

# Multiple periodic solutions for a delayed predator-prey system on time scales

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**Abstract**—This paper is devoted to a delayed periodic predator-prey system with non-monotonic numerical response on time scales. With the help of a continuation theorem based on coincidence degree theory, we establish easily verifiable criteria for the existence of multiple periodic solutions. As corollaries, some applications are listed. In particular, our results improve and generalize some known ones.

**Keywords**—Predator-prey system, periodic solution, time scale, delay, coincidence degree.

## I. INTRODUCTION

In the past decade, many authors have studied the existence of periodic solutions for population models governed by the differential and difference equations [1–7].

In [1], Chen studied the following periodic predator-prey system with type IV functional response:

$$\begin{cases} x'(t) = x(t) \left[ r(t) - a(t)x(t - \tau_1(t)) - \frac{b(t)y(t - \tau_2(t))}{\frac{x^2(t)}{m} + x(t) + n} \right], \\ y'(t) = y(t) \left[ -d(t) + \frac{c(t)x(t - \tau_3(t))}{\frac{x^2(t - \tau_3(t))}{m} + x(t - \tau_3(t)) + n} \right], \end{cases} \quad (1)$$

where  $x(t)$  and  $y(t)$  stand for the population density of prey and predator at time  $t$ , respectively. The function  $r(t) - a(t)v$  is the growth rate of the prey in the absence of the predator. The function  $d(t)$  is the death rate of the predator. The function  $(b(t)v)/(v^2/m + v + n)$ , called functional response of predator to prey, describes the change in the rate of exploitation of prey by a predator as a result of a change in the prey density. The function  $(c(t)v)/(v^2/m + v + n)$ , called numerical response of predator to prey, describes the change in reproduction rate with changing prey density. Using the method of coincidence degree, the author established sufficient conditions for the existence of multiple periodic solutions.

In [7], Zhang et al. discussed the discrete analogy of system (1):

$$\begin{cases} x(k+1) = x(k) \exp \left[ r(k) - a(k)x(k - \tau_1(k)) - \frac{b(k)y(k - \tau_2(k))}{\frac{x^2(k)}{m} + x(k) + n} \right], \\ y(k+1) = y(k) \exp \left[ -d(k) + \frac{c(k)x(k - \tau_3(k))}{\frac{x^2(k - \tau_3(k))}{m} + x(k - \tau_3(k)) + n} \right], \end{cases} \quad (2)$$

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and established sufficient conditions for the existence of multiple periodic solutions which is consistent with the ones in [1].

In [5], Hu et al. considered the following periodic predator-prey system with general non-monotonic functional response:

$$\begin{cases} x'(t) = x(t) [r(t) - a(t)x(t)] - b(t)f(x(t))y(t), \\ y'(t) = y(t) [-d(t) + c(t)f(x(t - \tau))], \end{cases} \quad (3)$$

and established sufficient conditions for the existence of multiple periodic solutions.

In [2], Ding and Jiang investigated the following generalized periodic Gause-type predator-prey system with non-monotonic numerical response:

$$\begin{cases} x'(t) = x(t) [f(t, x(t - \tau_1(t))) - g(t, x(t))y(t - \tau_2(t))], \\ y'(t) = y(t) [-d(t) + h(t, x(t - \tau_3(t)))] , \end{cases} \quad (4)$$

and established sufficient conditions for the existence of multiple periodic solutions that improve and generalize the ones for systems (1) and (3) in [1] and [5], respectively.

Recently, in order to unify differential and difference equations, people have done a lot of research about dynamic equations on time scales. In fact, continuous and discrete systems are very important in implementing and applications. But it is troublesome to study the existence and stability of periodic solutions for continuous and discrete systems, respectively. Therefore, it is meaningful to study that on time scale which can unify the continuous and discrete situations. For the theory of dynamic equations on time scales, we refer the reader to [8–10]. For the research on periodic solutions of dynamic equations on time scales describing population dynamics, one may consult [11–16], etc.

In this paper, we consider the following periodic predator-prey system with non-monotonic numerical response on the time scale  $\mathbb{T}$ :

$$\begin{cases} u_1^\Delta(t) = f(t, e^{u_1(t - \tau_1(t))}) - g(t, e^{u_1(t)}) e^{u_2(t - \tau_2(t))}, \\ u_2^\Delta(t) = -d(t) + h(t, e^{u_1(t - \tau_3(t))}). \end{cases} \quad (5)$$

$\mathbb{T}$  is a periodic time scale which has the subspace topology inherited from the standard topology on  $\mathbb{R}$ . The symbol  $\Delta$  stands for the delta-derivative. If  $\mathbb{T} = \mathbb{R}$  is the set of all real numbers, this delta derivative is equal to the usual derivative, and if  $\mathbb{T} = \mathbb{Z}$  is the set of all integers, this is equal to the usual forward difference.

**Remark 1.** Let  $x(t) = \exp[u_1(t)]$ ,  $y(t) = \exp[u_2(t)]$ . If  $\mathbb{T} = \mathbb{R}$ , then (5) reduces to (4). If  $\mathbb{T} = \mathbb{Z}$ , then (5) is reformulated

as

$$\begin{cases} x(k+1) = x(k) \exp [f(k, x(t - \tau_1(k))) \\ \quad - g(k, x(k)) y(k - \tau_2(k))], \\ y(k+1) = y(k) \exp [-d(k) + h(k, x(k - \tau_3(k)))]. \end{cases} \quad (6)$$

The main purpose of this paper is, by using the coincidence degree theory developed by Gaines and Mawhin [17], to derive a set of easily verifiable sufficient conditions for the existence of multiple periodic solutions of system (5). As corollaries, some applications are listed. In particular, our results improve and generalize some known ones.

