Thermoelastic Waves in Anisotropic Plates using Normal Mode Expansion Method with Thermal Relaxation Time

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Abstract—Analysis for the generalized thermoelastic Lamb waves, which propagates in anisotropic thin plates in generalized thermoelasticity, is presented employing normal mode expansion method. The displacement and temperature fields are expressed by a summation of the symmetric and antisymmetric thermoelastic modes in the surface thermal stresses and thermal gradient free orthotropic plate, therefore the theory is particularly appropriate for waveform analyses of Lamb waves in thin anisotropic plates. The transient waveforms excited by the thermoelastic expansion are analyzed for an orthotropic thin plate. The obtained results show that the theory provides a quantitative analysis to characterize anisotropic thermoelastic stiffness properties of plates by wave detection. Finally numerical calculations have been presented for a NaF crystal, and the dispersion curves for the lowest modes of the symmetric and antisymmetric vibrations are represented graphically at different values of thermal relaxation time. However, the methods can be used for other materials as well.

Keywords—Anisotropic, dispersion, frequency, normal, thermoelasticity, wave modes.

I. INTRODUCTION

WAVE propagation in an infinite elastic plate is one of the classical problems of continuum mechanics Achenbach [1]. Detecting damage in composite materials, several techniques have been developed; however Lamb wave methods have recently emerged as a consistent way to situate defects in such materials. Every technique implemented in the literature offers their own unique advantages in detecting certain types of defects. Lamb waves are a form of elastic perturbation that can propagate in a solid plate with free boundaries; there are two groups of waves, symmetric and anti-symmetric. An extensive theoretical and experimental set of data has been available in the literature on the properties of plate vibrations, particularly for elastic plates with traction-free faces. The temperature changes due to the elastic deformation cannot be ignored; therefore it is required to

determine the thermal and mechanical fields in the body concurrently.

The theory to include the effect of temperature change, known as the theory of thermoelasticity, has also been well established. According to the theory, the temperature field is coupled with the elastic strain field. In thermoelasticity, classical heat transfer, Fourier's conduction equation is extensively used in many engineering applications. The classical theory of thermoelasticity Nowacki, [2, 3] rests upon the hypothesis of the Fourier law of heat conduction, in which the temperature distribution is governed by a parabolic-type partial differential equation. Consequently, the theory predicts that a thermal signal is felt instantaneously everywhere in a body. This implies that an infinite speed of propagation of the thermal signal, which is impractical from the physical point of view, particularly for short-time. Thus, the use of Fourier's equation may result in discrepancies under some special conditions, such as low-temperature heat transfer, high-frequency or ultrahigh heat flux heat transfer, and so on. Fourier's equation should be modified to a non-Fourier equation in the above cases. In recent years, the non-Fourier equation has gradually become an important research topic for its potential importance in many engineering fields. In comparison to the conventional theory of coupled thermoelasticity based on a parabolic heat equation which predicts an infinite speed for the propagation of heat, theories involve a hyperbolic heat equation and are referred to as generalized thermoelasticity theories. Generalized thermoelasticity admits finite speed for the propagation of thermoelastic disturbances has received much attention in recent years. Lord and Shulman [4] (referred to as the LS theory), Green and Lindsay [5] (referred to as the GL theory) extended the coupled theory of thermoelasticity by introducing the thermal relaxation time(s) in the constitutive equations. These theories eliminates the paradox of infinite velocity of heat propagation, are the generalized theories of thermoelasticity. Banerjee and Pao [6] extended this theory to anisotropic heat conducting elastic materials. Dhaliwal and Sherief [7] treated the problem in more systematic manner. The literature dedicated to the hyperbolic thermoelastic models is quite large and extensive survey on the subject can be found in the review articles by Chandrasekharaiah [8, 9] and Chadwick [10]. They derived governing field equations of generalized thermoelastic media and proved that these equations have a unique solution. Verma [11, 12] studied

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thermo-mechanical coupling and dispersion of thermoelastic vibrations of plate with thermal relaxations. Verma, and Hasebe [13, 14, 15] investigated wave propagation problems in plates of general anisotropic media in generalized thermoelasticity. The propagation of thermoelastic waves in a plate under plane stress by using generalized theories of thermoelasticity has been studied by Massalas [16]. Authors Nayfeh and Nasser [17], , Massalas and Kalpakidis [18] have considered the propagation of generalized thermoelastic waves in plates of isotropic media. They have used the generalized theory of Lord and Schulman to study the characteristics of wave motion in a thin plate under plane stress state with mixed boundary conditions.

The normal mode expansion method has been proposed by Cheng and Zhang [19] for modeling the thermoelastic generation process of elastic waveforms in an isotropic plate. Cheng and Berthelot [20] have extended this method to Lamb wave propagation along two principal directions in an orthotropic plate. Cheng and Zhang [21] studied quantitative theory for modeling the laser generated transient ultrasonic Lamb waves, which propagates along arbitrary directions in orthotropic plates, by employing an expansion method of generalized Lamb wave modes.

