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Revealing Nonlinear Couplings between Oscillators from Time Series

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Abstract**—** Quantitative characterization of nonlinear directional couplings between stochastic oscillators from data is considered. We suggest coupling characteristics readily interpreted from a physical viewpoint and their estimators. An expression for a statistical significance level is derived analytically that allows reliable coupling detection from a relatively short time series. Performance of the technique is demonstrated in numerical experiments.

Keywords**—**Nonlinear time series analysis, directional couplings, coupled oscillators.

I. Introduction

THE key questions emerging in analysis of a complex system of any origin are whether interactions between its elements exist and how to quantify them. In particular, an architecture and strengths of couplings in an ensemble of oscillators determine possibility of their synchronization, e.g. [1]. Growing attention is currently paid to the detection and characterization of directional couplings from data, e.g. [2], [3], which is demanded in electronics [4], cardiology [5], [6], [7], neurophysiology [8], [9], etc.

A fruitful technique to reveal couplings between two oscillators is suggested in Ref. [2]. It is based on phase dynamics modeling and applies in case of a long time series or low noise level. Special corrections extending its applicability to shorter time series (several dozens of basic periods) are obtained in Ref. [3] and used to analyze complex processes from neurophysiology [10] and climatology [11]. However, the corrections apply only when a particular empirical phase model is used, namely, for a particular low-order polynomial in model equations. They are not suitable for other couplings.

Here, we suggest new quantitative characteristics of directional couplings based on the same idea of phase dynamics modeling and derive an analytic expression for a statistical significance level at which couplings are detected. The suggested estimators are suitable for couplings described with an arbitrary order of nonlinearity and apply to relatively short time series as illustrated in numerical experiments. Moreover, they can be easily extended to analyze couplings in ensembles oscillators.

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II. Previous characteristics

Phase dynamics of weakly coupled oscillators can be to a good approximation described with a set of stochastic differential equations [12]

$$d\phi_k/dt = \omega_k + G_k(\phi_k, \phi_j) + \xi_k(t), \qquad (1)$$

where $k, j = 1,2,\ k \neq j$, are phases of the oscillators, ω_k are their natural frequencies, ξ_k are independent zero-mean white noises with autocorrelation functions (ACFs) $\langle \xi_k(t)\xi_k(t') \rangle = \sigma^2_{\xi_k}\delta(t-t')$, $\sigma^2_{\xi_k}$ characterize noise intensities. The functions G_k are 2π-periodic with respect to both arguments and describe both couplings between the oscillators and their individual phase nonlinearities.

Let $\sigma^2_{\xi_k}$ and $|G_k|$ be reasonably small so that the contribution of the respective terms in (1) to the phase increment $\Delta\phi_k(t) = \phi_k(t+\tau) - \phi_k(t)$ is small in comparison with the ``linear increment" $\omega_k\tau$, where a time interval τ is of the order of a basic oscillation period or greater. Then, one can convert to difference equations via integration of Eq. (1) and get

$$\Delta\phi_k(t) = F_k(\phi_k(t), \phi_j(t)) + \varepsilon_k(t), \qquad (2)$$

where $k, j = 1,2,\ k \neq j$, $\varepsilon_k(t)$ are zero-mean noises, F_k are trigonometric polynomials

$$F_k(\phi_k, \phi_j, \mathbf{a}_k) = w_k + \sum_{(m,n)\in\Omega_k} (\alpha_{k,m,n}\cos(m\phi_k - n\phi_j) + \beta_{k,m,n}\sin(m\phi_k - n\phi_j)), \qquad (3)$$

$\mathbf{a}_k = (w_k, \{\alpha_{k,m,n}, \beta_{k,m,n}\}_{(m,n)\in\Omega_k})$ are vectors of their coefficients, and Ω_k are summation ranges, i.e. sets of pairs (*m*, *n*) showing which monomials are contained in F_k. The terms with $m = n = 1$ can be induced by linear coupling of the form $k_{2\to1}x_2$ or $k_{2\to1}(x_2 - x_1)$ in the “original equations” of the oscillators, while the terms with $n = 2$ can be due to a driving force which is quadratic with respect to the coordinate of the driving oscillator, e.g. $k_{2\to1}x_2^2$. Various combinations are also possible so that couplings in the phase dynamics equations can be described with a set of monomials of different orders with $n \neq 0$. The strongest driving arises from the “resonant terms”, i.e. that corresponding to the ratios $m/n \approx \omega_j/\omega_k$ in the equation for the *k*th oscillator phase. However, non-resonant terms can be also significant.