## II. PRELIMINARIES

In this section, we briefly give some elements of the time scale calculus, recall the continuation theorem from coincidence degree theory, and state an auxiliary result that will be used in this paper.

First, let us present some foundational definitions and results from the calculus on time scales so that the paper is self-contained. For more details, we refer the reader to [8–10].

A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of the real numbers  $\mathbb{R}$ , which inherits the standard topology of  $\mathbb{R}$ . Thus, the real numbers  $\mathbb{R}$ , the integers  $\mathbb{Z}$  and the natural numbers  $\mathbb{N}$  are examples of time scales, while the rational numbers  $\mathbb{Q}$  and the open interval  $(1, 2)$  are no time scales.

Let  $\omega > 0$ . Throughout this paper, the time scale  $\mathbb{T}$  is assumed to be  $\omega$ -periodic, i.e.,  $t \in \mathbb{T}$  implies  $t + \omega \in \mathbb{T}$ . In particular, the time scale  $\mathbb{T}$  under consideration is unbounded above and below.

For  $t \in \mathbb{T}$ , the forward and backward jump operators  $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$  are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively.

If  $\sigma(t) = t$ ,  $t$  is called right-dense (otherwise: right-scattered), and if  $\rho(t) = t$ , then  $t$  is called left-dense (otherwise: left-scattered).

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be rd-continuous if it is continuous at right-dense points in  $\mathbb{T}$  and its left-sided limits exist (finite) at left-dense points in  $\mathbb{T}$ . The set of rd-continuous functions is denoted by  $C_{rd}(\mathbb{T})$ .

For  $f : \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}$  we define  $f^\Delta(t)$ , the delta derivative of  $f$  at  $t$ , to be the number (provided it exists) with the property that, given any  $\varepsilon > 0$ , there is a neighborhood  $U$  of  $t$  (i.e.,  $U = (t - \delta, t + \delta) \cap \mathbb{T}$  for some  $\delta > 0$ ) in  $\mathbb{T}$  such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|,$$

for all  $s \in U$ .  $f$  is said to be delta differentiable if its delta derivative exists for all  $t \in \mathbb{T}$ . The set of functions  $f : \mathbb{T} \rightarrow \mathbb{R}$  that are delta differentiable and whose delta derivative are rd-continuous functions is denoted by  $C_{rd}^1(\mathbb{T})$ .

A function  $F : \mathbb{T} \rightarrow \mathbb{R}$  is called a delta antiderivative of  $f : \mathbb{T} \rightarrow \mathbb{R}$  provided  $F^\Delta(t) = f(t)$ , for all  $t \in \mathbb{T}$ . Then, we define the delta integral by

$$\int_a^b f(t) \Delta t = F(b) - F(a), \text{ for all } a, b \in \mathbb{T}.$$

**Lemma 1.** Every delta differentiable function is continuous.

**Lemma 2.** Every rd-continuous function has a delta antiderivative.

**Lemma 3.** If  $a, b, c \in \mathbb{T}$ ,  $\alpha, \beta \in \mathbb{R}$  and  $f, g \in C_{rd}(\mathbb{T})$ , then

- (a)  $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t$ ;
- (b)  $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t$ ;
- (c) if  $f(t) \geq 0$  for all  $a \leq t < b$ , then  $\int_a^b f(t) \Delta t \geq 0$ ;
- (d) if  $|f(t)| \leq g(t)$  on  $[a, b] := \{t \in \mathbb{T} : a \leq t < b\}$ , then  $|\int_a^b f(t) \Delta t| \leq \int_a^b g(t) \Delta t$ .

To facilitate the discussion below, we now introduce some notation to be used throughout this paper. Let

$$\begin{aligned} \kappa &= \min\{[0, +\infty) \cap \mathbb{T}\}, \quad I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}, \\ \hat{a} &= \frac{1}{\omega} \int_{I_\omega} a(t) \Delta t = \frac{1}{\omega} \int_\kappa^{\kappa+\omega} a(t) \Delta t, \\ \hat{A} &= \frac{1}{\omega} \int_{I_\omega} |a(t)| \Delta t = \frac{1}{\omega} \int_\kappa^{\kappa+\omega} |a(t)| \Delta t, \\ \tilde{\varphi}(v) &= \frac{1}{\omega} \int_{I_\omega} \varphi(t, v) \Delta t = \frac{1}{\omega} \int_\kappa^{\kappa+\omega} \varphi(t, v) \Delta t, \end{aligned}$$

where  $a \in C_{rd}(\mathbb{T})$  is an  $\omega$ -periodic function, i.e.  $a(t + \omega) = a(t)$  for all  $t \in \mathbb{T}$ ,  $\varphi : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$  is rd-continuous  $\omega$ -periodic in its first argument and continuous in its second argument.

Next, let us recall the continuation theorem in coincidence degree theory. To do so, we need to introduce the following notation.