In this paper, waves propagating along an arbitrary direction in a heat conducting orthotropic thermoelastic plate are presented by utilizing the normal mode expansion method in generalized theory of thermoelasticity with one thermal relaxation time. The displacement and temperature fields are expressed into the symmetric and antisymmetric wave modes for surface thermal stress and thermal gradient free heat conducting orthotropic plate. Technique used is suitable for analyses of the transient thermoelastic wave in the thin plates, as one need only to determine contributions of the few lower antisymmetric and symmetric modes. Theoretical results obtained are solved numerically for NaF plate and the dispersion curves for the few lower modes are presented. The three motions namely, longitudinal, transverse and thermal of the medium are found dispersive and coupled with each other due to the thermal and anisotropic effects. Due to the thermal and anisotropic effects phase velocity of the waves is get modified and is also influenced by the thermal relaxation time. Relevant results of previous investigations are deduced as special cases. Relevant results of previous investigations are deduced as special cases. Finally numerical solution of the frequency equations for a NaF crystal is carried out, and the dispersion curves for the lowest six modes of the symmetric and antisymmetric vibrations are represented graphically at different values of thermal relaxation time.

II. THEORETICAL BACKGROUND

We consider an infinite thermoelastic plate of finite thickness $2h$ of heat conducting orthotropic material. The coordinate axes x_1, x_2, and x_3 of the model are chosen as to be analogous with the principal axes x, y, and z of the material, with z being normal to the plate. The bottom and upper surfaces of the thermoelastic plate are $z = \pm h$, respectively, with $z = 0$ being the mid-plane of the plate. The

displacement field vector $\mathbf{u} = (u_1, u_2, u_3)$ satisfies equation of motion and heat conduction

$$\sum_{j=1}^{3}\left(\frac{\partial \tau_{ij}(\mathbf{u})}{\partial x_j}\right) + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad i = 1,2,3 \qquad (2.1)$$

$$K_{ij}T_{,ij} - \rho C_e\left(\dot{T} + \tau_0 \ddot{T}\right) = T_0 \beta_{ij}\left(\dot{u}_{i,j} + \tau_0 \ddot{u}_{i,j}\right) \qquad (2.2)$$

where

$$\tau_{ij} = C_{ijkl}e_{kl} - \beta_{ij}T, \quad \beta_{ij} = C_{ijkl}\alpha_{kl} \qquad (2.3)$$

the summation convention is implied; ρ is the density, t is the time, u_i is the displacement in the x_i direction, K_{ij} are the thermal conductivities, C_e and τ_0 are respectively the specific heat at constant strain, and thermal relaxation time, σ_{ij} and e_{ij} are the stress and strain tensor respectively; β_{ij} are thermal moduli; α_{ij} is the thermal expansion tensor; T is temperature; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl}, \text{ and}$$

$$\alpha_{ij} = \alpha_{ji}, \quad \beta_{ij} = \beta_{ji}, \quad K_{ij} = K_{ji} \qquad (2.4)$$

Te boundary conditions at surfaces $x_3 = \pm h$

$$\sum_{j=1}^{3} \tau_{ij}(\mathbf{u})n_j = 0, \quad (i = 1, 2, \text{ and } 3) \qquad (2.5)$$

$$\frac{\partial T}{\partial z} = 0 \qquad (2.6)$$

where $\mathbf{f} = (f_1, f_2, f_3)$ are the bulk force densities, ρ is the volume density, $(n_1, n_2, n_3) = (0, 0, \pm 1)$ are the normal vectors at the lower and upper surfaces, respectively, and $\tau_{ij}(\mathbf{u})$ $(i, j = 1, 2 \text{ and } 3)$ is the stress tensor. β_{ij}, α_{kl} are the thermal moduli and linear thermal expansion tensors; K_{ij} is the thermal conductivity tensor and ρ, C_e and τ_0 are respectively the density, specific heat at constant strain and thermal relaxation time of the layer. For orthotropic media the stresses and heat conduction equation for an orthotropic material in the symmetric coordinate system can be expressed by

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{13} \\ 2e_{12} \end{bmatrix} - \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} T \qquad (2.7)$$

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$$K_1 T_{,11} + K_2 T_{,22} + K_3 T_{,33} - \rho C_e \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) T$$

$$= T_0 \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \left[\beta_1 u_{1,1} + \beta_2 u_{2,2} + \beta_3 u_{3,3} \right] \qquad (2.8)$$

where

$$\beta_1 = c_{11}\alpha_1 + c_{12}\alpha_2 + c_{13}\alpha_3,$$
$$\beta_2 = c_{12}\alpha_1 + c_{22}\alpha_2 + c_{23}\alpha_3, \qquad (2.9)$$
$$\beta_3 = c_{13}\alpha_1 + c_{32}\alpha_2 + c_{33}\alpha_3.$$

