Applications of Entropy Measures in Field of Queuing Theory

R.K.Tuli

Abstract**—**In the present communication, we have studied different variations in the entropy measures in the different states of queueing processes. In case of steady state queuing process, it has been shown that as the arrival rate increases, the uncertainty increases whereas in the case of non-steady birth-death process, it is shown that the uncertainty varies differently. In this pattern, it first increases and attains its maximum value and then with the passage of time, it decreases and attains its minimum value.

Keywords**—**Entropy, Birth-death process, M/G/1 system, G/M/1 system, Steady state, Non-steady state

I. Introduction

IT is a known fact that in any stochastic process, the probability distribution changes with time and consequently, it becomes obvious that the entropy or uncertainty of a probability distribution also changes with time. It becomes therefore interesting to know how the uncertainty changes with time. In the usual analysis of queueing system, the birth-and-death process is the base of the system, according to which, the mean rate at which the entering customers occur must equal the mean rate at which leaving customers occur. The system must assume some kind of stability for obtaining a probabilistic model and the basic formulae obtained are reliable to the extent to which the conditions of the process are satisfied. If the real probability distribution of the states of the queueing system is known, the corresponding entropy may be effectively computed for measuring the amount of uncertainty about state of the system. But generally, we do not know this real probability distribution. The available information is summarized in mean values, mean arrival rates, mean service rates of the mean number of customers in the system.

Affendi and Kouvatsos [1] have used maximum entropy formalism to analyse the M/G/1 and G/M/1 queueing systems at equilibrium. The authors have obtained the solution for the number of jobs in the M/G/1 system and determined the corresponding service time distribution. Guiasu [2] used the maximum entropy conditions and obtained a probabilistic model for the queueing system for the known mean values. The authors also proved that in a steady state, for some one-server queueing system, when the expected number of customers is given, the maximum entropy condition gives the same probability distribution as the birth-and-death process applied to $M/M/1$ system.

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In simple birth-death process of queueing theory, let *pn(t)* denotes the probability of there being *n* persons in the population at time *t* and let *n0* denote the number of persons at time *t = 0*, then Medhi [5] has obtained an expression for *pn(t)* . In fact, if we define the probability generating function by

$$\phi(s,t)=\sum_{n=o}^{\infty} p_n(t)s^n \qquad (1)$$

we get the following result

$$\phi(s,t)=\left[\frac{(\lambda-\mu)s+\mu(x-1)}{(\lambda-\lambda x)s+(\lambda x-1)}\right]^{n_0},\ \lambda\neq\mu \qquad (2)$$

$$=\left[\frac{\lambda t-(\lambda t-1)s}{1-\lambda t-\lambda ts}\right]^{n_0},\ \lambda=\mu \qquad (3)$$

where

$$\left[\frac{\lambda t-(\lambda t-1)s}{1-\lambda t-\lambda ts}\right]^{n_0},\ \lambda=\mu \qquad (4)$$

By expanding $\phi(s,t)$ in power series of *s*, we can find *pn(t).* In a queuing system, let λ and μ denote arrival and service rates in the steady state case, then the following result is well known:

$$p_n=(1-\rho)\rho^n\ ,\ \ n=0,1,2.3,....;\rho=\frac{\lambda}{\mu} \qquad (5)$$

At any time *t*, the number of persons in the system can be 0, 1, 2,…, so that there is uncertainty about the number of persons in the system. We want to develop a measure of this uncertainty, which shows how this uncertainty varies with λ, μ and *t*. Taking these parameters into consideration, Kapur [4] has studied such types of variations by using various measures of entropy and obtained interesting results. Prabhakar and Gallager [7] have undertaken the study of queues which deal with two single server discrete-time queues. The authors have shown that when units arrive according to an arbitrary ergodic stationary arrival process, the corresponding departure process has an entropy rate no less than the entropy rate of the arrival process. Using this approach from the entropy standpoint, the authors have established connections with the time capacity of queues.

