. The attenuation or kernel function has the properties that it decays rapidly and the nonlocal effect vanishes when the two elements are too far from each other with respect to the influence distance $|\mathbf{x} - \mathbf{x}'|$. In the limit of local (classical) material, namely for $\tau \to 0$, the attenuation or kernel function has to become a Dirac delta function so that the nonlocal elasticity approach recovers to the classical model. To ensure this condition, it is sufficient to impose a normalized condition.

$$\int_{V_\infty} \alpha(\mathbf{x}, \mathbf{x}', \tau) dV' = 1 \tag{3}$$

in which V_∞ is the infinite domain embedding , if V is finite.

B. NL-FEM Procedure

In general, the standard finite element method (FEM) discretizes a structure into finite, countable elements within the physical domain. It converts a differential equation system into a system of algebraic equations with finite degrees of freedom (DOFs) instead of infinite degree of freedom. Let the domain V be discretized into N_e finite elements, according to Polizzotto [20], the global stiffness matrix can be expressed as [25].

$$\hat{\mathbf{K}}\mathbf{U} = \mathbf{F} \tag{4}$$

where

$$\left.\begin{aligned} \hat{\mathbf{K}} &= \sum_{n=1}^{N_e} \left[\zeta_1 \mathbf{K}_n^{loc} + \zeta_2 \mathbf{K}_{nm}^{nonloc} \right] \\ \mathbf{K}_n^{loc} &= C_n^T \mathbf{k}_n^{loc} C_m \\ \mathbf{K}_{nm}^{nonloc} &= C_n^T \mathbf{k}_{nm}^{nonloc} C_m \\ \mathbf{F} &= \sum_{n=1}^{N_e} C_n^T \mathbf{f}_n \end{aligned}\right\} \tag{5a-d}$$

with

$$\mathbf{k}_n^{loc} = \int_{V_n} \mathbf{B}_n^T(\mathbf{x}) \mathbf{D} \mathbf{B}_n(\mathbf{x}) dV \tag{6}$$

$$\mathbf{k}_{nm}^{nonloc} = \int_{V_n} \int_{V_m} \alpha(\mathbf{x}, \mathbf{x}', \tau) \mathbf{B}_n^T(\mathbf{x}) \mathbf{D} \mathbf{B}_m(\mathbf{x}') dV' dV \tag{7}$$

$$\mathbf{f} = \int_{V_n} \mathbf{N}_n^T(\mathbf{x}) \bar{\mathbf{b}}(\mathbf{x}) dV + \int_{S_{t(n)}} \mathbf{N}_n^T(\mathbf{x}) \bar{\mathbf{t}}(\mathbf{x}) dS \tag{8}$$

and $\mathbf{U}$ is the global displacement vector. As the nonlocal strain at the field point $\mathbf{x}$ has two parts, the n-th contribution of the total strain energy with "local elastic" part (of volume fraction ζ_1) whose stiffness matrix is given by matrix $\mathbf{k}_n^{loc}$ and a "nonlocal elastic" part (of volume fraction ζ_2) whose stiffness matrix is given by the set of element matrices $\mathbf{k}_{nm}^{nonloc}$. The later set includes the direct- or self-stiffness matrix (i.e. $\mathbf{k}_{nm}^{nonloc}$ for $n = m$) and as many indirect- or cross-stiffness matrices (i.e. $\mathbf{k}_{nm}^{nonloc}$ for $n \neq m$) as many elements are in the adopted mesh. The main difference between the standard FEM and NL-FEM is only the construction of nonlocal element stiffness $\mathbf{k}_{nm}^{nonloc}$ and each $\mathbf{k}_{nm}^{nonloc}$ represents the "nonlocal effects" of the m-th element on the n-th one.

In Fig. 1, the evaluation of $\mathbf{k}_{nm}^{nonloc}$ is presented for single DOFs with 2 nodes per element and three Gauss sampling points are used for numerical integrations. Introducing a natural coordinate system, say $\xi \in [0,1]$ and using the coordinate transformation with $\mathbf{J}(\xi)$, the Jacobian matrix (analogous relations holding for $\xi' \in [0,1]$), "(7)" can be written as

$$\mathbf{k}_{nm}^{nonloc} = \int_0^1 \int_0^1 A_{11} \left[\det(\mathbf{J}(\xi')) d\xi' \right] \left[\det(\mathbf{J}(\xi')) d\xi' \right] \tag{9}$$