By the two-dimensional Fourier transform for variables x_1 and x_2

$$(u_i, T)(x_1, x_2, x_3)$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\tilde{u}_i, \tilde{T})(x_1, x_2, x_3) \exp[i(k_1 x_1 + k_2 x_2) dk_1 dk_2 , \qquad (2.10)$$

Equations (2.1), (2.2) and (2.5), (2.6) are reduced to forms of the matrix operators

$$\rho \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} = \mathbf{a}(\tilde{\mathbf{u}}) + \tilde{f} \qquad (2.11a)$$

$$K_1 \tilde{T}_{,11} + K_2 \tilde{T}_{,22} + K_3 \tilde{T}_{,33} - \rho C_e \left(\frac{\partial \tilde{T}}{\partial t} + \tau_0 \frac{\partial^2 \tilde{T}}{\partial t^2} \right)$$
$$= T_0 \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \left[\beta_1 \tilde{u}_{1,1} + \beta_2 \tilde{u}_{2,2} + \beta_3 \tilde{u}_{3,3} \right] \qquad (2.11b)$$

$$\mathbf{b}(\tilde{\mathbf{u}}) = \mathbf{0}, \qquad (2.12)$$

$$\frac{\partial \tilde{T}}{\partial z} = 0 \text{ at } z = \pm h \qquad (2.13)$$

where $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^t$ is the displacement column vector, $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)^t$ are the body force column vectors, respectively, the superscript t represents the transpose, and the elements a_{ij} and b_{ij} of the 3×3 matrix operators $\mathbf{a} = (a_{ij})$ and $\mathbf{b} = (b_{ij})$ are follows:

$$a_{11} = -(c_{11}k_1^2 + c_{66}k_2^2) + c_{55}\frac{\partial^2}{\partial z^2},$$

$$a_{12} = -(c_{12} + c_{66})k_1 k_2$$

$$a_{13} = a_{31} = i(c_{13} + c_{55})k_1 \frac{\partial}{\partial z}$$

$$a_{14} = i\beta_1 k_1,$$

$$a_{22} = -(c_{66}k_1^2 + c_{22}k_2^2) + c_{44}\frac{\partial^2}{\partial z^2},$$

$$a_{23} = a_{32} = i(c_{23} + c_{44})k_2 \frac{\partial}{\partial z},$$

$$a_{24} = i\beta_2 k_2,$$

$$a_{33} = -(c_{55}k_1^2 + c_{44}k_2^2) + c_{33}\frac{\partial^2}{\partial z^2},$$

$$a_{34} = \beta_3 \frac{\partial}{\partial z}$$

$$a_{41} = -T_0 \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \beta_{11} k_1$$

$$a_{42} = -T_0 \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \beta_{22} k_2$$

$$a_{43} = -T_0 \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \beta_{33} \frac{\partial}{\partial z} \qquad (2.14)$$

$$a_{44} = -(K_1 k_1^2 + K_2 k_2^2) + K_3 \frac{\partial^2}{\partial z^2} - \rho C_e \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right)$$

$$b_{11} = c_{55}\frac{\partial}{\partial z}, \quad b_{12} = b_{21} = 0, \quad b_{13} = ic_{55}k_1, \quad b_{14} = 0$$

$$b_{22} = c_{44}\frac{\partial}{\partial z}, \quad b_{23} = ic_{44}k_2, \quad b_{24} = 0$$

$$b_{31} = ic_{13}k_1, \quad b_{32} = i_{23}k_2, \quad b_{33} = c_{33}\frac{\partial}{\partial z}, \quad b_{34} = T\beta_{33}$$

$$b_{41} = b_{42} = b_{43} = 0, \quad b_{44} = \frac{\partial}{\partial z} \qquad (2.15)$$

where a_{j4}, a_{4i} and b_{j4}, i, $j = 1, 2, 3, 4$ are corresponding to (2.11b) and (2.13).