In the literature of information theory, there exist many well known measures of entropy, each with its own merits and limitations. All these measures have been obtained by the motivation of Shannon's [12] fundamental measure of entropy. Some of these measures are due to Renyi [11], Kapur

[3, 4], Sharma and Mittal [13], Nanda and Paul [6], Rathie [8], Rathie and Taneja [9], Rao,Yunmei and Wang [10] etc. These measures can be good contributors for studying the behaviour of uncertainty in the different states of the queueing system, which is the theme of the present paper.

In section 2, the variations of different entropy measures have been studied in steady state whereas the variations in the non-steady state of queuing theory have been presented in section 3.

II. Applications Of Entropy Measures For Studying Variations In The Steady State

In this section, we have studied the variations of different probabilistic measures of entropy in the steady state queuing processes. For this purpose, we have considered the following cases:

A. Variations in Sharma and Mittal's [13] measure of entropy

We know that Sharma and Mittal's [13] measure of entropy is given by

$$S_1^\alpha(\lambda,\mu)=\frac{1}{2^{1-\alpha}-1}\left\{\exp_2\left((\alpha-1)\sum_{n=0}^{\infty}p_i\log p_i\right)-1\right\}$$

$$=\frac{1}{2^{1-\alpha}-1}\left\{\exp_2\left(\begin{matrix}(\alpha-1)\\ \sum_{n=0}^{\infty}\rho^n(1-\rho)\log\rho^n(1-\rho)\end{matrix}\right)-1\right\}$$

$$=\frac{1}{2^{1-\alpha}-1}\left\{\exp_2\left(\begin{matrix}(\alpha-1)(1-\rho)\\ \sum_{n=0}^{\infty}\rho^n\log\rho^n(1-\rho)\end{matrix}\right)-1\right\}$$

$$\frac{1}{2^{1-\alpha}-1}\left[\exp_2\left\{\begin{matrix}(\alpha-1)\\ \left(\frac{\rho\log\rho}{(1-\rho)}+\log(1-\rho)\right)\end{matrix}\right\}-1\right] \tag{6}$$

Now, taking limit as $\alpha\to1$, equation (6) becomes

$$S_1(\lambda,\mu)=\lim_{\alpha\to1}\frac{1}{2^{1-\alpha}-1}\left[\exp_2\left\{\begin{matrix}(\alpha-1)\\ \left(\frac{\rho\log\rho}{(1-\rho)}+\log(1-\rho)\right)\end{matrix}\right\}-1\right]$$

$$=-\left[\frac{\rho\log\rho+(1-\rho)\log(1-\rho)}{(1-\rho)}\right]$$

Differentiating equation (6) w.r.t. ρ , we get

$$\frac{d}{d\rho}S_1^\alpha(\lambda,\mu)$$

$$=\frac{(\alpha-1)}{2^{1-\alpha}-1}\left(\frac{\log\rho}{(1-\rho)^2}\right)$$

$$\left[\exp_2\left\{(\alpha-1)\left(\frac{\rho\log\rho}{(1-\rho)}+\log(1-\rho)\right)\right\}\right] \tag{7}$$

Now, taking limit as $\alpha\to1$, equation (7) becomes

$$\frac{d}{d\rho}S_1(\lambda,\mu)$$

$$=\lim_{\alpha\to1}\frac{(\alpha-1)}{2^{1-\alpha}-1}\left(\frac{\log\rho}{(1-\rho)^2}\right)$$

$$\left[\exp_2\left\{(\alpha-1)\left(\frac{\rho\log\rho}{(1-\rho)}+\log(1-\rho)\right)\right\}\right]$$

$$=-\frac{\log\rho}{(1-\rho)^2}>0$$

which means that in steady-state queuing process, the uncertainty increases monotonically from 0 to as increases from 0 to unity. Thus, in the present case, we see that the uncertainty measure increases if the traffic intensity increases.