III. NEW CHARACTERISTICS

We introduce intensity of the coupling $j \to k$ (i.e. from the *j*th oscillator to the *k*th one) as follows. Let us consider statistical properties of the phase increment $\Delta\phi_k$, i.e. the left-hand side of Eq. (2), under the above conditions of weak coupling, weak nonlinearity, and low noise level. Its mean value is $\langle \Delta\phi_k \rangle = w_k \approx \omega_k \tau$. Due to noise and driving from the other oscillator, $\Delta\phi_k$ somewhat varies about the mean value (phase modulation). Stationary probability distribution of the wrapped phases $(\phi_1 \bmod 2\pi, \phi_2 \bmod 2\pi)$ is almost uniform over the square $[0,2\pi)\times[0,2\pi)$ and monomials in F_k are mutually orthogonal in this domain with the uniform weight function. Therefore, taking expectation of squared values for both sides of Eq. (2) gives

$$\left\langle (\Delta\phi_k)^2 \right\rangle = w_k^2 + \frac{1}{2}\sum_{(m,n)\in\Omega_k}\left(\alpha_{k,m,n}^2 + \beta_{k,m,n}^2\right) + \sigma_{\varepsilon_k}^2 , \qquad (4)$$

where angle brackets stand for expectation and $\sigma_{\varepsilon_k}^2$ is variance of the noise ε_k. The terms with $n \neq 0$ describe the driving $j \to k$. We denote their sum as $c_{j\to k}$ and call it "coupling intensity":

$$c_{j\to k} = \frac{1}{2}\sum_{(m,n)\in\Omega_k,\, n\neq 0}\left(\alpha_{k,m,n}^2 + \beta_{k,m,n}^2\right). \qquad (5)$$

The rest of the terms describe individual phase dynamics and are denoted b_k:

$$b_k = \frac{1}{2}\sum_{(m,n)\in\Omega_k,\, n=0}\left(\alpha_{k,m,n}^2 + \beta_{k,m,n}^2\right). \qquad (6)$$

Recalling $\langle \Delta\phi_k \rangle = w_k$, one can see that the variance of the phase increment $\sigma_{\Delta\phi_k}^2 = \left\langle (\Delta\phi_k)^2 \right\rangle - \langle \Delta\phi_k \rangle^2$ is determined by the three terms:

$$\sigma_{\Delta\phi_k}^2 = b_k + c_{j\to k} + \sigma_{\varepsilon_k}^2 . \qquad (7)$$

The quantity $c_{j\to k}$ can be normalized in different ways. Thus, $c_{j\to k}/\sigma_{\Delta\phi_k}^2$ shows a portion of phase increment variance induced by the driving $j \to k$. It is less than unity but may almost reach it if individual phase nonlinearity and noise are weak in comparison with the driving $j \to k$. $c_{j\to k}$ depends on the parameter τ which is a time scale. Experience shows that it is reasonable to take τ equal to a basic oscillation [2], [3].

Everything is analogous for an ensemble consisting of more than two oscillators. The only difference is that more terms describing possible influences of different oscillators are present in the right-hand side of Eq. (2).

IV. ESTIMATION FROM TIME SERIES

In data analysis, one has time series from two systems, $\{x_1(t_1),\ldots,x_1(t_{N_x})\}$, $\{x_2(t_1),\ldots,x_2(t_{N_x})\}$, $t_i = i\Delta t$, where Δt is a sampling interval and N_x is a time series length. Phase dynamics equations are unknown, therefore, one cannot use the formula (5) directly. It is necessary to get statistical estimates of coupling characteristics. For that, one computes time series of the oscillation phases $\{\phi_1(t_1),\ldots,\phi_1(t_N)\}$, $\{\phi_2(t_1),\ldots,\phi_2(t_N)\}$, see e.g. [1]. Further, an empirical phase dynamics model is constructed and coupling characteristics estimates are obtained from the estimates of the model coefficients.

According to a traditional approach to empirical modeling, one assumes that observed phase dynamics can be described with equations (2) and functions (3) and selects some set of monomials in (3) which is assumed large enough to describe nonlinearities of the processes. Polynomial coefficients are estimated by minimizing $S(w_k,\alpha_k,\beta_k) = \frac{1}{N-\tau/\Delta t}\times$

$$\sum_{i=1}^{N-\tau/\Delta t}(\Delta\phi_k(t_i) - w_k - \alpha_k\cos(\phi_j-\phi_k) - \beta_k\sin(\phi_j-\phi_k))^2 .$$

Estimators of the noise variance and coefficients are denoted with hats and read

$$\hat{\sigma}_{\varepsilon_k}^2 = \min_{\mathbf{a}_k} S(\mathbf{a}_k) , \qquad (8)$$

$$\hat{\mathbf{a}}_k = \arg\min_{\mathbf{a}_k} S(\mathbf{a}_k) . \qquad (9)$$

Coupling intensity estimators can be expressed via $\hat{\mathbf{a}}_k$ analogously to Eq. (5):

$$\hat{c}_{j\to k} = \frac{1}{2}\sum_{(m,n)\in\Omega_k,\, n\neq 0}\left(\hat{\alpha}_{k,m,n}^2 + \hat{\beta}_{k,m,n}^2\right). \qquad (10)$$