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III. MODES EXPANSION

To solve (2.11)-(2.13), the method of expansion in normal modes is employed by considering eigen function series $\{\boldsymbol{\omega}_m, \omega_m, m = 1, 2, 3, ...\}$ by the eigen value problem of the operator $\boldsymbol{a}$ under the boundary operator $\boldsymbol{b}$

$$\mathbf{a}[\boldsymbol{\omega}_m] = -\rho \omega_m^2 \boldsymbol{\omega}_m, \quad -h < z < h \qquad (3.1)$$

$$K_{ij}\tilde{T}_{,ij} - \rho C_e (i - \tau_0 \omega_m)\omega_m \boldsymbol{\omega}_m \tilde{T}$$
$$= T_0 \beta_{ij} \omega_m \boldsymbol{\omega}_m (i - \tau_0 \omega_m)[\tilde{u}_{i,j} + \tau_0 \tilde{u}_{i,j}] \qquad (3.2)$$

$$\mathbf{b}[\boldsymbol{\omega}_m] = 0, \frac{\partial \tilde{T}}{\partial z} = 0, z = \pm h \qquad (3.3)$$

where m is the eigen frequency corresponding to the eigen mode $\boldsymbol{\omega}_m = [\omega_{1m}, \omega_{2m}, \omega_{3m}, \omega_{4m}]^t$. Because the operator $\boldsymbol{a}$ is a self-adjoint operator under the boundary operator $\boldsymbol{b}$, the eigen-function series $\{\boldsymbol{\omega}_m\}$ forms an orthogonal set with the weighting function ρ Eringen, and Suhubi [22],

$$\int_{-h}^{h} \rho \boldsymbol{\omega}_n \boldsymbol{\omega}_m^{*t} dz = \delta_{mn} \qquad (3.4)$$

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On the other hand, it has been proven that $\{\boldsymbol{\omega}_m\}$ is also a complete function series Eringen, and Suhubi [22], so that the displacement column vector $\tilde{\mathbf{u}}$ can be expanded by the generalized Fourier series

$$(\tilde{\mathbf{u}}, \ \tilde{T})(k_1, k_2, x_3, t)$$
$$= \sum_m \xi_m(k_1, k_2, \omega_m, t) \ \boldsymbol{\omega}_m(k_1, k_2, \omega_m, x_3), \qquad (3.5)$$

where $\xi_m(t)$ are the generalized Fourier coefficients

$$\xi_m = \frac{1}{\omega_m} \int_0^t \sin \omega_m(t-\tau) \Big[\tilde{\mathbf{s}} \boldsymbol{\omega}_m^{*t} \big|_{z=\pm h} + \int_v \tilde{\mathbf{f}} \boldsymbol{\omega}_m^{*t} dv \Big] d\tau \qquad (3.6)$$

where v represent the body of the plate and the superscript "*" represents complex conjugation.

IV. THERMOELASTIC WAVE MODES

The shear horizontal mode polarized parallel to the plate surface is not coupled to the dilatational and flexural modes, which simplifies greatly the Lamb wave motion. In orthotropic thermoelastic materials coefficients are c_{11}, c_{13}, c_{33}, c_{23}, c_{55}, and, K_1, K_3 and β_1, β_3, the Lamb waves propagate in the principal directions (in the principal axis) of coordinate system. The Lamb wave propagation in antisymmetric directions is more complex than that along the principal directions, because there will no longer be a family of shear horizontal modes independent of the dilatational, flexural and thermal modes in the antisymmetric directions. All partial waves are coupled, and the thermoelastic free-plate modes can only be classified as symmetric and antisymmetric modes with respect to the median plane. The interest in a transversely isotropic thermoelastic plate has been motivated. In fact, (3.1)-(3.2) mean that the eigen-functions of the operator $\boldsymbol{a}$, are the generalized thermoelastic Lamb wave modes and the relations between ω_m and (k_1, k_2) are dispersion equations of the Lamb wave modes. The eigen-functions can be classified as antisymmetric and symmetric modes with respect to the median plane ($z = 0$) Cheng and Zhang [6],

$$\omega_{jm}^a = i \sum_{l=p,q,r,s} H_{jl} \alpha_l e_l \sin(\alpha_l z) \qquad j = 1, 2$$
$$(4.1)$$

$$\omega_{3m}^a = \sum_{l=p,q,r,s} H_{3l} e_l \cos(\alpha_l z) \qquad (4.2)$$

$$\omega_{4m}^a = i \sum_{l=p,q,r,s} \alpha_l e_l \sin(\alpha_l z) \qquad (4.3)$$

and for antisymmetric modes

$$\omega_{jm}^s = -i \sum_{l=p,q,r,s} H_{jl} \alpha_l e_l \cos(\alpha_l x_3), \quad j = 1, 2 \qquad (4.4)$$

$$\omega_{3m}^s = \sum_{l=p,q,r,s} H_{3l} e_l \sin(\alpha_l z) \qquad (4.5)$$

$$\omega_{4m}^s = -i \sum_{l=p,q,r,s} \alpha_l e_l \cos(\alpha_l z) \qquad (4.6a)$$

for symmetric modes. The generalized Rayleigh–Lamb

equations for determining the relations between eigen-frequency ω_m and the wave numbers (k_1 and k_2) can be obtained by combining (4.1)-(4.6) with (3.3)

$$\det[t_{ij}(\omega_m, k_1, k_2)] = 0, \qquad (4.6b)$$

with the elements of 4×4 matrix $\mathbf{t} = (t_{ij})$ ($l = p, q, r$ and s)

$$t_{ij} = \alpha_l^2 H_{jl} + k_j, \quad j = 1, 2 \text{ and } 3$$
$$t_{3l} = [c_{13} k_1 H_{1l} + c_{23} k_2 H_{2l} + c_{33} H_{3l} + \beta_3] \alpha_l \tan(\alpha_l h)^{\pm 1}, \qquad (4.7)$$