where the following relations hold

$$\left.\begin{array}{l}A_{11}=\left\{\alpha\left(\left|\mathbf{x}(\xi)-\mathbf{x}'(\xi')\right|,\tau\right)\mathbf{B}_n^T(\xi)\mathbf{D}\mathbf{B}_m(\xi')\right\}\\ \mathbf{B}_n(\xi):=\mathbf{B}_n\left[\mathbf{x}(\xi)\right]\\ \mathbf{B}_m(\xi'):=\mathbf{B}_m\left[\mathbf{x}'(\xi')\right]\\ dV=dx=\det\left(\mathbf{J}(\xi)\right)d\xi\\ dV'=dx'=\det\left(\mathbf{J}(\xi')\right)d\xi'\end{array}\right\} \quad (10)$$

and $\alpha\left(\left|\mathbf{x}(\xi)-\mathbf{x}'(\xi')\right|,\tau\right)$ which indicates the value of attenuation or kernel function pertinent to points ξ and ξ' in the natural coordinate system of elements $\#n$ and $\#m$ respectively, has to be computed considering the Euclidean distance between these points in the absolute coordinate system $|\mathbf{x}-\mathbf{x}'|$.

By applying the Gauss quadrature rule for numerical integration, "(9)" finally yields

$$\mathbf{k}_{nm}^{nonloc}=\sum_{i=1}^{3}\sum_{j=1}^{3}\Phi_l\mathbf{B}_n^T(\xi_i)\mathbf{D}\mathbf{B}_m(\xi_j')w_iw_j'\Phi_r \quad (11)$$

with

$$\left.\begin{array}{l}\Phi_l=\alpha\left(\left|\mathbf{x}(\xi_j')-\mathbf{x}(\xi_i)\right|,\tau\right)\\ \Phi_r=\det\left(\mathbf{J}(\xi_j')\right)\det\left(\mathbf{J}(\xi_i)\right)\end{array}\right\} \quad (11a,b)$$

In which: $\alpha\left(\left|\mathbf{x}(\xi_j')-\mathbf{x}(\xi_i)\right|,\tau\right)$ is the attenuation or kernel function associated to the modified Gauss points with Cartesian coordinates $\mathbf{x}$ and $\mathbf{x}'$; and w_i w_j are the modified Gauss weights. The readers are referred to a complete survey on NL-FEM reported by Pasino *et al.* [20].

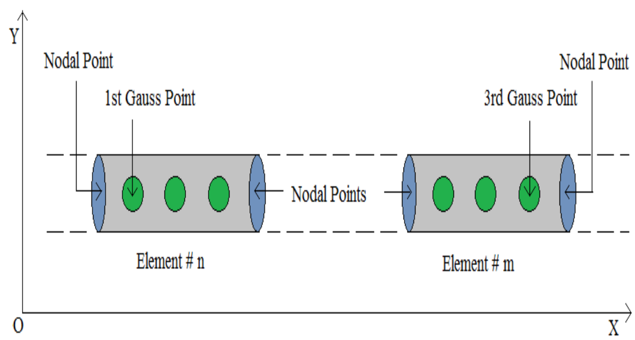

Fig. 1 Evaluation of the nonlocal global stiffness matrix: element global DOF numbers for two elements $\#n$ and $\#m$ in the mesh

III. Analytical Solution

Consider a nanorod/tube with length L and subjected to a combined distributed torque $T(x)$ and fixed end torque T_0 is shown in Fig. 2.

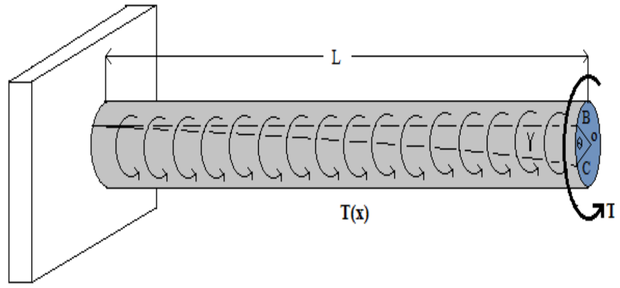

Fig. 2 Torsion of a fixed-free nanorod, where line AB is deformed to AC, with angle of twist θ and shear strain γ

The rod/tube is made of a nonlocal homogenous linear elastic material whose constitutive behavior compiles with the Eringen model given in Section 2.1. The variation of the strain energy of the system is given by

$$\delta U=\int_V\sigma_{r\theta}\delta\gamma dV=\left[T_{r\theta}\delta\theta\right]_0^L-\int_0^L\left(\frac{\partial T_{r\theta}}{\partial x}\delta\theta\right)dx \quad (12)$$

with

$$T_{r\theta}=\int_A r_0\sigma_{r\theta}dA;\gamma=r_0\frac{d\theta}{dx} \quad (13a,b)$$

where $T_{r\theta}$ is called the stress resultant or torque. In the presence of twisting moment $T(x)$ and an end torque T_0, the variation of work done by this combined load is given by

$$\delta W=\left[T_0\delta\theta\right]_0^L+\int_0^L T(x)\delta\theta dx \quad (14)$$