B. Variations in Rathie's [8] measure of entropy

We know that Rathie's [8] measure of entropy is given by

$$S_2^{\alpha,\beta}(\lambda,\mu)=\frac{1}{1-\alpha}\left[\log\frac{\sum_{n=0}^{\infty}p_n^{\alpha+\beta-1}}{\sum_{n=0}^{\infty}p_n^{\beta}}\right]$$

$$=\frac{1}{1-\alpha}\left[\log\sum_{n=0}^{\infty}p_n^{\alpha+\beta-1}-\log\sum_{n=0}^{\infty}p_n^{\beta}\right]$$

$$=\frac{1}{1-\alpha}\left[\begin{matrix}\log\sum_{n=0}^{\infty}(\rho^n(1-\rho))^{\alpha+\beta-1}\\ -\log\sum_{n=0}^{\infty}(\rho^n(1-\rho))^{\beta}\end{matrix}\right]$$

$$=\frac{1}{1-\alpha}\left[\log\frac{(1-\rho)^{\alpha+\beta-1}}{1-\rho^{\alpha+\beta-1}}-\log\frac{(1-\rho)^{\beta}}{1-\rho^{\beta}}\right]$$

$$=\begin{matrix}-\log(1-\rho)\\ +\frac{\log(1-\rho^{\beta})-\log(1-\rho^{\alpha+\beta-1})}{1-\alpha}\end{matrix} \tag{8}$$

Now, taking limit as $\alpha\to\beta$, equation (8) gives the following result:

$$S_2^{\beta}(\lambda,\mu)=$$

$$-\log(1-\rho)+\frac{\log(1-\rho^{\beta})-\log(1-\rho^{2\beta-1})}{1-\beta} \qquad (9)$$

Again, taking limit as $\beta\to 1$, equations (9) becomes

$$S_2(\lambda,\mu)=$$

$$-\log(1-\rho)+\underset{\beta\to 1}{lt}\frac{\log(1-\rho^{\beta})-\log(1-\rho^{2\beta-1})}{1-\beta}$$

$$= -\left[\frac{(1-\rho)\log(1-\rho)+\rho\log\rho}{(1-\rho)}\right] \qquad (10)$$

Differentiating equation (8) w.r.t. ρ, we get

$$\frac{d}{d\rho}S_2^{\alpha,\beta}(\lambda,\mu)=$$

$$\frac{1}{1-\rho}+\frac{1}{1-\alpha}\left[-\frac{\beta\rho^{\beta-1}}{1-\rho^{\beta}}+\frac{(\alpha+\beta-1)\rho^{\alpha+\beta-2}}{1-\rho^{\alpha+\beta-1}}\right] \qquad (11)$$

Taking limit as $\alpha\to\beta$, equations (11) becomes

$$\frac{d}{d\rho}S_2^{\beta}(\lambda,\mu)=$$

$$\frac{1}{1-\rho}+\frac{1}{1-\beta}\left[-\frac{\beta\rho^{\beta-1}}{1-\rho^{\beta}}+\frac{(2\beta-1)\rho^{2\beta-2}}{1-\rho^{2\beta-1}}\right] \qquad (12)$$

Now, taking limit as $\beta\to 1$, equation (12) gives

$$\frac{d}{d\rho}S_2(\lambda,\mu)=\frac{2-2\rho+\log\rho-2+2\rho-2\log\rho}{(1-\rho)^2}$$

$$=\frac{\log\rho-2\log\rho}{(1-\rho)^2}$$

$$=-\frac{\log\rho}{(1-\rho)^2}>0$$

Thus, we see that in the steady state queuing case, the uncertainty measure increases monotonically from 0 to ∞ as ρ increases from 0 to unity, which shows that as the arrival rate increases relatively to service rate, uncertainty increases.

Hence, we conclude that in all these cases, the variations of entropy remain same, that is, entropy always increases monotonically and in this queueing process, this result is most desirable.

