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Existence of solution for singular two-piont boundary value problem of second-order differential equation

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Abstract—In this paper , by constructing a special set and utilizing fixed point theory in coin , we study the existence of solution of singular two points boundary value problem for second-order differential equation , which improved and generalize the result of related paper .

Keywords—singular differential equation , boundary value problem, coin , fixed point theory .

I. Introduction

THE theory of singular differential equation is emerging as an important area of investigation since it is much richer than the corresponding theory of concerning equation without singular . Recently,some existence results concerning the general boundary value problem of singular differential equation have been obtained ([1-3]) , in thesis [4].wang proved the existence of solution for the general boundary value problem for the second-order differential equation:

$$\begin{cases} x^{''}(t)=f(t,x(t)), \\ \alpha x(0)-\beta x^{'}(0)=0, \\ \gamma x(1)+\delta x^{'}(1)=0, \end{cases}$$

however , the f is a function without the term x' , motivated by the work of Wang , in this paper we study the following second-order differential equation

$$\begin{cases} x^{''}(t)=f(t,x(t)x^{'}(t)), \\ \alpha x(0)-\beta x^{'}(0)=0, \\ \gamma x(1)+\delta x^{'}(1)=0, \end{cases} \quad (1)$$

where $f\in C[J\times R^2,R], J=[0,1], f(..)$ may be singular at $t=0,1$, that is $\lim_{t\to 0}\|f(t..)\|=\infty$, $\lim_{t\to 1}\|f(t..)\|=\infty$, let $C[J,R]=\{x: J\to R \mid x$ is continuous in $J\}$, and $C^1[J,R]=\{x\in C[J,R]\mid x^{'}$ is continuous in $J\}$, It is easy to prove that $C[J,R]$ is a Banach space with norm $\| x\|=\max_{t\in J}|x(t)|$, $C^1[J,R]$ is also a Banach space with norm $\|x\|_1=\max_{t\in J}\{|x(t)|,|x^{'}(t)|\}$, A map $x\in C^1[J,R]\bigcap C^2[J,R]$ is called a solution of (1) if it satisfies all equations of (1) . For convenience sake , we list the definition and preliminary lemmas .

Definition 1.1 let E be a real Banach space , if P is a convex close set and satisfied the following conditions : $(1)x\in P,\lambda\geq 0\Rightarrow \lambda x\in P;(2)x\in P,-x\in P\Rightarrow x=\theta,\theta$ is element zero of E , we call P is a coin in E .

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Lemma 1.1 Assume that $\Delta=\alpha\gamma+\alpha\delta+\beta\gamma$,

$$G(t,s)=\begin{cases} \frac{1}{\Delta}(\beta+\alpha t)(\delta+\gamma(1-s)), 0\leq t\leq s\leq 1, \\ \frac{1}{\Delta}(\beta+\alpha s)(\delta+\gamma(1-t)), 0\leq s\leq t\leq 1. \end{cases}$$

Let $y\in C[J,E]$, then

$$\begin{cases} x^{''}(t)=y, \\ \alpha x(0)-\beta x^{'}(0)=0, \\ \gamma x(1)+\delta x^{'}(1)=0. \end{cases}$$

has a unique solution in $C^2[J,R]$ given by $x(t)=\int_0^1 G(t,s)y(s)ds$. We also easily obtain $G(t,s)\leq G(s,s)=e(s)$ and $G(t,s)\leq G(t,t)=e(t), 0<t,s<1$.

Lemma 1.2 ([5])Let K be a coin in a Banach space , $\underline{0<r<R}, B(\theta,R)=\{x\in K\mid \|x\|\underline{\leq R}\}, \overline{K_R}=\overline{B(\theta,R)\bigcap K}$. Suppose that operator $A:\overline{K_R}\to K$ is a completely continuous such that following conditions are satisfied : for $x\in K, \|x\|=R, \|A(x)\|\leq\|x\|$, and for $\underline{x\in K}, \|x\|=r\ \|A(x)\|\geq\|x\|$, then A has a fixed point in $\overline{K_R}\setminus K_r$.

II. Conclusion

Theorem2.1 Let $f:(0,1)\times R^2\to R$ be a continuous function , suppose that the following conditions are satisfied :

(H_1) For all $(x,y)\in R^2$ and $t\in(0,1)$, $f(t,x,y)=p_1(t)q_1(x)+p_2(t)q_2(y)+r(t)$, where $p_i,q_i\in C[(0,1),R]$, and $\int_0^1 e(s)p_i(s)ds<+\infty (i=1,2), r(t)\in C[J,R]$ and $\int_0^1 e(s)r(s)ds<+\infty$.