The quantity $\hat{c}_{j\to k}$ is always non-negative. Even for uncoupled oscillators, i.e. $c_{j\to k} = 0$, the estimator $\hat{c}_{j\to k}$ is almost surely greater than zero. Thus, a reliable conclusion about the presence of coupling cannot be made from the relationship $\hat{c}_{j\to k} > 0$. We suggest to make it only if $\hat{c}_{j\to k}$ is *significantly* greater than zero. To get a quantitative criterion, let us determine a distribution law for the estimator $\hat{c}_{j\to k}$ when coupling and nonlinearity are absent, i.e. for a system (1) with $G_1 \equiv G_2 \equiv 0$. In this case, estimators $\hat{\alpha}_{k,m,n}, \hat{\beta}_{k,m,n}$ are statistically independent and identically distributed according to Gaussian law with zero mean [3]. Let us denote their variance $\sigma_{k,m,n}^2$ and recall that the sum of *M* independent Gaussian random quantities with zero mean and unit variance is distributed according to the χ^2 law with *M* degrees of freedom. Then, the quantity $\chi_{j\to k}^2 = \sum_{m,n\in\Omega_k\ (n\neq 0)} \frac{\hat{\alpha}_{k,m,n}^2 + \hat{\beta}_{k,m,n}^2}{\sigma_{k,m,n}^2}$ is distributed according to the χ^2 law with M_k degrees of freedom, where M_k is the number of terms in the right-hand side of (10). Variances $\sigma^2_{k,m,n}$ are *a priori* unknown, but their estimators $\hat{\sigma}^2_{k,m,n}$ are obtained in [3].

Thus, one gets a criterion to judge whether a conclusion about the presence of the driving $j \to k$ is statistically significant. Namely, if the quantity $\hat{\chi}^2_{j\to k}$ computed from a time series exceeds $100(1-p)\%$ -quantile of the χ^2 law with M_k degrees of freedom, then one makes such a conclusion at a significance level *p*, i.e. probability of a random erroneous coupling detection is *p*. The less the value of *p*, the more reliable the conclusion. In practice, one usually takes $p = 0.05$ or $p = 0.01$. Probability distribution function F_{χ^2, M_k} for the χ^2 law with M_k degrees of freedom is tabulated so that one can assess significance level at which $\hat{\chi}^2_{j\to k}$ differs from zero as $\hat{p}_{j\to k} = 1 - F_{\chi^2, M_k}(\hat{\chi}^2_{j\to k})$.

In practice, it may be useful to compare intensities of the driving $j \to k$ under different conditions. However, it is not straightforward with the estimator $\hat{c}_{j\to k}$ since it is biased: Its bias equals to the sum of variances $\sigma^2_{k,m,n}$ similarly to [3] and may strongly differ for different noise levels. An unbiased estimator can be derived as in [3] and reads

$$\hat{C}_{j\to k} = \frac{1}{2} \sum_{(m,n)\in\Omega_k,\, n\neq 0} \left(\hat{\alpha}^2_{k,m,n} + \hat{\beta}^2_{k,m,n} - 2\hat{\sigma}^2_{k,m,n} \right). \qquad (14)$$

To summarize, the suggested coupling estimators and the formula for a significance level apply under the following conditions.

1) Oscillation phases are well-defined.

2) Weak interdependence between simultaneous values of the phases. This condition can be checked via estimating phase synchronization indices $\hat{\rho}_{m,n} = \left| \left\langle e^{i(m\phi_1(t) - n\phi_2(t))} \right\rangle_t \right|$ and requiring $\hat{\rho}_{m,n} < \rho_c$ for all $(m,n) \in \Omega_k$ (*m*, *n*). The threshold value $\rho_c = 0.45$ is found empirically.

3) ACF of the noises ε_k decays linearly from 1 to 0 over an interval of time lags [0, τ] and equals zero for greater lags. This is because $\varepsilon_k(t) \approx \int_t^{t+\tau} \xi(t')dt'$ under the above conditions of low noise level, weak nonlinearity, and weak coupling.

V. Numerical simulations

The technique is applied here to time series from unidirectionally coupled van der Pol oscillators:

$$\frac{d^2x_1}{dt^2} - (0.2 - x_1^2)\frac{dx_1}{dt} + \omega_1^2 x_1 = \xi_1 + k_{2\to 1} x_2^2,$$
$$\frac{d^2x_2}{dt^2} - (0.2 - x_2^2)\frac{dx_2}{dt} + \omega_2^2 x_2 = \xi_2,$$

where $\xi_{1,2}$ are independent white noises. Parameters of the oscillators: $\sigma_\xi = 0.2$, $\omega_1 = 1.05$, $\omega_2 = 0.5$, coupling is described by the resonant quadratic term.