Based on the assumption of nonlocal elasticity, the nonlocal stress tensor at a point $\mathbf{x}$ can be expressed as a result of the weighted average of the contributions of the stress field within the continuum in the following expression (see [10],[12],[13] for detail)

$$\boldsymbol{\sigma}(\mathbf{x})=\int_V\alpha(\mathbf{x},\mathbf{x}')\boldsymbol{\sigma}'(\mathbf{x}')dV(\mathbf{x}') \quad (15)$$

where $\boldsymbol{\sigma}(\mathbf{x})$ and $\boldsymbol{\sigma}(\mathbf{x}')$ are the nonlocal and classical stress, respectively. The weight is specified by a nonlocal modulus $\alpha(\mathbf{x},\mathbf{x}')$ which depends on a dimensionless nanoscale τ. According to Eringen [14], a differential form of the nonlocal constitutive relation ("(15)") for a one-dimensional elastic thin structure can be expressed as

$$\sigma_{r\theta}-(e_0a)^2\frac{d^2\sigma_{r\theta}}{dx^2}=\sigma'_{r\theta}=G\gamma=Gr_0\frac{d\theta}{dx} \quad (16)$$

where $\sigma_{r\theta}$ and $\sigma'_{r\theta}$ are the nonlocal and classical shear stresses of structure respectively; and x is the axial coordinate. For static equilibrium, the variational principle requires that

$$\delta(U - W) = 0 \tag{17}$$

Substituting δU and δW from "(12)" and "(13)" into "(17)" and integrating by parts, and collecting the co-efficient of $\delta\theta$, the following equation of motion interms of torque is obtained.

$$\frac{\partial T_{r\theta}}{\partial x} + T(x) = 0 \tag{18}$$

The boundary conditions are

$$T_{r\theta} = T_0 \text{ or } \theta = 0 \text{ at } x = 0, L \tag{19a,b}$$

and the corresponding stress resultant is obtained as

$$T_{r\theta} - (e_0 a)^2 \frac{d^2 T_{r\theta}}{dx^2} = GJ \frac{\partial \theta}{\partial x} \tag{20}$$

where $J = \int_A r_0^2 dA$ is the polar second moment of area. Again combining "(18)" and "(20)", the nonlocal equation of motion can be expressed interms of deformation angle $\delta\theta$ as

$$GJ \frac{\partial^2 \theta}{\partial x^2} = -T + (e_0 a)^2 \frac{\partial^2 T}{\partial x^2} \tag{21}$$

in dimensionless forms, "(21)" can also be written as

$$\frac{\partial^2 \theta}{\partial \bar{x}^2} = -\bar{T} + \tau^2 \frac{\partial^2 \bar{T}}{\partial \bar{x}^2} \tag{22}$$

and the boundary conditions, "(19a,b)" become

$$\bar{T}_{r\theta} = \bar{T}_0 \text{ or } \theta = 0 \text{ at } \bar{x} = 0, 1 \tag{23a,b}$$

where

$$\bar{T}_0 = \frac{T_0 L}{GJ}; \bar{T} = \frac{TL^2}{GJ}; \bar{x} = \frac{x}{L} \tag{24a-c}$$

IV. Example of Fixed-Fixed Nanorods/Nanotubes

For a fully fixed nanorod, the boundary conditions are

$$\theta\big|_{\bar{x}=0} = 0; \theta\big|_{\bar{x}=1} = 0 \tag{25a,b}$$

A. When the Distributed Load $\bar{T}(\bar{x}) = 1$

Substituting $\bar{T} = 1$ into "(22)" and solving using the boundary conditions in "(25a,b)" yield

$$\theta(\bar{x}) = \frac{(\bar{x} - \bar{x}^2)}{2} \tag{26}$$

which is independent of nonlocal nanoscale τ. Similarly, if $\bar{T} = \bar{x}$, the solution of "(22)" can be written in the form

$$\theta(\bar{x}) = \frac{(\bar{x} - \bar{x}^3)}{6} \tag{27}$$

which is also independent of nonlocal nanoscale τ. Here the corresponding NL-FEM solution is obtained by applying a kernel function

$$\alpha(\mathbf{x}, \mathbf{x}', \tau) = \frac{1}{L\sqrt{\pi\tau}} \exp\left(-\frac{|x - x'|^2}{L^2 \tau}\right) \tag{28}$$