Here the "+1" and "-1" in the element t_{3l} correspond to antisymmetric and symmetric modes, respectively. The parameters in (4.1)-(4.6) are discussed as follows:

The partial wave numbers α_l ($l = p, q, r$ and s) are the roots of the determinant equation

$$\det[g_{ij}(\alpha)] = 0 \qquad (4.8)$$

Equation (4.8) is deduced by substituting the partial waves $v_{jm} = v_{jm}^0 \exp(i\alpha z)$, ($j = 1, 2, 3$ and 4) into (3.1) and (3.3). The elements of 4×4 matrix $g(\alpha) = [g_{ij}(\alpha)]$ are

$$g_{11}(\alpha) = (\rho \omega_m^2 - a_{11}) - c_{55} \alpha^2,$$
$$g_{12}(\alpha) = g_{21}(\alpha) = -a_{12},$$
$$g_{13}(\alpha) = g_{31}(\alpha) = -a_{13}\alpha, \ g_{14}(\lambda) = -a_{14}$$
$$g_{22}(\lambda) = (\rho \omega_m^2 - a_{22}) - c_{44} \lambda^2,$$
$$g_{23}(\alpha) = g_{23}(\alpha) = -a_{23}\alpha, g_{24}(\alpha) = -a_{24}$$
$$g_{33}(\alpha) = (\rho \omega_m^2 - a_{33}) - c_{33} \alpha^2, \ g_{34}(\alpha) = -a_{34}\alpha$$
$$g_{41}(\alpha) = -a_{41}, g_{42}(\alpha) = -a_{42}, g_{43}(\alpha) = -a_{43}\alpha \qquad (4.9)$$
$$g_{44}(\alpha) = (\rho \omega_m^2 C_e \tau - a_{44}) - K_{33} \alpha^2$$

The coefficients H_{1l}, H_{2l} and H_{3l} are the partial wave amplitudes, defined by

$$H_{jl} = \frac{D_j}{D}, \qquad j = 1, 2, 3 \qquad (4.10)$$

where

$$D = g_{11} g_{22} g_{33} - g_{33} a_{12}^2 + \left(2a_{12} a_{13} a_{23} - a_{13}^2 g_{22} - a_{23}^2 g_{11}\right) \alpha^2$$

$$D_1 = -\begin{pmatrix} g_{33} g_{22} a_{14} + g_{33} g_{12} a_{24} \\ + \left(a_{13} a_{23} a_{24} - a_{14} a_{23}^2 + ia_{13} a_{34} g_{22} + ia_{12} a_{23} a_{34}\right) \alpha^2 \end{pmatrix}$$

$$D_2 = -\begin{pmatrix} g_{33} g_{11} a_{24} + a_{12} a_{14} g_{33} \\ + \left(a_{13} a_{14} a_{23} - a_{24} a_{13}^2 + ia_{23} a_{34} g_{11} + ia_{12} a_{13} a_{34}\right) \alpha^2 \end{pmatrix}$$

$$D_3 = -\begin{pmatrix} a_{12} a_{13} a_{24} + a_{12} a_{14} a_{23} + a_{13} a_{14} g_{22} \\ + a_{23} a_{24} g_{11} + ia_{34} g_{11} g_{22} - ia_{12}^2 a_{34} \end{pmatrix} \alpha$$

$$D = g_{11} g_{22} g_{33} - g_{33} a_{12}^2$$
$$+ \left(2a_{12} a_{13} a_{23} - a_{13}^2 g_{22} - a_{23}^2 g_{11}\right) \alpha^2$$

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The amplitudes e_l and f_l (l = 1 and 2) depend on e_r and f_r by following equations

$$\sum_{l=p,q,r,s} t_{jl} \cos(\alpha_l h) \, e_l = -e_r t_{jr} \cos(\alpha_r h), \quad j=1, \, 2,3 \qquad (4.11)$$

$$\sum_{l=p,q,r,s} t_{jl} \sin(\alpha_l h) \, f_l = -f_r t_{jr} \sin(\alpha_r h), \quad j=1, \, 2,3 \qquad (4.12)$$

The e_r or f_r is determined by the normalized condition

$$\int_{-h}^{h} \rho \, \omega_m^{a,s} (\omega_m^{a,s})^{*t} \, dz = 1 \qquad (4.13)$$

Finally, one can obtain a normalized eigen function series for symmetric and antisymmetric modes.

