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Computations of Bezier Geodesic-like Curves on Spheres

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***Abstract*—**It is an important problem to compute the geodesics on a surface in many fields. To find the geodesics in practice, however, the traditional discrete algorithms or numerical approaches can only find a list of discrete points. The first author proposed in 2010 a new, elegant and accurate method, the geodesic-like method, for approximating geodesics on a regular surface. This paper will present by use of this method a computation of the Bezier geodesic-like curves on spheres.

***Keywords*—** Geodesics, Geodesic-like curve, Spheres, Bezier.

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I. Introduction

COMPUTATION of geodesics on parametric surface play a crucial role in many fields such as computer aid geometric design (CAGD), computer aid design (CAD), computer graphics and robotics etc. (Paluszny[6], Ravi Kumar et.al.[8, 9], Sanchez-Reyes and Dorado[10], Sprynski et.al.[11]) In the classical differential geometry, there are several ways to define a geodesic on a curved surface. First, a curve γ on a regular surface is a geodesic if its osculating plane at each point on the curve contains the normal vector of the surface at that point. Here the osculating plane at a point $\gamma(s_0)$ is the plane generated by the tangent vector and normal vector of γ at $\gamma(s_0)$. For example, a great circle on the unit sphere with center at the origin is a geodesic since the osculating plan at each point at the great circle passes through the origin. Secondly, a geodesic γ on a surface is a curve with zero geodesic curvature at each point of γ or equivalently a solution of geodesic equations (1). It is easily to know that a great circle is a geodesic $\gamma(s)$ on the unit sphere centered at the origin since the second derivative $\gamma''(s)$ is parallel to the position vector $\gamma(s)$, and hence is perpendicular to the tangent vector $\gamma'(s)$. Moreover, in the calculus of variation, a geodesic curve is a critical point of the energy function in equation (2). Therefore, the geodesics joining two given points on the surface have extreme lengths or energies. The readers can find more details about geodesics in Do Carmo[1].

However, in practice, the analytical approaches are complicated and difficult to find geodesics on a parametric surface. Some references are as follows. In 2000, Hotz and Hagen[3] improved the method of estimating geodesic from the first fundamental form, the second fundamental form and numerical method. Emin Kasap et. al. [4] also presented a numerical method for computation of geodesic with fixed endpoints in 2005. They improve these problems from the geodesic equations (1) and finite-difference. There are many different kinds of methods also estimate the geodesic or shortest path by discrete geodesic in CAGD and CAD. These methods approach geodesic on tessellated surfaces[13], polygonal surfaces[7] and triangular meshes[5,12].

A new method to the geodesic problem via the geodesic-like curves is proposed by the first author in 2010[2]. The geodesic-like curve is a mathematical curve, like as the Bezier curve and the B-spline curve. The geodesic-like method is a powerful tool for improving the geodesic problems and computation of geodesics. This paper will review the geodesic-like method and compute the system of geodesic-like equations on a sphere. .

II. Geodesics and Bezier Geodesic-like Curves

This section will review the definitions of geodesics and Bezier geodesic-like curves on surfaces. Let S be a regular surface with a parameterization b, $b:U\subset R^2\to S$, where (U,b) is a system of coordinate on S, and $\gamma(s)=(x_1(s),x_2(s))$ be a curve from $[0,1]$ to U. Note that b is a diffeomorphism between U and $b(U)$. Then γ is a geodesic curve in (U,b) on S if it is a solution of the following system of geodesic equations

$$\frac{d^2}{ds^2}x_k+\sum_{i,j=1}^{2}\Gamma_{i,j}^{2}\frac{dx_i}{ds}\frac{dx_j}{ds}=0 \tag{1}$$

, where $k=1,2$ and $\Gamma_{i,j}^{2}$ is the Christoffel symbol. From calculus of variation, a geodesic curve is equivalent to be a critical point of the energy variations. It is known that the energy function of γ is

$$E(\gamma)=\frac{1}{2}\int_0^1\left\|\frac{d}{ds}b(\gamma(s))\right\|^2 ds \tag{2}$$

Then the Bezier geodesic-like curves will be defined by this notion.

A proper variation of $b\circ\gamma$ is a differential map

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$h:[0,1]\times[-\varepsilon,\varepsilon]\to U$ such that

$$\begin{cases} h(s,0)=b\circ\gamma(s) & s\in[0,1] \\ h(0,t)=b\circ\gamma(0) & t\in[-\varepsilon,\varepsilon] \\ h(1,t)=b\circ\gamma(1) & t\in[-\varepsilon,\varepsilon] \end{cases}. \tag{3}$$

Therefore, $h_t(s)=h(s,t)$ for $(s,t)\in[0,1]\times[-\varepsilon,\varepsilon]$ is a family of curves with the same terminal points $b(\gamma(0))$ and $b(\gamma(1))$. Then the energy of h_t is represented by

$$E(t)=\int_0^1\left\|\frac{\partial(b\circ h)}{\partial s}\right\|^2 ds \quad \text{for} \quad t\in[-\varepsilon,\varepsilon]. \tag{4}$$

It is known that the geodesics can be characterized by the critical points of the energy functional. For simplicity, the geodesics can be defined as follows.