The NL-FEM solutions with corresponding classical solutions ("(26)" and "(27)") are shown in Fig.3 (a,b) for $\bar{T} = 1$ and $\bar{T} = \bar{x}$ respectively. It is clear from the Fig. 3 (a) and Fig. 3 (b) that the analytical solutions underestimate the angular deflection in the presence of nonlocal parameter and 12.3% and 14.45% increment for NL-FEM are observed form the Fig. 3 (a) at $\bar{x} = 0.5$ and Fig. 3 (b) at about $\bar{x} = 0.57$ respectively.

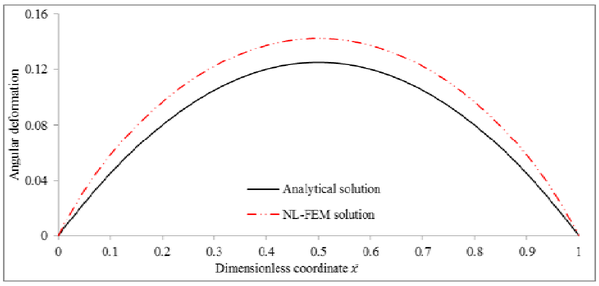

Fig. 3 (a) Angular deformation for a fixed-fixed nanorod for $\tau = 0.1$ and $\bar{T} = 1$

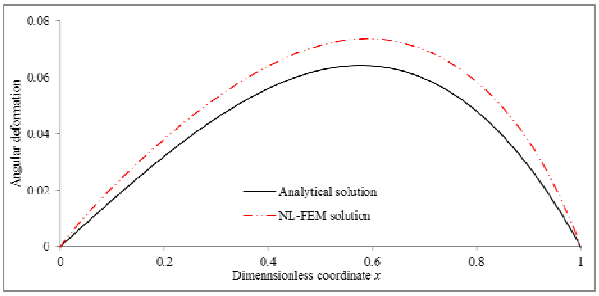

Fig. 3 (b) Angular deformation for a fixed-fixed nanorod for $\tau = 0.1$ and $\bar{T} = \bar{x}$

B. When the Distributed Load $\overline{T} = \overline{x}^2$

Substituting $\overline{T} = \overline{x}^2$ into "(22)" and solving using the boundary conditions in "(25a,b)" yield

$$\theta(\overline{x}) = \left(\frac{\overline{x}}{12} - \frac{\overline{x}^4}{12}\right) - \tau^2\left(\overline{x} - \overline{x}^2\right) \tag{29}$$

And the relation ("(29)") with corresponding NL-FEM solution is presented in Fig.4. Fig.4 illustrates that the analytical solution declines the maximum angular deflection 6.3% while NL-FEM solution increases 16% at about $\overline{x} = 0.63$.

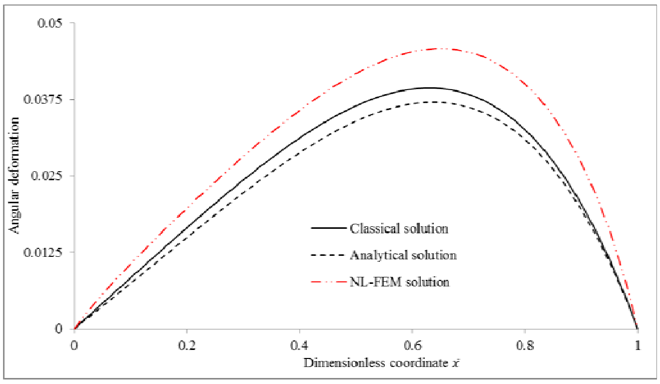

Fig. 4 Angular deformation for a fixed-fixed nanorod for $\tau = 0.1$ and $\overline{T}(\overline{x}) = \overline{x}^2$

C. When the Distributed Load $\overline{T}(\overline{x}) = \sin(\overline{x})$

Substituting $\overline{T}(\overline{x}) = \sin(\overline{x})$ into "(22)" and solving using the boundary conditions in "(25a,b)" yield

$$\theta(\overline{x}) = \left(1 + \tau^2\right)\left(\sin(\overline{x}) - 0.841470985\overline{x}\right) \tag{30}$$

The effect of nonlocal nanoscale on angular deflection for the analytical solution ("(30)") with corresponding NL-FEM solution for $\tau = 0.1$ is presented in Fig.5. It is observed from the Fig.5 that the maximum rotation occurs at about $\overline{x} = 0.57$ and analytical solution increased the maximum rotation is about 0.99% while the numerical solution is about 12.67%.