Let  $X, Y$  be real Banach spaces,  $L : \text{Dom}L \subset X \rightarrow Y$  be a linear mapping, and  $N : X \rightarrow Y$  be a continuous mapping.

The mapping  $L$  is said to be a Fredholm mapping of index zero, if  $\dim \text{Ker}L = \text{codim Im}L < +\infty$  and  $\text{Im}L$  is closed in  $Y$ .

If  $L$  is a Fredholm mapping of index zero, then there exist continuous projectors  $P : X \rightarrow X$  and  $Q : Y \rightarrow Y$ , such that  $\text{Im}P = \text{Ker}L$ ,  $\text{Ker}Q = \text{Im}L = \text{Im}(I - Q)$ . It follows that the restriction  $L_P$  of  $L$  to  $\text{Dom}L \cap \text{Ker}P : (I - P)X \rightarrow \text{Im}L$  is invertible. Denote the inverse of  $L_P$  by  $K_P$ .

The mapping  $N$  is said to be  $L$ -compact on  $\bar{\Omega}$ , if  $\Omega$  is an open bounded subset of  $X$ ,  $QN(\bar{\Omega})$  is bounded and  $K_P(I - Q)N : \bar{\Omega} \rightarrow X$  is compact.

Since  $\text{Im}Q$  is isomorphic to  $\text{Ker}L$ , there exists an isomorphism  $J : \text{Im}Q \rightarrow \text{Ker}L$ .

Here we state the Gaines-Mawhin theorem, which is a main tool in the proof of our main result.

**Lemma 4** (Continuation theorem [17, p.40]). Let  $\Omega \subset X$  be an open bounded set,  $L$  be a Fredholm mapping of index zero and  $N$  be  $L$ -compact on  $\bar{\Omega}$ . Assume

- (a) for each  $\lambda \in (0, 1)$ ,  $x \in \partial\Omega \cap \text{Dom}L$ ,  $Lx \neq \lambda Nx$ ;
- (b) for each  $x \in \partial\Omega \cap \text{Ker}L$ ,  $QNx \neq 0$ ;
- (c)  $\deg(JQN, \Omega \cap \text{Ker}L, 0) \neq 0$ .

Then  $Lx = Nx$  has at least one solution in  $\bar{\Omega} \cap \text{Dom}L$ .

In order to achieve the priori estimation in the case of dynamic equations on a time scale  $\mathbb{T}$ , we also require the following inequality which is proved in [18, Theorem 2.4].

**Lemma 5.** Let  $t_1, t_2 \in I_\omega$ , and  $t \in \mathbb{T}$ . If  $\varphi \in C_{rd}^1(\mathbb{T})$  is an  $\omega$ -periodic real function, then

$$\varphi(t) \leq \varphi(t_1) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |\varphi^\Delta(t)| \Delta t$$

and

$$\varphi(t) \geq \varphi(t_2) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |\varphi^\Delta(t)| \Delta t.$$

### III. EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS

We are now in a position to state and prove our result on the existence of multiple periodic solutions of system (5). For the sake of generality, we make the following fundamental assumptions for system (5):

- (H<sub>1</sub>)  $\tau_i(t) : \mathbb{T} \rightarrow \mathbb{R}$  is an  $\omega$ -periodic continuous function such that  $t - \tau_i(t) \in \mathbb{T}$  for  $i = 1, 2, 3$ , and  $t \in \mathbb{T}$ .
- (H<sub>2</sub>)  $d(t) : \mathbb{T} \rightarrow \mathbb{R}$  is an  $\omega$ -periodic continuous function such that  $\hat{d} > 0$ .
- (H<sub>3</sub>)  $f(t, v), g(t, v), h(t, v) : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathbb{R}$  are continuous functions and  $\omega$ -periodic in  $t$ ,  $(\partial f / \partial v)(t, v), (\partial g / \partial v)(t, v)$  and  $(\partial h / \partial v)(t, v)$  are also continuous functions.
- (H<sub>4</sub>) There exists a positive constant  $\alpha$  such that  $\tilde{f}(v) > 0$  for  $v \in (0, \alpha)$ . There exists a continuous  $\omega$ -periodic function  $r(t)$  such that  $\hat{r} > 0$  and  $f(t, v) \leq r(t)$  for  $t \in \mathbb{T}, v > 0$ .
- (H<sub>5</sub>) There exists a positive constant  $c_0$  such that  $0 < g(t, v) \leq c_0$  for  $t \in \mathbb{T}, v > 0$ .
- (H<sub>6</sub>)  $h(t, 0) = 0, \lim_{v \rightarrow +\infty} h(t, v) = 0$ . There exists a positive constant  $v_0$  such that  $(v - v_0)(\partial h / \partial v)(t, v) < 0$  for  $t \in \mathbb{T}, v > 0$  and  $v \neq v_0$ , and  $\hat{d} < \sup_{v \geq 0} \tilde{h}(v)$ .

By (H<sub>3</sub>) and (H<sub>6</sub>), we have

$$(v - v_0)\tilde{h}'(v) = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} (v - v_0) \frac{\partial h}{\partial v}(t, v) \Delta t < 0,$$

$$\lim_{v \rightarrow +\infty} \tilde{h}(v) = 0,$$

then  $\tilde{h}(v)$  is strictly increasing on  $[0, v_0]$  and strictly decreasing on  $[v_0, +\infty)$ . By this, (H<sub>3</sub>) and (H<sub>6</sub>), one can easily see that equation  $\tilde{h}(v) = \hat{d}$  has two distinct positive solutions, namely,  $v_-, v_+$ . Without loss of generality, we suppose that  $v_- < v_+$ , then  $v_- < v_0 < v_+$ .