The equations are integrated with the Euler technique and step $h = 0.01$. The sampling interval is selected so to provide about 10 data points per a basic oscillation period: $\Delta t = 0.6$. Accordingly, the value of τ in coupling estimation is set equal to $\tau = 10\Delta t$. For each value of coupling coefficient, an ensemble of 100 time realizations is generated to assess the performance of the suggested technique statistically. The length of each time series is moderate and corresponds approximately to 100 mean oscillation periods: $N = 1000$. Coupling characteristics are estimated from each time series in an ensemble and either the presence of the driving $j \to k$ is inferred (a positive conclusion) or not (a negative one) at a given significance level $p = 0.05$. Positives $2 \to 1$ are correct and positives $1 \to 2$ are false. The rate of positives is denoted $\nu_{j\to k} = Q_{j\to k}/N_s$, where $Q_{j\to k}$ is the number of time series in an ensemble leading to positive conclusions. The suggested estimators are regarded applicable if the rate of *false* positives $\nu_{j\to k}$ does not exceed *p* up to acceptable variations. Also, it is desirable for the technique to be sensitive, i.e. to give a considerable rate of *correct* positives.

Coupling estimators are computed for the three model functions (3):

1) "linear" coupling, i.e. the set of indices Ω_k contains only $(m,n) = (1,1)$;

2) quadratic nonlinearity is included, i.e. Ω_k contains (1,1), (1,2), (2,1);

3) cubic nonlinearity is taken into account, i.e. Ω_k contains the pairs (1,1), (1,2), (2,1), (1,3), (3,1).

The size of the set Ω_k is denoted $|\Omega_k|$. The number of degrees of freedom for the χ^2 law in testing for statistical significance is then $M_k = 2|\Omega_k|$. Ω_1 and Ω_2 are the same.

The signals $x_{1,2}(t)$ are taken as observables. The phases are computed via the Hilbert transform. Estimation results demonstrating applicability of the technique are shown in Fig.1. The rate of false positives does not exceed an acceptable level (Fig.1, a) for any $|\Omega_k|$. Probability to detect an existing coupling rises with $k_{2\to 1}$ for $|\Omega_k| = 2$ (Fig.1,b) while couplings are missed with $|\Omega_k| = 1$. The choice of unnecessary $|\Omega_k| = 3$ reduces sensitivity in comparison with $|\Omega_k| = 2$. Coupling intensity estimator quadratically rises with coupling coefficient for $|\Omega_k| > 1$ (Fig.1,c) as expected according to (14).

Application of the suggested estimators to ensembles of nonlinear stochastic oscillators and deterministically chaotic systems can be found in [13].

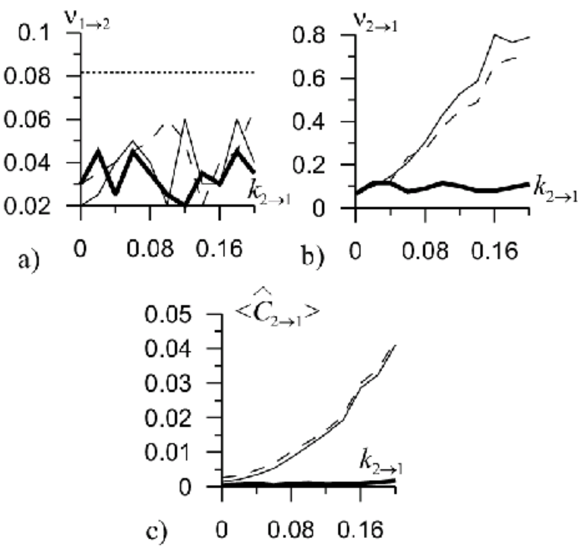

Fig. 1 Estimation results: a) the rates of false positives $\nu_{1\to 2}$ (thick line for $|\Omega_k| = 1$, thin for $|\Omega_k| = 2$, dashed for $|\Omega_k| = 3$, dotted for an acceptable error level); b) the rates of correct positives $\nu_{2\to 1}$; c) averaged coupling strength estimators

VI. Conclusion

New characteristics of directional couplings between oscillators are suggested in this work along with their estimators. They are based on the well-known idea of empirical phase dynamics modeling but, in contrast to the previously used quantities, make clear physical sense and are suitable for an arbitrary coupling nonlinearity. To assure reliability of coupling detection for relatively short time series and noisy oscillators, an analytic expression for a statistical significance level is derived. Criteria for practical applicability of the technique include well-defined phases, low degree of synchrony between the oscillators, moderate intensity and certain correlation properties of the noises in their phase dynamics. Performance of the technique is shown in numerical experiments with exemplary oscillators and that suggests possibility of its wide practical applications.

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