III. Applications Of Entropy Measures For The Study Of Variations In The Non-Steady State

In this section, we have studied the variations of different measures of entropy in the non-steady state queuing processes. For this purpose, we first of all develop the following results:

Equation (3) gives

$$\sum_{n=0}^{\infty}p_n(t)s^n=\frac{\lambda t}{1+\lambda t}\left\{1-\frac{\lambda t}{1+\lambda t}s\right\}\left\{1-\frac{\lambda t}{1+\lambda t}s\right\}^{-1}$$

$$=\frac{\lambda t}{1+\lambda t}\left\{1-\frac{\lambda t}{1+\lambda t}s\right\}\sum_{n=o}^{\infty}\left\{1-\frac{\lambda t}{1+\lambda t}s\right\}^{-1}$$

so that

$$p_n(t)=\frac{\lambda t}{1+\lambda t}\left(\frac{\lambda t}{1+\lambda t}\right)^{n}-\left(\frac{\lambda t-1}{\lambda t}\right)\left(\frac{\lambda t}{1+\lambda t}\right)^{n-1}$$

$$=\frac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}},\ n\ge 1$$

Also $p_0(t)=\dfrac{\lambda t}{1+\lambda t}$

Thus, we have the following expression for the probability of n persons at any time t:

$$p_n(t)=\begin{cases}\dfrac{(\lambda t)^{n-1}}{(1+\lambda t)^{n+1}},\ n\ge 1\\[2ex] \dfrac{\lambda t}{1+\lambda t}, n=0\end{cases} \qquad (13)$$

Now, we study the different variations by taking into consideration the different probabilistic measures of entropy:

A. Variation in Sharma and Mittal's [13] entropy

We know that Sharma and Mittal's [13] entropy of degree α, β is given by

$$S_1^{\alpha}(\lambda,t)=\frac{1}{2^{1-\alpha}-1}\left\{\exp_2\left((\alpha-1)\sum_{n=0}^{\infty}p_i\log p_i\right)-1\right\}$$

$$=\frac{1}{2^{1-\alpha}-1}\left\{\exp_2\left((\alpha-1)S\right)-1\right\} \qquad (14)$$

Where

$$S=\frac{\lambda t}{1+\lambda t}\log\frac{\lambda t}{1+\lambda t}+\frac{1}{(1+\lambda t)^2}\log\frac{1}{(1+\lambda t)^2}$$

$$+\frac{\lambda t}{(1+\lambda t)^3}\log\frac{\lambda t}{(1+\lambda t)^3}+...$$

$$=\log\lambda t\left[\frac{\lambda t}{1+\lambda t}+\frac{\lambda t}{(1+\lambda t)^2}+\frac{2(\lambda t)^2}{(1+\lambda t)^4}+...\right]$$

$$-\log(1+\lambda t)\left[\frac{\lambda t}{1+\lambda t}+\frac{2}{(1+\lambda t)^2}+\frac{3\lambda t}{(1+\lambda t)^3}+...\right]$$

$$=\log\lambda t\left[\frac{2\lambda t}{1+\lambda t}\right]-\log(1+\lambda t)\left[\frac{\lambda t}{1+\lambda t}+\frac{2+\lambda t}{1+\lambda t}\right]$$

$$=\frac{2(\lambda t)\log(\lambda t)-2(1+\lambda t)\log(1+\lambda t)}{1+\lambda t}$$

From equation (14), we get

$$S_1^{\alpha}(\lambda,t) = \frac{1}{2^{1-\alpha}-1}\left\{\exp_2\left(2(\alpha-1)\frac{\begin{bmatrix}(\lambda t)\log(\lambda t)\\ -(1+\lambda t)\log(1+\lambda t)\end{bmatrix}}{1+\lambda t}\right)-1\right\} \quad (15)$$

Taking limit as $\alpha \to 1$, equation (15) gives

$$S_1(\lambda,t) = \frac{2[(1+\lambda t)\log(1+\lambda t)-(\lambda t)\log(\lambda t)]}{1+\lambda t} \quad (16)$$

Which is a result developed by Kapur [4].

Now, differentiating (16) w.r.t. λt, we get

$$\frac{d}{d(\lambda t)}S_1^{\alpha}(\lambda,t) = \frac{2(\alpha-1)}{2^{1-\alpha}-1}\left\{\exp_2\left(2(\alpha-1)\frac{\begin{bmatrix}(\lambda t)\log(\lambda t)\\ -(1+\lambda t)\log(1+\lambda t)\end{bmatrix}}{1+\lambda t}\right)\right\}\frac{\log(\lambda t)}{(1+\lambda t)^2} \quad (17)$$

Now, taking limit as $\alpha \to 1$, equation (17) gives

$$\frac{d}{d(\lambda t)}S_1(\lambda,t) = -\frac{2\log(\lambda t)}{(1+\lambda t)^2} > 0 \ iff\ \lambda t < 1$$

which means that the uncertainty increases if λt <1 and decreases if $\lambda t \geq 1$.