**Theorem 1.** In addition to (H<sub>1</sub>)–(H<sub>6</sub>), suppose further that the following hold:

- (H<sub>7</sub>)  $v_- e^{(\hat{R} + \hat{r})\omega} < v_+$ ,
- (H<sub>8</sub>)  $v_+ e^{(\hat{R} + \hat{r})\omega/2} < \alpha$ .

Then system (5) has at least two  $\omega$ -periodic solutions.

*Proof:* In order to apply Lemma 4 to system (5), let

$$X = Y = \{u = (u_1(t), u_2(t))^T \in C(\mathbb{T}, \mathbb{R}^2) : u_i(t + \omega) = u_i(t), i = 1, 2\},$$

and

$$\|u\| = \|(u_1(t), u_2(t))^T\| = \max_{t \in I_\omega} |u_1(t)| + \max_{t \in I_\omega} |u_2(t)|,$$

then  $X$  and  $Y$  are Banach spaces with the norm  $\|\cdot\|$ . Set

$$L \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u_1^\Delta(t) \\ u_2^\Delta(t) \end{bmatrix},$$

$$N \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} f(t, e^{u_1(t-\tau_1(t))}) - g(t, e^{u_1(t)}) e^{u_2(t-\tau_2(t))} \\ -d(t) + h(t, e^{u_1(t-\tau_3(t))}) \end{bmatrix}.$$

With these notations system (5) can be written in the form

$$Lu = Nu, \quad u \in X.$$

Obviously,  $\text{Ker}L = \mathbb{R}^2$ ,  $\text{Im}L = \{(u_1(t), u_2(t))^T \in Y : \int_{\kappa}^{\kappa+\omega} u_i(t) \Delta t = 0, i = 1, 2\}$  is closed in  $Y$ , and  $\dim \text{Ker}L = \text{codim} \text{Im}L = 2$ . Therefore  $L$  is a Fredholm mapping of index zero. Now define two projectors  $P : X \rightarrow X$  and  $Q : Y \rightarrow Y$  as

$$P \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = Q \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}, \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \in X = Y,$$

then  $P$  and  $Q$  are continuous projectors such that

$$\text{Im}P = \text{Ker}L, \quad \text{Ker}Q = \text{Im}L = \text{Im}(I - Q).$$

Furthermore, through an easy computation we find that the generalized inverse  $K_P$  of  $L_P$  has the form

$$K_P : \text{Im}L \rightarrow \text{Dom}L \cap \text{Ker}P,$$

$$K_P(u) = \int_{\kappa}^t u(s) \Delta s - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t u(s) \Delta s \Delta t.$$

Then  $QN : X \rightarrow Y$  and  $K_P(I - Q)N : X \rightarrow X$  read as

$$QN u = \frac{1}{\omega} \begin{bmatrix} \int_{\kappa}^{\kappa+\omega} [f(t, e^{u_1(t-\tau_1(t))}) - g(t, e^{u_1(t)}) \times e^{u_2(t-\tau_2(t))}] \Delta t \\ \int_{\kappa}^{\kappa+\omega} [-d(t) + h(t, e^{u_1(t-\tau_3(t))})] \Delta t \end{bmatrix},$$

$$K_P(I - Q)Nu = \int_{\kappa}^t Nu(s) \Delta s - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} \int_{\kappa}^t Nu(s) \Delta s \Delta t - \frac{1}{\omega} \left( t - \kappa - \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} (t - \kappa) \Delta t \right) \int_{\kappa}^{\kappa+\omega} Nu(s) \Delta s.$$

Clearly,  $QN$  and  $K_P(I - Q)N$  are continuous. By using the Arzela-Ascoli theorem, it is not difficult to prove that  $\overline{K_P(I - Q)N(\bar{\Omega})}$  is compact for any open bounded set  $\Omega \in X$ . Moreover,  $QN(\bar{\Omega})$  is bounded. Therefore  $N$  is  $L$ -compact on  $\bar{\Omega}$  with any open bounded set  $\Omega \subset X$ .

In order to apply Lemma 4, we need to find at least two disjoint open bounded subsets in  $X$ . Corresponding to the operator equation  $Lu = \lambda Nu, \lambda \in (0, 1)$ , we have

$$\begin{cases} u_1'(t) = \lambda [f(t, e^{u_1(t-\tau_1(t))}) - g(t, e^{u_1(t)}) e^{u_2(t-\tau_2(t))}], \\ u_2'(t) = \lambda [-d(t) + h(t, e^{u_1(t-\tau_3(t))})]. \end{cases} \quad (7)$$

Suppose that  $(u_1(t), u_2(t))^T \in X$  is a solution of (7) for a certain  $\lambda \in (0, 1)$ . Integrating (7) on both sides from  $\kappa$  to