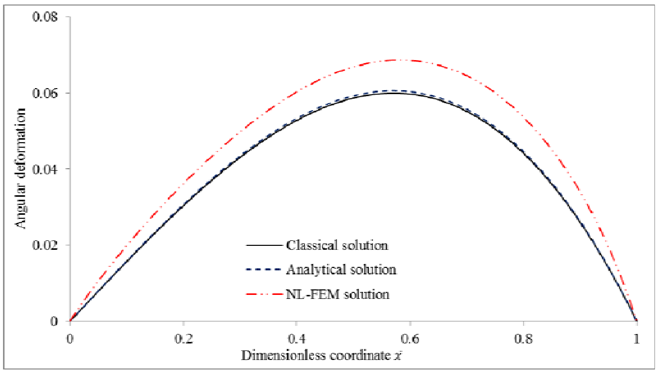

Fig. 5 Angular deformation for a fixed-fixed nanorod for $\tau = 0.1$ and $\overline{T} = \sin(\overline{x})$

D. When the Distributed Load $\overline{T} = \sin(n\pi\overline{x})$

Substituting $\overline{T} = \sin(n\pi\overline{x})$ into Eq. (22) and solving using the boundary conditions in Eq. (25a,b) yield

$$\theta(\overline{x}) = \left(1 + (n\pi\tau)^2\right)\frac{\sin(n\pi\overline{x})}{(n\pi)^2} \tag{31}$$

It is clear from "(31)" that the maximum rotation occurs when $\sin(n\pi\overline{x}) = 1$. For $n = 1$, the maximum occurs at $\overline{x} = 0.5$. The analytical solution ("(31)") with corresponding NL-FEM solution for $\tau = 0.1$ and $n = 1$ is presented in Fig. 6. Again, Fig. 6 demonstrates that the angular deflection increases with increasing nonlocal effect in both cases. It is also noted that 8.98% increment is found for the analytical solution while the numerical solution is about 13.5%.

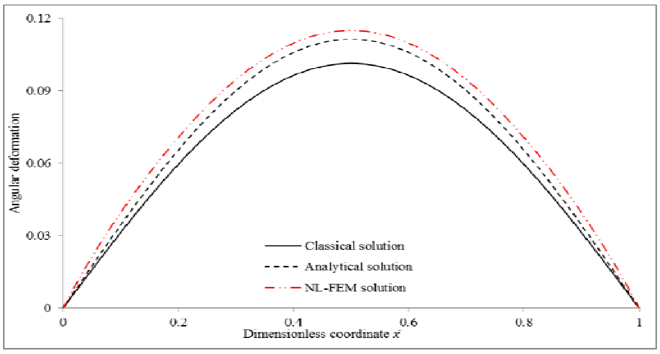

Fig. 6 Angular deformation for a fixed-fixed nanorod for $\tau = 0.1$ and $\overline{T} = \sin(\pi\overline{x})$

V. Example of Fixed-Free Nanorod/Nanotube without End Torque

For a fixed-free nanorod/nanotube with distributed torque ($\overline{T}(\overline{x})$), the boundary conditions are

$$\theta\big|_{\overline{x}=0} = 0; \left.\frac{\partial\theta}{\partial\overline{x}}\right|_{\overline{x}=1} = 0 \tag{32a,b}$$

A. When the Distributed Load $\bar{T} = 1$

Substituting $\bar{T} = 1$ into "(22)" and solving using the boundary conditions in "(32a,b)" yield

$$\theta(\bar{x}) = \left(\bar{x} - \frac{\bar{x}^2}{2}\right) \tag{33}$$

Which is independent of nonlocal nanoscale τ. The analytical solution ("(33)") with corresponding NL-FEM solution for $\tau = 0.1$ is presented in Fig. 7 (a). Similarly result, for $\bar{T} = \bar{x}$ and $\tau = 0.1$ is also presented in Fig. 7 (b). From the Fig. 7 (a) and Fig. 7 (b), it can be seen that the NL-FEM solutions increase the rotation 4.58% and 3.24% at $\bar{x} = 1$ for $\bar{T} = 1$ and $\bar{T} = \bar{x}$ respectively.