(H_2)There exist constant $a>0$ such that

$$\int_0^1 e(t)p_1(t)\max_{x\in[g_a(t),a]}q_1(x)dt + \int_0^1 e(t)p_2(t)\max_{y\in[g_a(t),a]}q_2(y)dt + \int_0^1 e(t)r(t)dt<a,$$

where

$$G_a(t)=\begin{cases} at, 0\leq t\leq\frac{1}{2}, \\ a(1-t), \frac{1}{2}\leq t\leq 1. \end{cases}$$

(H_3)There exist constant $b \in (0,a)$ such that

$$\| \int_0^1 Me(t) \min_{x\in[g_b(t),b]} f(t,x,x^{'})dt \| \geq b,$$

where $M = \min\{\frac{\beta+\theta\alpha}{\beta+\alpha}, \frac{\delta+\theta\gamma}{\delta+\gamma}\}$, then the problem (1) has at least one solution $x \in C^2[0,1]$.

Proof : Operator $A : C^1[0,1] \to C^1[0,1]$ is defined as follows : $Ax(t) = \int_0^1 G(t,s)f(s,x(s),x^{'}(s))ds$. It is easy to see $x(t)$ is a solution of (1) if and only if $x(t)$ is a fixed point of the operator $Ax(t) = x(t)$. Let $E = C^1[0,1], K = \{x \in C^1[0,1], \min_{t\in(\frac{1}{4}\frac{3}{4})} x(t) \geq M \| x(t) \|\}$, it is easy to see K is a coin in E .

First , we'll prove that operator A is a completely continuous one mapping K into K . Let D be a arbitrary bounded set of $C^1[0,1]$, for all $x \in D$, $\sup q_i[0, \| x \|]$ is defined as follow: $\sup q_i[0, \| x \|_1] = \sup\{q_i(y) \mid y \in [0, \| x \|_1]\}(i = 1,2)$. therefore ,

$$\begin{aligned} Ax(t) &= \int_0^1 G(t,s)f(s,x(s),x^{'}(s))ds \\ &\leq \int_0^1 e(s)p_1(s)q_1(x(s))ds \\ &+ \int_0^1 e(s)p_2(s)q_2(x^{'}(s))ds + \int_0^1 e(s)r(s)ds, \end{aligned}$$

hence,

$$\begin{aligned} \| Ax \| &\leq \int_0^1 e(s)f(s,x(s),x^{'}(s))ds \\ &\leq \sup q_1[0, \| x \|_1] \int_0^1 e(s)p_1(s)ds \\ &+ \sup q_2[0, \| x \|_1] \int_0^1 e(s)p_2(s)ds + \int_0^1 e(s)r(s)ds. \end{aligned}$$

In view of (H_1) , we easily obtain that AD are all uniformly bounded on J. On the other hand , according to Lebesgue control collect theory , we know that A is continuous on J. For $\frac{1}{4} < t < \frac{3}{4}$, we have

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{\varphi(t)}{\varphi(s)}, s \leq t, \\ \frac{\psi(t)}{\psi(s)}, t \leq s, \end{cases} \geq \begin{cases} \frac{\delta+\theta\gamma}{\delta+\gamma}, s \leq t, \\ \frac{\beta+\theta\alpha}{\beta+\alpha}, t \leq s, \end{cases}$$

where

$$\varphi(t) = (\gamma + \delta - \gamma t), \psi(t) = (\beta + \alpha t), 0 \leq t \leq 1,$$

so we get $\frac{G(t,s)}{G(s,s)} \geq M$, therefore $G(t,s) \geq Me(s)$.

If $x \in K$, then

$$\begin{aligned} \min_{\frac{1}{4}<t<\frac{3}{4}} Ax(t) &= \min_{\frac{1}{4}<t<\frac{3}{4}} \int_0^1 G(t,s)f(s,x(s),x^{'}(s))ds \\ &\geq M \int_0^1 e(s)f(s,x(s),x^{'}(s))ds \\ &\geq M \| Ax \|, \end{aligned} \tag{2}$$

so $Ax \in K$, which imply that $AK \subset K$. For any $x \in [0,+\infty)$, we can define $P_{i,n}(t)(i = 1,2)$ as follow :

$$P_{i,n}(t) = \begin{cases} \min\{p_i(t), p_i(\frac{1}{n})\}, & 0 \leq t \leq \frac{1}{n}, \\ p_i(t), & \frac{1}{n} \leq t \leq \frac{n-1}{n}, \\ \min\{p_i(t), p_i(\frac{n-1}{n})\}, & \frac{n-1}{n} \leq t \leq 1. \end{cases}$$