$\kappa + \omega$ ] leads to

$$\begin{aligned} & \int_{\kappa}^{\kappa+\omega} \lambda \left[ f \left( t, e^{u_1(t-\tau_1(t))} \right) - g \left( t, e^{u_1(t)} \right) e^{u_2(t-\tau_2(t))} \right] \Delta t \\ &= \int_{\kappa}^{\kappa+\omega} u_1'(t) \Delta t = 0, \\ & \int_{\kappa}^{\kappa+\omega} \lambda \left[ -d(t) + h \left( t, e^{u_1(t-\tau_3(t))} \right) \right] \Delta t \\ &= \int_{\kappa}^{\kappa+\omega} u_2'(t) \Delta t = 0. \end{aligned}$$

That is

$$\begin{aligned} & \int_{\kappa}^{\kappa+\omega} f \left( t, e^{u_1(t-\tau_1(t))} \right) \Delta t \\ &= \int_{\kappa}^{\kappa+\omega} g \left( t, e^{u_1(t)} \right) e^{u_2(t-\tau_2(t))} \Delta t, \quad (8) \\ & \int_{\kappa}^{\kappa+\omega} h \left( t, e^{u_1(t-\tau_3(t))} \right) \Delta t = \int_{\kappa}^{\kappa+\omega} d(t) \Delta t = \hat{d}\omega. \quad (9) \end{aligned}$$

By (8), we have

$$\hat{r}\omega = \int_{\kappa}^{\kappa+\omega} \left[ r(t) - f \left( t, e^{u_1(t-\tau_1(t))} \right) + g \left( t, e^{u_1(t)} \right) e^{u_2(t-\tau_2(t))} \right] \Delta t, \quad (10)$$

It follows from (7), (9), (10), and (H<sub>4</sub>)–(H<sub>6</sub>) that

$$\begin{aligned} & \int_{\kappa}^{\kappa+\omega} |u_1'(t)| \Delta t \\ & \leq \lambda \int_{\kappa}^{\kappa+\omega} \left| f \left( t, e^{u_1(t-\tau_1(t))} \right) - g \left( t, e^{u_1(t)} \right) \right. \\ & \quad \left. \times e^{u_2(t-\tau_2(t))} \right| \Delta t \\ & < \int_{\kappa}^{\kappa+\omega} \left[ r(t) - f \left( t, e^{u_1(t-\tau_1(t))} \right) + g \left( t, e^{u_1(t)} \right) \right. \\ & \quad \left. \times e^{u_2(t-\tau_2(t))} \right] \Delta t + \int_{\kappa}^{\kappa+\omega} |r(t)| \Delta t \\ &= (\hat{R} + \hat{r})\omega, \quad (11) \end{aligned}$$

$$\begin{aligned} & \int_{\kappa}^{\kappa+\omega} |u_2'(t)| \Delta t \\ & \leq \lambda \int_{\kappa}^{\kappa+\omega} \left| -d(t) + h \left( t, e^{u_1(t-\tau_3(t))} \right) \right| \Delta t \\ & < \int_{\kappa}^{\kappa+\omega} |d(t)| \Delta t + \int_{\kappa}^{\kappa+\omega} h \left( t, e^{u_1(t-\tau_3(t))} \right) \Delta t \\ &= (\hat{D} + \hat{d})\omega. \quad (12) \end{aligned}$$

Since  $(u_1(t), u_2(t))^T \in X$ , there exist  $\xi_i, \eta_i \in I_{\omega} (i = 1, 2)$  such that

$$u_i(\xi_i) = \min_{t \in I_{\omega}} u_i(t), \quad u_i(\eta_i) = \max_{t \in I_{\omega}} u_i(t), \quad i = 1, 2. \quad (13)$$

By (9), (13) and the monotonicity of  $h$  and  $\tilde{h}$ , we will show that  $u_1(\xi_1)$  and  $u_1(\eta_1)$  can not simultaneously lie in  $(-\infty, \ln v_-)$ ,  $(\ln v_-, \ln v_+)$  or  $(\ln v_+, +\infty)$ . In fact, if

$u_1(\xi_1) \leq u_1(\eta_1) < \ln v_-$ , then

$$\begin{aligned} \hat{d} &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h \left( t, e^{u_1(t-\tau_3(t))} \right) \Delta t \\ &\leq \tilde{h} \left( e^{u_1(\eta_1)} \right) < \tilde{h}(v_-) = \hat{d}. \end{aligned}$$

This is a contradiction. If  $\ln v_+ < u_1(\xi_1) \leq u_1(\eta_1)$ , then

$$\begin{aligned} \hat{d} &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h \left( t, e^{u_1(t-\tau_3(t))} \right) \Delta t \\ &\leq \tilde{h} \left( e^{u_1(\xi_1)} \right) < \tilde{h}(v_+) = \hat{d}. \end{aligned}$$

This is also a contradiction. If  $\ln v_- < u_1(\xi_1) \leq u_1(\eta_1) < \ln v_+$ , then

$$\begin{aligned} \hat{d} &= \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} h \left( t, e^{u_1(t-\tau_3(t))} \right) \Delta t \\ &\geq \min \left\{ \tilde{h} \left( e^{u_1(\xi_1)} \right), \tilde{h} \left( e^{u_1(\eta_1)} \right) \right\} > \tilde{h}(v_{\pm}) = \hat{d}. \end{aligned}$$

This is also a contradiction. Consequently, the distribution of  $u_1(\xi_1)$  and  $u_1(\eta_1)$  only have following two cases.