Definition 1

Let S be a regular surface with parameterization b. A curve $\gamma:[0,1]\to U$ is a geodesic on S if there is a proper variation h of $b\circ\gamma$ such that $E'(0)=0$ where h and E is defined in equations (3) and (4), respectively.

In 2010, the first author proposed a new method in [2] for estimating the geodesic from equation (4). By constructing the proper variation by the Bezier curves, the critical point of the energy function is called the Bezier geodesic-like curve.

Definition 2

Let $b(u,v)$, $b:U\to R^3$ where U is an open set in R^2, be a parameterization of a regular surface S. The Bezier geodesic-like curve $\gamma(s)$ of degree n, $\gamma:[0,1]\to U$, between (u_0,v_0) and (u_n,v_n) is a Bezier curve with terminal points (u_0,v_0) and (u_n,v_n) on U whose energy attend minimal. That is, $\gamma(s)=\sum_{i=0}^{n}B_i^n(s)(\tilde{u}_i,\tilde{v}_i)$ is a critical point of the following equation (5).

$$E(u_1,\cdots,u_{n-1},v_1,\cdots,v_{n-1}) = \int_0^1\left\|\frac{\partial}{\partial s}b(\sum_{i=0}^{n}B_i^n(s)(u_i,v_i))\right\|^2 ds \tag{5}$$

Since, by the following Weierstrass theorem, any piecewise differential curve can be approximate the B-spline curves, a Bezier geodesic-like curve approaches a geodesic on a surface when its degree tends to infinity.

Weierstrass theorem

Let $f:[0,1]\to R$ be a continuous function. Then the sequence of Bernstein polynomials

$$p_n(s)=\sum_{i=0}^{n}B_i^n(s)f(\frac{i}{n}) \tag{6}$$

, where $B_i^n(s)=C_i^n(1-s)^{n-i}s^i$ is the Bernstein polynomial, converges uniformly to f as $n\to\infty$.

The set of control points of geodesic-like curve is a critical points of $E(u_i,v_j)$, that is the solution of the system of

$$\begin{cases} E_{u_i}=\dfrac{\partial}{\partial u_i}E=0 & \text{for each } i=1\cdots n-1 \\ E_{v_j}=\dfrac{\partial}{\partial v_j}E=0 & \text{for each } j=1\cdots n-1 \end{cases}. \tag{7}$$

After some computations, the system of equations (7) can be rewritten as

$$\begin{cases} \int_0^1 <(b_{uu}u'+b_{uv}v')B_i^n(s)+b_u(\dfrac{d}{ds}B_i^n(s)),b_uu'+b_vv'>ds=0 \\ \int_0^1 <(b_{uv}u'+b_{vv}v')B_i^n(s)+b_v(\dfrac{d}{ds}B_i^n(s)),b_uu'+b_vv'>ds=0 \end{cases} \tag{8}$$

where $u=u(s)=\sum_{i=0}^{n}B_i^n(s)u_i$ and $v=v(s)=\sum_{i=0}^{n}B_i^n(s)v_i$.

Obviously, the solutions of the system of equations (7) are the Bezier geodesic-like curves on a surface. The system of equations (8) is called the system of geodesic-like equations.

III. The Bezier geodesic-like curves on shperes

In this section, a computation of the Bezier geodesic-like curves on a sphere via the system of equation (8) will be presented. Let $b(u,v)$ be the parameterization from $U=(0,2\pi)\times(0,2\pi)$ to the unit sphere S and let $\gamma(s)$ be a curve in U, $\gamma(s)=(u(s),v(s))$. Let

$$b(u,v)=\begin{bmatrix}\cos u\cos v\\ \cos u\sin v\\ \sin u\end{bmatrix} \tag{9}$$

, where $u,v\in(0,2\pi)$. The first partial derivatives of b with respect to u,v are

$$b_u=\begin{bmatrix}-\sin u\cos v\\ -\sin u\sin v\\ \cos u\end{bmatrix}, \tag{10}$$

$$b_v=\begin{bmatrix}-\cos u\sin v\\ \cos u\cos v\\ 0\end{bmatrix} \tag{11}$$

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And the second partial derivatives of b are

$$b_{uu} = \begin{bmatrix} -\cos u \cos v \\ -\cos u \sin v \\ -\sin u \end{bmatrix},$$

$$b_{uv} = \begin{bmatrix} \sin u \sin v \\ -\sin u \cos v \\ 0 \end{bmatrix},$$

$$b_{vv} = \begin{bmatrix} -\cos u \cos v \\ -\cos u \sin v \\ 0 \end{bmatrix}.$$