V. SPECIAL CASES

A. Higher symmetry materials

Results for higher symmetry materials such as transversely isotropic, cubic, and isotropic can be obtained as special cases with the following restrictions for transverse isotropy symmetry:

$$c_{33} = c_{22}, \ c_{13} = c_{12},$$
$$c_{55} = c_{66}, \ c_{22} - c_{23} = 2c_{44},$$
$$\alpha_{22} = \alpha_{33}, k_{22} = k_{33}$$

for cubic symmetry,

$$c_{11} = c_{22} = c_{33}, c_{12} = c_{13} = c_{23},$$
$$c_{44} = c_{55} = c_{66}, \alpha_{11} = \alpha_{22} = \alpha_{33}, \ .$$
$$k_{11} = k_{22} = k_{33}$$

for isotropic symmetry,

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu,$$
$$c_{12} = c_{13} = c_{23} = \lambda,$$
$$c_{44} = c_{55} = c_{66} = \mu,$$
$$\alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha,$$
$$k_{11} = k_{22} = k_{33} = k$$

B. Thermo-mechanical coupling constant is zero

This case corresponds to the situation when the strain and temperature fields are not coupled with each other. In this case the thermo-mechanical coupling constant $\beta_{ij} = 0$ are identically zero, and elastic waves decouple from thermal wave which is influenced by the thermal relaxation.

C. Coupled thermoelasticity

When $\tau_0 = 0$, i. e. no thermal relaxation time, this is the case of coupled thermoelasticity, proceeding on the same lines, we arrived at equations of the form that is in agreement with the corresponding result for coupled thermoelasticity.

VI. NUMERICAL AND DISCUSSION

The fundamental way to describe the propagation of Lamb waves in a material is with their dispersion curves, which plots the phase velocity versus the wave. In the principal directions of the transversely isotropic material, there exist four types of free plate modes; namely, the pure shear horizontal, dilatational, flexural modes, and thermal. The shear horizontal mode polarized parallel to the plate surface is not coupled to the dilatational and flexural modes, which simplifies greatly the Lamb wave motion. For a transversely isotropic plate, the (x_1 -x_3) plane is isotropic so that there are only five independent stiffness constants. Computation for the symmetric and antisymmetric modes have been carried out for a single crystal NaF, for which, the basic physical data is given in Banerjee and Pao [2].

Dispersion curves for seven lowest modes for symmetric and antisymmetric vibrations are shown in Figures (1) to (6) for different values of thermal relaxation times $\tau_0 = 1.8 \cdot 10^{-7} \sec s$, $\tau_0 = 1.8 \cdot 10^{-6} \sec s$ and $\tau_0 = 1.8 \cdot 10^{-5} \sec s$ respectively by employing the normal mode expansion of generalized thermoelastic Lamb wave modes, dispersion curves in the forms of variations of phase velocity (dimensionless) with wave number (dimensionless) are constructed at different values of thermal relaxation times for crystal of NaF. Each of figure display coupled three wave speeds corresponding to quasi-longitudinal, quasi-transverse and quasi-thermal at zero wave number limits, for the higher value wave numbers higher modes appear in both cases (symmetric and antisymmetric thermoelastic modes) with ξ increases. One of the thermoelastic mode seems to be associated with quick change in the slope of the mode. Lower modes are found to highly influence by the thermal relaxation times at low values of wave number both in symmetric and antisymmetric thermoelastic modes, while in higher modes, change is observed at high values of wave number. The corresponding results for the symmetric and antisymmetric vibrations in coupled thermoelasticity are shown in Figure (7) and Figure (8) .

VII. CONCLUSION

Transient thermoelastic Lamb wave analysis in generalized thermoelasticity with one thermal relaxation time propagating along arbitrary directions in heat conducting orthotropic plates is presented by using normal mode expansion method. The displacement field and temperature are expressed by a summation of the symmetric and antisymmetric modes in the stress and thermal gradient free surface of plate. The method used is appropriate for waveform analyses of thermoelastic Lamb wave in thin plates because one needs only to compute the lower modes. Numerical analyses indicate that the method will provide a useful technique to characterize anisotropic properties of orthotropic thin plates in thermal environment. However, it is available to consider not only the higher frequency components of the lowest Lamb wave modes, but also the higher-order Lamb wave modes for thicker plates in

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the numerical analysis.