**Case 1.**  $u_1(\xi_1) \leq \ln v_- \leq u_1(\eta_1)$ . By Lemma 5, we obtain from (11) that for  $t \in I_{\omega}$

$$\begin{aligned} u_1(t) &\geq u_1(\eta_1) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_1'(t)| \Delta t \\ &> \ln v_- - \frac{\hat{R} + \hat{r}}{2} \omega := \beta_1, \quad (14) \end{aligned}$$

$$\begin{aligned} u_1(t) &\leq u_1(\xi_1) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_1'(t)| \Delta t \\ &< \ln v_- + \frac{\hat{R} + \hat{r}}{2} \omega := \beta_2. \quad (15) \end{aligned}$$

**Case 2.**  $u_1(\xi_1) \leq \ln v_+ \leq u_1(\eta_1)$ . By Lemma 5, we also obtain from (11) that for  $t \in I_{\omega}$

$$\begin{aligned} u_1(t) &\geq u_1(\eta_1) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_1'(t)| \Delta t \\ &> \ln v_+ - \frac{\hat{R} + \hat{r}}{2} \omega := \beta_3, \quad (16) \end{aligned}$$

$$\begin{aligned} u_1(t) &\leq u_1(\xi_1) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_1'(t)| \Delta t \\ &< \ln v_+ + \frac{\hat{R} + \hat{r}}{2} \omega := \beta_4. \quad (17) \end{aligned}$$

By (H<sub>7</sub>), we know

$$\beta_1 < \ln v_- < \beta_2 < \beta_3 < \ln v_+ < \beta_4. \quad (18)$$

Denote

$$\begin{aligned} \tilde{f}^M &= \max_{v \in [e^{\beta_1}, e^{\beta_4}]} \tilde{f}(v), \quad \tilde{f}^m = \min_{v \in [e^{\beta_1}, e^{\beta_4}]} \tilde{f}(v), \\ g^m &= \min_{t \in I_{\omega}, v \in [e^{\beta_1}, e^{\beta_4}]} g(t, v). \end{aligned}$$

By (H<sub>3</sub>), (H<sub>5</sub>), and (H<sub>8</sub>), one can easily see that  $\tilde{f}^M, \tilde{f}^m$  and  $g^m$  are positive constants. Noticing that

$$e^{\beta_1} \leq e^{u_1(t-\tau_1(t))} \leq e^{\beta_4},$$

it follows from (8), (13), and (H<sub>5</sub>) that

$$\begin{aligned} \tilde{f}^m &\leq \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} f(t, e^{u_1(t-\tau_1(t))}) \Delta t \\ &\leq \frac{e^{u_2(\eta_2)}}{\omega} \int_{\kappa}^{\kappa+\omega} g(t, e^{u_1(t)}) \Delta t \leq c_0 e^{u_2(\eta_2)}, \end{aligned}$$

which implies

$$u_2(\eta_2) \geq \ln \tilde{f}^m - \ln c_0. \quad (19)$$

Similarly, we also have

$$\begin{aligned} \tilde{f}^M &\geq \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} f(t, e^{u_1(t-\tau_1(t))}) \Delta t \\ &\geq \frac{e^{u_2(\xi_2)}}{\omega} \int_{\kappa}^{\kappa+\omega} g(t, e^{u_1(t)}) \Delta t \geq g^m e^{u_2(\xi_2)}, \end{aligned}$$

which implies

$$u_2(\xi_2) \leq \ln \tilde{f}^M - \ln g^m. \quad (20)$$

By Lemma 5, we obtain from (12), (19) and (20) that for  $t \in I_\omega$

$$\begin{aligned} u_2(t) &\geq u_2(\eta_2) - \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2'(t)| \Delta t \\ &> \ln \tilde{f}^m - \ln c_0 - \frac{\hat{D} + \hat{d}}{2} \omega := \beta_5. \end{aligned} \quad (21)$$

$$\begin{aligned} u_2(t) &\leq u_2(\xi_2) + \frac{1}{2} \int_{\kappa}^{\kappa+\omega} |u_2'(t)| \Delta t \\ &< \ln \tilde{f}^M - \ln g^m + \frac{\hat{D} + \hat{d}}{2} \omega := \beta_6. \end{aligned} \quad (22)$$

In view of (21) and (22) we have

$$\max_{t \in I_\omega} |u_2(t)| \leq \max\{|\beta_5|, |\beta_6|\} := \beta_7. \quad (23)$$

Clearly,  $\beta_1, \beta_2, \beta_3, \beta_4$  and  $\beta_7$  are independent of  $\lambda$ .