Let $p_{1,n}(t)q_1(x) + p_{2,n}(t)q_2(x^{'}) + r(t) = f_n(t,x,x^{'})$, it is easy to see for any $t \in [0,1]$,

$$f_n(t,x,x^{'}) = \begin{cases} \min\{f(t,x,x^{'}), f(\frac{1}{n},x,x^{'})\}, 0 \leq t \leq \frac{1}{n}, \\ f(t,x,x^{'}), & \frac{1}{n} \leq t \leq 1-\frac{1}{n}, \\ \min\{f(t,x,x^{'}), f(1-\frac{1}{n},x,x^{'})\}, 1-\frac{1}{n} \leq t, \end{cases}$$

correspondingly we can define

$$A_n x(t) = \int_0^1 G(t,s)f_n(s,x(s),x^{'}(s))ds, n \geq 2. \tag{3}$$

Obviously , for any $n \geq 2$ we can see that $f_n(t,x,x^{'})$ is continuous in $[0,1] \times [0,+\infty) \times [0,+\infty)$ and $f_n(t,x,x^{'}) \leq f(t,x,x^{'}), P_{i,n}(t) \leq p_i(t)(i = 1,2)$, A_n is relatively compact in K. For $R > 0$, let $B_R = \{x \in K | \|x\| \leq R\}$, now we prove that A_n is approximate to A in B_R.

$$\begin{aligned} &| A_n x(t) - Ax(t) | \\ &\leq \int_0^{\frac{1}{n}} G(s,s)(f(s,x(s),x^{'}(s)) - f_n(s,x(s),x^{'}(s)))ds \\ &+ \int_{\frac{n-1}{n}}^1 G(s,s)(f(s,x(s),x^{'}(s)) - f_n(s,x(s),x^{'}(s)))ds \\ &\leq \int_0^{\frac{1}{n}} e(s)(p_1(s) - p_{1,n}(s))q_1(x(s))ds \\ &+ \int_0^{\frac{1}{n}} e(s)(p_2(s) - p_{2,n}(s))q_2(x^{'}(s))ds \\ &+ \int_{\frac{n-1}{n}}^1 e(s)(p_1(s) - p_{1,n}(s))q_1(x(s))ds \\ &+ \int_{\frac{n-1}{n}}^1 e(s)(p_2(s) - p_{2,n}(s))q_2(x^{'}(s))ds \\ &\leq \max q_1[0,\|x\|](\int_0^{\frac{1}{n}} + \int_{\frac{n-1}{n}}^1)e(s)(p_1(s) - p_{1,n}(s))ds \\ &+ \max q_2[0,\|x\|](\int_0^{\frac{1}{n}} + \int_{\frac{n-1}{n}}^1)e(s)(p_2(s) - p_{2,n}(s))ds. \end{aligned} \tag{4}$$

By condition $0 < \int_0^1 e(s)p_i(s)ds < +\infty$ and $0 < p_{i,n}(s) \leq p_i(s)$, we can get $| A_n x(t) - Ax(t) | \leq \epsilon(n \to \infty)$ which imply that A_n is approximate to A , so A is relatively compact in K.

Finally , we show that A has a fixed point . Let $\partial B_a = \partial B(\theta,a) \bigcap K$, $\forall x \in \partial B_a, \|x\| = a$, A is a convex function, therefore $x(s) \in [g_a(s),a], s \in [0,1]$, by condition H_2

$$\begin{aligned}
\|Ax\| &= \| \int_0^1 G(t,s) f(s, x(s), x^{'}(s)) \| \\
&\leq \| \int_0^1 e(s)(p_1(s) q_1(x(s)) ds \| \\
&\quad + \| \int_0^1 e(s)(p_2(s) q_2(x^{'}(s)) ds \| \\
&\quad + \| \int_0^1 e(s) r(s) ds \| \\
&\leq \int_0^1 e(s) p_1(s) \max_{x \in [g_a(s), a]} q_1(x) ds \\
&\quad + \int_0^1 e(s) p_2(s) \max_{y \in [g_a(s), a]} q_2(y) ds \\
&\quad + \int_0^1 e(s) r(s) ds \\
&< a. \qquad (5)
\end{aligned}$$

On the other hand , $\forall x \in \partial B_b, \| x \| = b, x(s) \in [g_b(s), b]$,

$$\begin{aligned}
\|Ax\| &= \| \int_0^1 G(t,s) f(s, x(s), x^{'}(s)) ds \| \\
&\geq \int_0^1 M e(s) \min_{x \in [g_b(s), b]} f(s, x(s), x^{'}(s)) ds \\
&\geq b. \qquad (6)
\end{aligned}$$

By lemma 2 , we can see that (1) has at least one solution .

As an example,we consider the following problem

$$\begin{cases} x^{''}(t) = t^{-\frac{1}{2}} x + (1-t)^{-\frac{1}{2}} x^{'} + t, t \in (0,1) \\ x(0) = x(1) = 0, \end{cases} \qquad (7)$$

it is easy to see $(H_1) - (H_3)$ are all satisfied , according to Theorem 2.1 , the problem (7) has at least a solution .

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