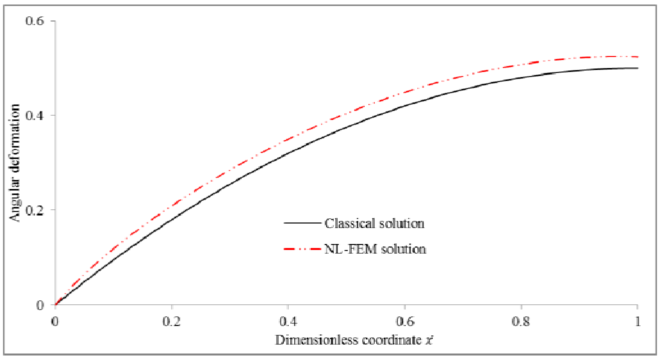

Fig. 7 (a) Angular deformation for a fixed-free nanorod for $\tau = 0.1$ and $\bar{T} = 1$

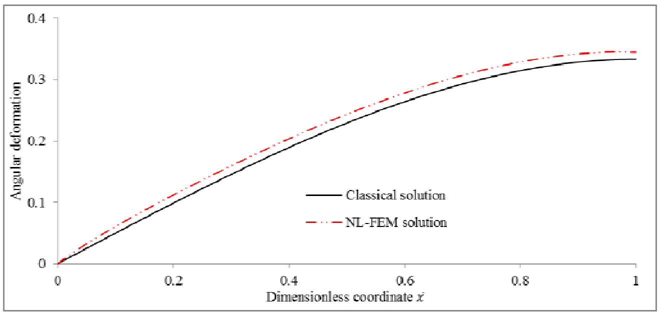

Fig. 7 (b) Angular deformation for a fixed-free nanorod for $\tau = 0.1$ and $\bar{T} = \bar{x}$

B. When the Distributed Load $\bar{T} = \bar{x}^2$

Substituting $\bar{T} = \bar{x}^2$ into "(22)" and solving using the boundary conditions in "(32a,b)" yield

$$\theta(\bar{x}) = \left(\frac{\bar{x}}{3} - \frac{\bar{x}^4}{12}\right) - \tau^2\left(2\bar{x} - \bar{x}^2\right) \tag{34}$$

The analytical solution ("(34)") with corresponding NL-FEM solution for $\tau = 0.1$ is present in Fig.8. From the Fig. 8, it is clear that the analytical solution reduces the rotation about 4.16% while the NL-FEM solution increases the rotation 2.52% at $\bar{x} = 1$.

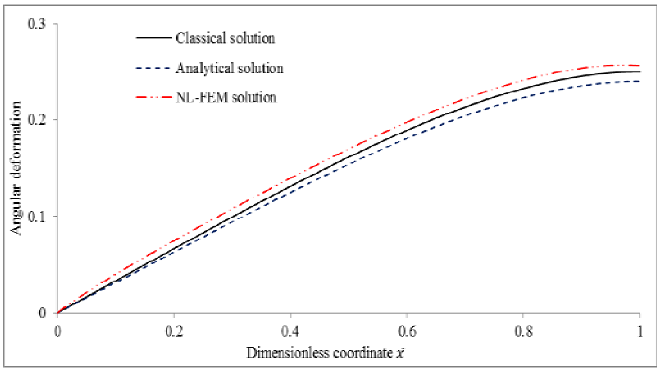

Fig. 8 Angular deformation for a fixed-free nanorod for $\tau = 0.1$ and $\bar{T} = \bar{x}^2$

C. When the Distributed Load $\bar{T} = \sin(\bar{x})$

Substituting $\bar{T} = \sin(\bar{x})$ into "(22)" and solving using the boundary conditions in "(32a,b)" yield

$$\theta(\bar{x}) = \left(1 + \tau^2\right)\left(\sin(\bar{x}) - 0.540302306\bar{x}\right) \tag{35}$$

The analytical solution "(35)" with corresponding NL-FEM solution is presented in Fig. 9. From the Fig. 9, it is seen that both the solutions enhance the angular deflection in the presence of nonlocal nanoscale τ and 0.99% and 3.5% enhancement are observed for analytical solution and NL-FEM solution respectively at $\bar{x} = 1$.