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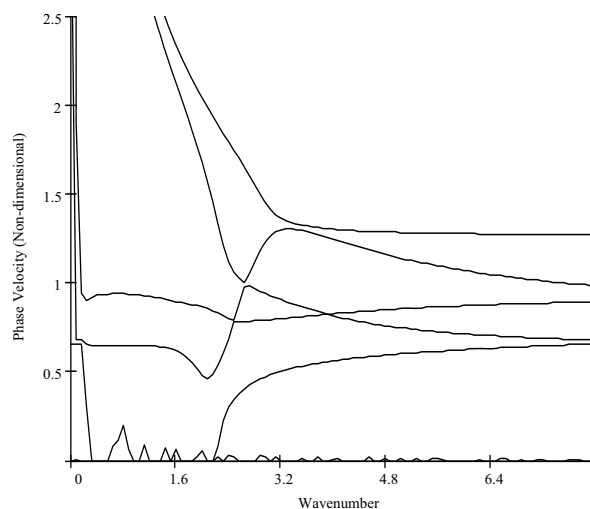
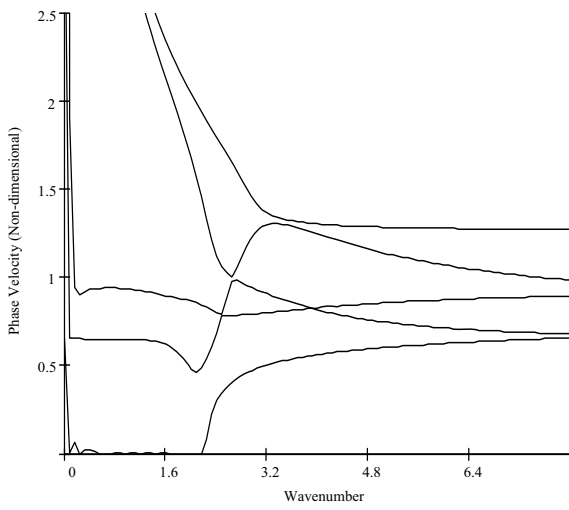Generalized Thermoelastic Symmetric Modes

Fig. 1 Dispersion curves for Symmetric thermoelastic modes for
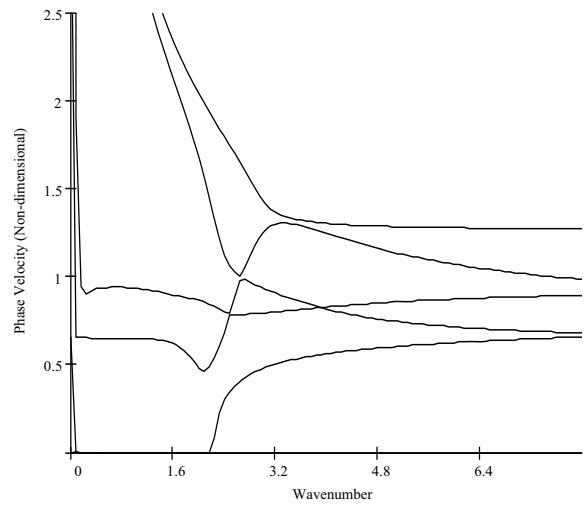$$\tau_0 = 1.8.10^{-7} \sec s$$
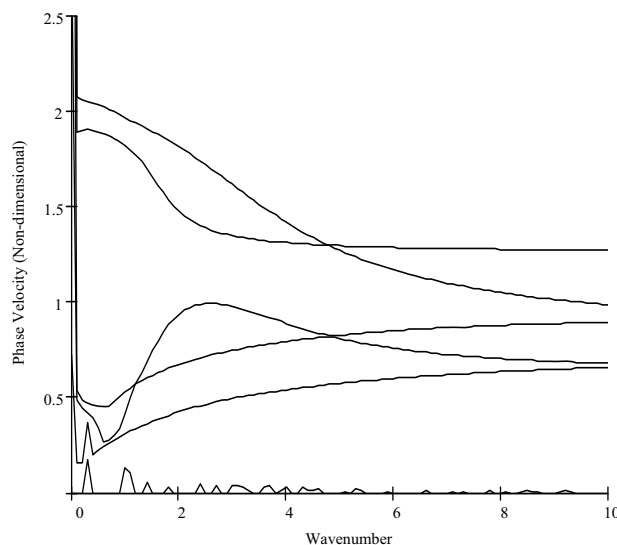
Generalized Thermoelastic Antisymmetric Mode

Fig. 2 Dispersion curves for Antisymmetric thermoelastic modes for
$$\tau_0 = 1.8.10^{-7} \sec s$$

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Generalized Thermoelastic Symmetric Modes

Fig. 3 Dispersion curves for Symmetric thermoelastic modes for $\tau_0 = 1.8.10^{-6}\sec s$