Then

$$b'(s) = \frac{d}{dt} b(\gamma(s)) = \begin{bmatrix} -u'\sin u \cos v - v'\cos u \sin v \\ -u'\sin \sin v + v'\cos u \cos v \\ u'\cos u \end{bmatrix}. \quad (12)$$

Hence the energy of $b(s)$ forms as

$$E(\gamma) == \int_0^1 (u'(s))^2 + (v'(s))^2 \cos^2 u(s) ds\,. \quad (13)$$

Assume that the curve $\gamma(s)$ is a Bezier curve $\gamma(s) = \sum_{i=0}^{n} B_i^n(s)(u_i, v_i)$. The first partial derivatives of the energy function $E(u_1, u_2, \cdots, u_{n-1}, v_1, \cdots, v_{n-1})$ are

$$\frac{\partial E}{\partial u_i} = \int_0^1 \frac{\partial}{\partial u_i}((u'(s))^2 + (v'(s))^2 \cos^2 u(s))) ds \quad (14)$$

and

$$\frac{\partial E}{\partial v_i} = \int_0^1 \frac{\partial}{\partial v_i}((u'(s))^2 + (v'(s))^2 \cos^2 u(s))) ds\,. \quad (15)$$

Since $\frac{\partial}{\partial u_i} u(s) = B_i^n(s)$, $\frac{\partial}{\partial v_i} u(s) = 0$, $\frac{\partial}{\partial u_i} v(s) = 0$ and $\frac{\partial}{\partial v_i} v(s) = B_i^n(s)$, the equations (14) and (15) can be rewritten as

$$\frac{\partial E}{\partial u_i} = \int_0^1 (\frac{dB_i^n(s)}{dt} u'(s) + (v'(s))^2 (B_i^n(s) \cos u(s) \sin u(s))) ds \quad (16)$$

and

$$\frac{\partial E}{\partial v_i} = \int_0^1 (\frac{dB_i^n(s)}{ds} v'(s) \cos^2 u(s)) ds\,. \quad (17)$$

The system of geodesic-like equations on sphere with polar coordinate system forms as

$$\begin{cases} \int_0^1 (\frac{dB_i^n(s)}{ds} u'(s) + B_i^n(s)(v'(s))^2 \cos u(s) \sin u(s)) ds &= 0 \\ \int_0^1 (\frac{dB_i^n(s)}{ds} v'(s) \cos^2 u(s)) ds &= 0 \end{cases} \quad (18)$$

Suppose that $c(s)$ is a horizontal vertical straight line, that is $u(s) = (1-s)p + sq$ and $v(s) = x$, where x is a constant and $p, q \in (0, 2\pi)$. Since $u'(s) = q - p$ and $v'(s) = 0$, the system of equation (18) becomes

$$\begin{cases} \int_0^1 (\frac{dB_i^n(s)}{ds} \cdot (q-p) + 0) ds = (q-p) \int_0^1 \frac{dB_i^n(s)}{ds} ds = 0 \\ \int_0^1 (\frac{dB_i^n(s)}{ds} \cdot 0) ds = 0 \end{cases}. \quad (19)$$

If $c(s) = (0, s)$, then the system of equation (18) is also satisfied.

These imply that the horizontal line $c(t) = (tp + (1-t)q, x)$ and the vertical line pass through (0,0) are geodesic-like curves on spheres with polar coordinate system. Actually, these curves are the geodesics on sphere.

Let $b(u,v) = (u, v, \sqrt{1-u^2-v^2})$, where $u^2 + v^2 < 1$, be another coordinate system of the unit sphere be. Using a similar method as above, the straight lines pass through $(0,0)$ on the set $\{(u,v) \mid u^2 + v^2 < 1\}$ are geodesic-like curves on sphere.

IV. Conclusion

This note reviews the geodesic-like method and uses it to compute the Bezier geodesic-like curves on spheres. Although considering only the polar coordinate of the unit sphere, it can be applied to another coordinate system. In fact, if a geodesic γ (as in Definition 1) is a Bezier curve, then it is also a Bezier geodesic-like curve. On the other hand, let γ_m be a given Bezier geodesic-like curve of degree m. If $\gamma_m = \gamma_n$ for any Bezier geodesic-like curve γ_n of degree $n > m$ passing through the same endpoints as γ_m does, then γ_m is a geodesic. It is because that a geodesic has minimal energy and a geodesic-like curve approaches a geodesic.

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