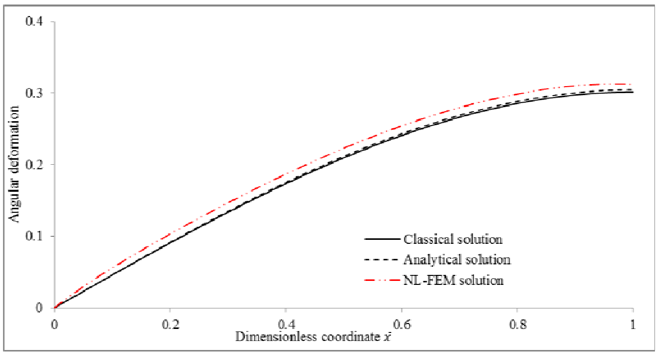

Fig. 9 Angular deformation for a fixed-free nanorod for $\tau = 0.1$ and $\bar{T} = \sin(\bar{x})$

D. When the Distributed Load $\bar{T} = \sin(n\pi\bar{x})$

Substituting $\bar{T} = \sin(n\pi\bar{x})$ into "(22)" and solving using the boundary conditions in "(32a,b)" yield

$$\theta(\bar{x}) = \left(1 + (n\pi\tau)^2\right)\left(\frac{\sin(n\pi\bar{x})}{(n\pi)^2} - \frac{(-1)^n \bar{x}}{n\pi}\right) \tag{36}$$

The analytical solution ("(36)") with corresponding NL-FEM solution for $\tau = 0.1$ and $n = 1$ is presented in Fig. 10. It is observed from the Fig.10 that the analytical solution increased the maximum rotation is about 9.87% while the numerical solution is about 5.84%.

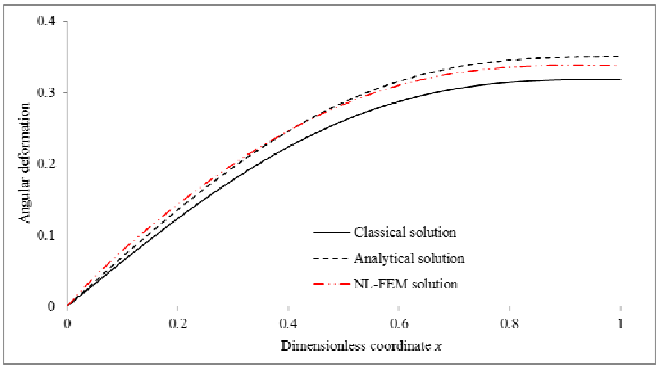

Fig. 10 Angular deformation for a fixed-free nanorod for $\tau = 0.1$ and $\bar{T} = \sin(n\pi\bar{x})$

VI. Example of Fixed-Free NANOROD/Nanotube without Distributed Load

For a Fixed-Free nanorod/nanotube with end torque ($\bar{T}_0$), the boundary conditions are

$$\theta\big|_{\bar{x}=0} = 0; \bar{T}_{r\theta}\big|_{\bar{x}=1} = T_0 \tag{37}$$

According to Peddieson et al. [21], any beam/column which does not acted by the distributed load is governed by the local equation which is scale free. So, the classical result in this case is given by

$$\theta(\bar{x}) = \bar{T}_0\bar{x} \tag{38}$$

By applying the kernel function "(28)", the NL-FEM solution and corresponding classical solution for $\tau = 0.1$ is shown in Fig.11. From the Fig. 11 it is clear that the NL-FEM solution increases the rotation about 5.54% at $\bar{x} = 1$.

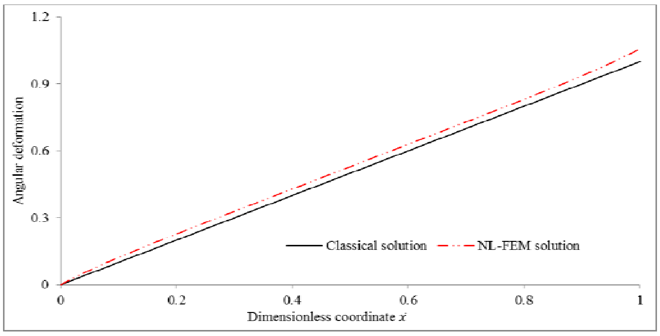

Fig. 11 Angular deformation for a fixed-free nanorod for $\zeta_1 = 0.5$, $\tau = 0.1$, and $\bar{T}_0 = 1$

VII. Conclusion

Based on the NL-FEM method, the torsional static for circular nanostructures of 1D nonlocal homogeneous linear elastic material is investigated in the presence of combined distributed torque and fixed end torque. Besides, the nonlocal equation of motion and boundary conditions are derived by means of variational principle and differential constitutive relation of Eringen. The NL-FEM solution for various end constrains with their corresponding analytical solutions are derived and discussed in details. It is observed that the NL-FEM solution is consistent and a nonlocal nanoscale is found to induce increases in angular rotation which clearly prove the reliability and effectiveness of numerical techniques.