Generalized Thermoelastic Symmetric Modes

Fig. 5 Dispersion curves for Symmetric thermoelastic modes for $\tau_0 = 1.8.10^{-5}\sec s$
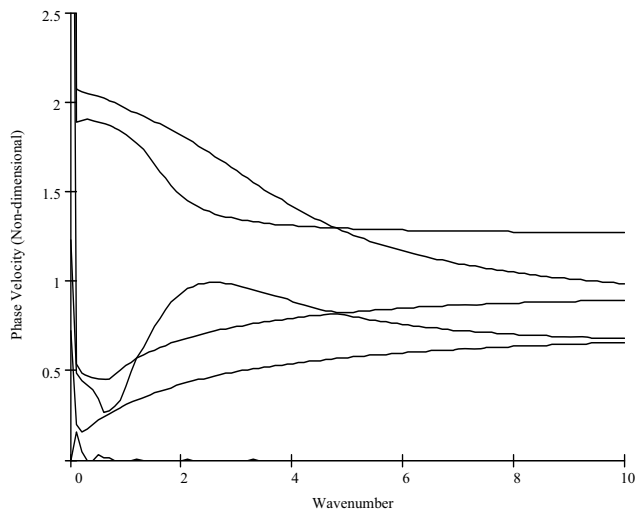
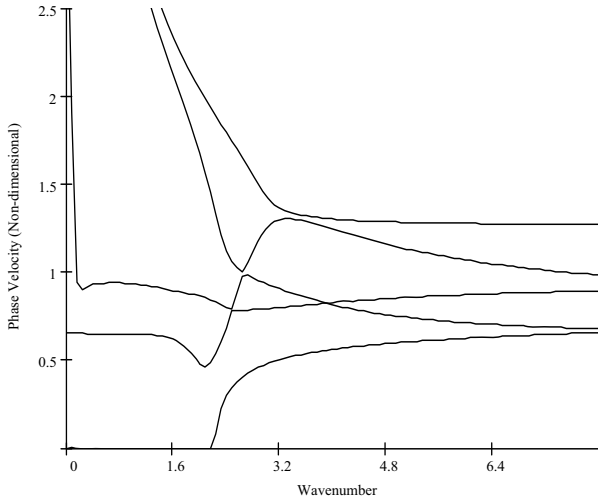

Generalized Thermoelastic Antisymmetric Modes

Fig. 4 Dispersion curves for Antisymmetric thermoelastic modes for $\tau_0 = 1.8.10^{-6}\sec s$
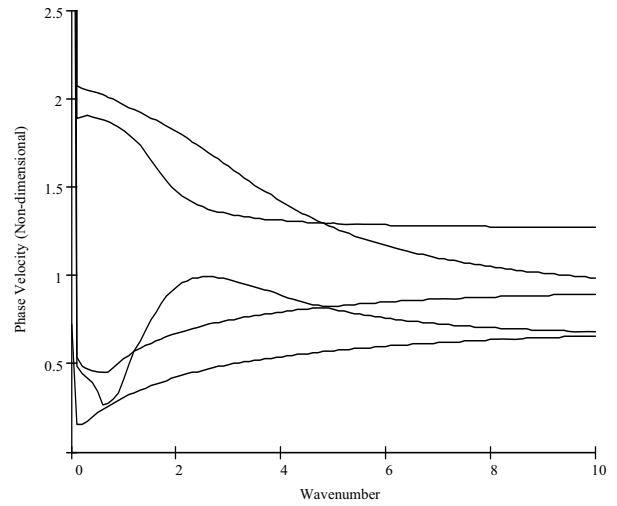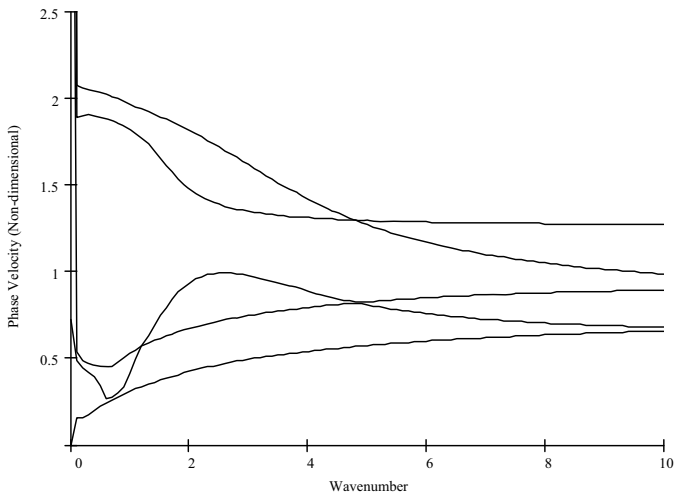

Generalized Thermoelastic Antisymmetric Modes

Fig. 6 Dispersion curves for Antisymmetric thermoelastic modes for $\tau_0 = 1.8.10^{-5}\sec s$

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Symmetric modes in coupled thermoelasticity

Fig. 7 Dispersion curves for Symmetric Thermoelastic modes for
$\tau_0 = 0$

Antisymmetric modes in coupled thermoelasticity

Fig. 8 Dispersion curves for Antisymmetric thermoelastic modes for
$\tau_0 = 0$