By the monotonicity of  $\tilde{h}$ , and (H<sub>4</sub>)–(H<sub>8</sub>), it is easy to show that algebraic equations

$$\begin{cases} \tilde{f}(e^{u_1}) - \tilde{g}(e^{u_1})e^{u_2} = 0, \\ -\hat{d} + \tilde{h}(e^{u_1}) = 0, \end{cases} \quad (24)$$

has two distinct solutions  $u_\pm = (\ln v_\pm, \ln \tilde{f}(v_\pm) - \ln \tilde{g}(v_\pm))^T$ . Choose  $\beta_0$  such that

$$\beta_0 > \max \left\{ \left| \ln \frac{\tilde{f}(v_-)}{\tilde{g}(v_-)} \right|, \left| \ln \frac{\tilde{f}(v_+)}{\tilde{g}(v_+)} \right| \right\}. \quad (25)$$

We now take

$$\begin{aligned} \Omega_- &= \{(u_1(t), u_2(t))^T \in X : \\ &\quad u_1(t) \in (\beta_1, \beta_2), |u_2(t)| < \beta_7 + \beta_0\}, \\ \Omega_+ &= \{(u_1(t), u_2(t))^T \in X : \\ &\quad u_1(t) \in (\beta_3, \beta_4), |u_2(t)| < \beta_7 + \beta_0\}. \end{aligned}$$

Then both  $\Omega_-$  and  $\Omega_+$  are bounded open subsets of  $X$ . It follows from (18) and (25) that  $u_\pm \in \Omega_\pm$  and  $\bar{\Omega}_- \cap \bar{\Omega}_+ = \emptyset$ . With the help of (14)–(18), (23) and (25), it is easy to see that  $\Omega_\pm$  satisfies condition (a) in Lemma 4. When

$(u_1(t), u_2(t))^T \in \partial\Omega_\pm \cap \text{Ker}L = \partial\Omega_\pm \cap \mathbb{R}^2$ ,  $(u_1(t), u_2(t))^T$  is a constant vector in  $\mathbb{R}^2$ . Thus, we have

$$QN \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \tilde{f}(e^{u_1}) - \tilde{g}(e^{u_1})e^{u_2} \\ -\hat{d} + \tilde{h}(e^{u_1}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This proves that condition (b) in Lemma 4 is satisfied.

Taking  $J = I : \text{Im}Q \rightarrow \text{Ker}L, (u_1, u_2)^T \rightarrow (u_1, u_2)^T$ , a direct calculation shows that

$$\deg(JQN, \Omega_\pm \cap \text{Ker}L, 0) = \text{sign}\{\tilde{f}(v_\pm)\tilde{h}'(v_\pm)\} = \mp 1 \neq 0.$$

Hence,  $\Omega_\pm$  satisfies all the requirements in Lemma 4. Consequently, system (5) has at least two  $\omega$ -periodic solutions  $u^*(t)$  and  $u^\dagger(t)$  in  $\text{Dom}L \cap \bar{\Omega}_-$  and  $\text{Dom}L \cap \bar{\Omega}_+$ , respectively. This completes the proof. ■

By Remark 1, we know that (4) and (6) have at least two  $\omega$ -periodic solutions strictly positive components if (5) has at least two  $\omega$ -periodic solutions.

**Corollary 1.** Suppose that (H<sub>1</sub>)–(H<sub>8</sub>) hold, then both (4) and (6) have at least two  $\omega$ -periodic solutions with strictly positive components.

**Remark 2.** In their Theorem 2.1, Ding and Jiang [2] proved that (4) has at least two  $\omega$ -periodic solutions with strictly positive components if (H<sub>1</sub>)–(H<sub>6</sub>),

$$(H_7') \quad v_- e^{2(\hat{R}+\hat{r})\omega} < v_+ \quad \text{and} \quad (H_8') \quad v_+ e^{(\hat{R}+\hat{r})\omega} < \alpha$$

hold. Obviously, the conditions (H<sub>7</sub>') and (H<sub>8</sub>') imply (H<sub>7</sub>) and (H<sub>8</sub>), respectively. Hence, our Corollary 1 improves Theorem 2.1 of [2].

#### IV. APPLICATIONS

In this section, we will list some applications of our above results.

**Example 1.** Consider the following system:

$$\begin{cases} u_1^\Delta(t) = r(t) - a(t)e^{u_1(t-\tau_1(t))} - \frac{b(t)e^{u_2(t-\tau_2(t))}}{e^{2u_1(t)}/m + e^{u_1(t)} + n}, \\ u_2^\Delta(t) = -d(t) + \frac{c(t)e^{u_1(t-\tau_3(t))}}{e^{2u_1(t-\tau_3(t))}/m + e^{u_1(t-\tau_3(t))} + n}, \end{cases} \quad (26)$$

which is a special case of (5) by letting

$$\begin{aligned} f(t, v) &= r(t) - a(t)v, \quad g(t, v) = \frac{b(t)}{v^2/m + v + n}, \\ h(t, v) &= \frac{c(t)v}{v^2/m + v + n}, \end{aligned}$$

where  $r, d, \tau_1, \tau_2, \tau_3 \in C(\mathbb{T}, \mathbb{R}^+)$  are  $\omega$ -periodic functions,  $m$  and  $n$  are positive constants. By Theorem 1, we get the following result.

**Theorem 2.** Suppose that

$$\begin{aligned} (1) \quad \hat{c} &> \max \left\{ \hat{d} \left( 1 + 2\sqrt{\frac{n}{m}} \right), \hat{d} \left( 1 + 2\sqrt{\frac{n}{m}} e^{(\hat{R}+\hat{r})\omega/2} \right) \right. \\ &\quad \left. - \frac{1}{m} \sqrt{m^2(\hat{c} - \hat{d})^2 - 4mn\hat{d}^2} \right\}, \\ (2) \quad 2\hat{r}\hat{d} &> \hat{a} \left[ m(\hat{c} - \hat{d}) + \sqrt{m^2(\hat{c} - \hat{d})^2 - 4mn\hat{d}^2} \right] e^{(\hat{R}+\hat{r})\omega/2} \end{aligned}$$

hold, then system (26) has at least two  $\omega$ -periodic solutions.