References

[1] H. Gleiter, "Nanocristalline materials," Progress in Materials Science, vol. 33, pp. 223-315, 1989.
[2] M. E. Gurtin, A. Murdoch, "A continuum theory of elastic material surfaces," Archive for Rational Mechanics Analysis, vol. 57 (4), pp. 291-323, 1975.
[3] C. W. Lim, Z. R. Li, L. H. He, "Size dependent, non-uniform elastic field inside a nano-scale spherical inclusion due to interface stress," International Journal of Solids and Structures, vol. 43 (17), pp. 5055-5065, 2006.
[4] E. Kröner, "Elasticity theory of materials with long range cohesive forces," International Journal of Solids and Structures, vol. 3, pp. 731-742, 1967.
[5] I. A. Kunin, "The theory of elastic media with microstructures and the theory of dislocaltion," In: Kröner E. ed. Mechanices of Geeneralized Continua, Proceedings of IUTAM Symposium 1967. New York: Springer, 1968.
[6] J. A. Krumhansl, "Some considerations on the relations between solid state physics and generalized continuum mechanics," In: Kröner E. ed. Mechanices of Geeneralized Continua. Berlin: Springer-Verlag, 1968, pp. 298-331.
[7] D. G. B. Edelen, "Protoelastic bodies with large deformations," Archive for Rational Mechanics Analysis, vol. 34 pp. 283-300, 1969.
[8] D. G. B. Edelen, N. Laws, "On the thermodynamics of systems with nonlocality," Archive for Rationale Mechanics and Analysis, vil. 43, pp. 24-35, 1971.
[9] D. G. B. Edelen, A. E. Green, N. Laws, "Nonlocal continuum mechanics," Archive for Rationale Mechanics and Analysis, vol. 43, pp. 36-44, 1971.
[10] A. C. Eringen, "Nonlocal polar elastic continua," International Journal of Engineering Science, vol. 10, pp. 1-16, 1972.
[11] A.C. Eringen, "Linear theory of nonlocal elasticity and dispersion of plane waves," International Journal of Engineering Science, vol. 10, pp. 425-435, 1972.
[12] A. C. Eringen, "Nonlocal Continuum Field Theories," New York, Springer, 2002.
[13] A. C. Eringen, D. B. G. Edelen, "On nonlocal elasticity," International Journal of Engineering Science. Vol. 10, pp. 233-248, 1972.
[14] A. C. Eringen, "On differential equations of nonlocal elasticity and solution of screw dislocaltion and surface waves," Journal of Applied Physics, vol. 54, pp.4703-4710, 1983.
[15] J. Peddieson, G.R. Buchanan, R. P. McNitt, "Application of nonlocal continuum models to nanotechnology," International Journal of Engineering Science, vol. 41, pp. 305-312, 2003.
[16] L. J. Sudak, "Column buckling of multi-walled carbon nanotubes using nonlocal continuum mechanics," Journal of Applied Physics, vol. 94 (11), pp. 7281-7287, 2003.
[17] C. M. Wang, Y.Y. Zhang, S. S. Ramesh, S. Kitipornchai, "Buckling analysis of micro- and nano-rods/tubes based on nonlocal Timoshenko beam theory," Journal of Physics D: Applied Physics, vol. 39 (17), pp. 3904-3909, 2006.
[18] Y.Y. Zhang, C.M. Wang, W.H. Duan, Y. Xiang, Z. Zong, "Assessment of continuum mechanics models in predicting buckling of single-walled

carbon nanotubes," Nanotechnology, vol. 20 (39), pp. 395707(8pp), 2009.
[19] T. H. Thai, "A nonlocal beam theory for bending buckling, and vibration of nanobeams," International Journal of Engineering Science, vol. 52, pp. 56-64, 2012.
[20] C. Polizzotto, "Nonlocal elasticity and related variational principles," International Journal of Solids and Structures, vol. 38, pp. 7359-7380, 2001.
[21] A. C. Eringen, B. S. Kim, "Stress concentration at the tip of a crack," Mechanics Research Communications, vol. 1, pp. 233-237, 1974.
[22] A. C. Eringen, C. G. Speziale, B.S. Kim, "Crack-tip problem in nonlocal elasticity," Journal of Mechanics, Physics and Silods, vol. 25, pp. 339-355, 1977.
[23] A. C. Eringen, "Theory of nonlocal elasticity and some applications," Res Mechaica, vol. 21, pp. 313-342, 1987.
[24] S. B. Altan, "Existance in nonlocal elasticity," Archive Mechanics, vol. 41, pp. 25-36, 1989.
[25] A. A. Pasano, A. Sofi, P. Fuschi, "Nonlocal integral elasticity: 2D finite element based solutions," International Journal of Solids and Structures, vol. 46, pp. 3836-3849, 2009.