**Corollary 2.** Suppose that the assumptions (1) and (2) hold, then both (1) and (2) have at least two  $\omega$ -periodic solutions with strictly positive components.

**Remark 3.** In his Theorem 2.2, Chen [1] proved that system (1) has at least two  $\omega$ -periodic solutions with strictly positive component under the conditions

$$(1') \hat{c} > \hat{d} \left( 1 + 2\sqrt{\frac{n}{m}} \right) e^{(\hat{R}+\hat{r})\omega},$$

$$(2') 2\hat{r}\hat{d} > \hat{a} \left[ m(\hat{c}e^{(\hat{R}+\hat{r})\omega} - \hat{d}) + \sqrt{m^2(\hat{c}e^{(\hat{R}+\hat{r})\omega} - \hat{d})^2 - 4mnd\hat{d}^2} \right] e^{(\hat{R}+\hat{r})\omega}.$$

Obviously, (1') implies (1). Notice that

$$m(\hat{c}e^{(\hat{R}+\hat{r})\omega} - \hat{d}) + \sqrt{m^2(\hat{c}e^{(\hat{R}+\hat{r})\omega} - \hat{d})^2 - 4mnd\hat{d}^2} > \left[ m(\hat{c} - \hat{d}) + \sqrt{m^2(\hat{c} - \hat{d})^2 - 4mnd\hat{d}^2} \right] e^{(\hat{R}+\hat{r})\omega},$$

thus (2') also implies (2). Hence, our Corollary 2 improves the Theorem 2.2 in [1]. Similarly, our Corollary 2 also improves Theorem 2.1 of [7].

**Example 2.** Consider the following system:

$$\begin{cases} u_1^\Delta(t) = \frac{r(t) - a(t)e^{u_1(t-\tau_1(t))}}{n + e^{u_1(t-\tau_1(t))}} - \frac{b(t)e^{u_2(t-\tau_2(t))}}{m^2 + e^{2u_1(t)}}, \\ u_2^\Delta(t) = -d(t) + \frac{c(t)e^{u_1(t-\tau_3(t))}}{m^2 + e^{2u_1(t-\tau_3(t))}}, \end{cases} \quad (27)$$

which is a special case of (5) by letting

$$f(t, v) = \frac{r(t) - a(t)v}{n + v}, \quad g(t, v) = \frac{b(t)}{m^2 + v^2}, \\ h(t, v) = \frac{c(t)v}{m^2 + v^2},$$

where all functions and constants are defined as above, the prey population follows Smith [19] model. By Theorem 1, we have the following result.

**Theorem 3.** Suppose that

$$(1) \hat{c} > \max \left\{ 2m\hat{d}e^{(\hat{R}+\hat{r})\omega/2} - \sqrt{\hat{c}^2 - 4m^2\hat{d}^2}, 2m\hat{d} \right\},$$

$$(2) 2\hat{r}\hat{d} > \hat{a} \left( \hat{c} + \sqrt{\hat{c}^2 - 4m^2\hat{d}^2} \right) e^{(\hat{R}+\hat{r})\omega/2},$$

hold, then system (27) has at least two  $\omega$ -periodic solutions.

**Example 3.** Consider the following system:

$$\begin{cases} u_1^\Delta(t) = r(t) + a(t)e^{u_1(t-\tau_1(t))} - e(t)e^{2u_1(t-\tau_1(t))} \\ \quad - \frac{b(t)e^{u_2(t-\tau_2(t))}}{m^2 + e^{2u_1(t)}}, \\ u_2^\Delta(t) = -d(t) + \frac{c(t)e^{u_1(t-\tau_3(t))}}{m^2 + e^{2u_1(t-\tau_3(t))}}, \end{cases} \quad (28)$$

which is a special case of (5) by letting

$$f(t, v) = r(t) + a(t)v - e(t)v^2, \quad g(t, v) = \frac{b(t)}{m^2 + v^2}, \\ h(t, v) = \frac{c(t)v}{m^2 + v^2},$$

where functions  $r, a, b, c, d, \tau_1, \tau_2, \tau_3$ , and constant  $m$  are defined as above,  $e$  is positive continuous  $\omega$ -periodic function, the prey population follows Allee effect [20] model. By Theorem 1, we have the following result.

**Theorem 4.** Suppose that

$$(1) \hat{c} > \max \left\{ 2m\hat{d}e^{(\hat{Q}+\hat{q})\omega/2} - \sqrt{\hat{c}^2 - 4m^2\hat{d}^2}, 2m\hat{d} \right\},$$

$$(2) 2\rho\hat{d} > \left( \hat{c} + \sqrt{\hat{c}^2 - 4m^2\hat{d}^2} \right) e^{(\hat{Q}+\hat{q})\omega/2},$$

hold, where

$$\rho = \frac{\hat{a} + \sqrt{\hat{a}^2 + 4\hat{r}\hat{e}}}{2\hat{e}}, \quad q(t) = \frac{a^2(t) + 4r(t)e(t)}{4e(t)},$$

then system (28) has at least two  $\omega$ -periodic solutions.

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