Also from equation (16), the maximum uncertainty occurs when λt =1 and in this case,

Max $S_1(\lambda,t) = 2\log 2$

Further when $t = 0$, the uncertainty is zero and when $t \to \infty$, we have

$$\underset{t\to\infty}{Lt}\ S_1(\lambda,t) = \underset{t\to\infty}{Lt}\ \frac{2[(1+\lambda t)\log(1+\lambda t)-\lambda t\log\lambda t]}{(1+\lambda t)} = 0$$

Thus, in this case, the uncertainty starts with zero value at time $t = 0$ and ends with zero value as time $t \to \infty$ and in between, it attains the maximum value at λt =1,that is, at t = $\frac{1}{\lambda}$.

B. Variation in Rathie's [8] entropy

We know that Rathie's [13] entropy of degree α, β, is given by

$$S_2^{\alpha,\beta}(\lambda,t) = \frac{1}{1-\alpha}\log\frac{\sum_{n=0}^{\infty}p_n^{\alpha+\beta-1}}{\sum_{n=0}^{\infty}p_n^{\beta}} = \frac{1}{1-\alpha}\left[\log\sum_{n=0}^{\infty}p_n^{\alpha+\beta-1} - \log\sum_{n=0}^{\infty}p_n^{\beta}\right]$$

$$= \frac{1}{1-\alpha}\left[\log\left\{\frac{1+(\lambda t)^{\alpha+\beta-1}\left\{(1+\lambda t)^{\alpha+\beta-1}-(\lambda t)^{\alpha+\beta-1}\right\}}{(1+\lambda t)^{\alpha+\beta-1}\left\{(1+\lambda t)^{\alpha+\beta-1}-(\lambda t)^{\alpha+\beta-1}\right\}}\right\} - \log\left\{\frac{1+(\lambda t)^{\beta}\left\{(1+\lambda t)^{\beta}-(\lambda t)^{\beta}\right\}}{(1+\lambda t)^{\beta}\left\{(1+\lambda t)^{\beta}-(\lambda t)^{\beta}\right\}}\right\}\right] \quad (18)$$

Now, taking limit as $\alpha \to \beta$, equation (18) becomes

$$S_2^{\beta}(\lambda,t) = \frac{1}{1-\beta}\left[\log\left\{\frac{1+(\lambda t)^{2\beta-1}((1+\lambda t)^{2\beta-1}-(\lambda t)^{2\beta-1})}{(1+\lambda t)^{2\beta-1}((1+\lambda t)^{2\beta-1}-(\lambda t)^{2\beta-1})}\right\} - \log\left\{\frac{1+(\lambda t)^{\beta}((1+\lambda t)^{\beta}-(\lambda t)^{\beta})}{(1+\lambda t)^{\beta}((1+\lambda t)^{\beta}-(\lambda t)^{\beta})}\right\}\right]$$

$$= \frac{1}{1-\beta}\begin{bmatrix}\log\left\{1+(\lambda t)^{2\beta-1}\left\{(1+\lambda t)^{2\beta-1}-(\lambda t)^{2\beta-1}\right\}\right\}\\ -\log\left\{(1+\lambda t)^{2\beta-1}\left\{(1+\lambda t)^{2\beta-1}-(\lambda t)^{2\beta-1}\right\}\right\}\\ -\log\left\{1+(\lambda t)^{\beta}\left\{(1+\lambda t)^{\beta}-(\lambda t)^{\beta}\right\}\right\}\\ +\log\left\{(1+\lambda t)^{\beta}\left\{(1+\lambda t)^{\beta}-(\lambda t)^{\beta}\right\}\right\}\end{bmatrix} \quad (19)$$

Now, taking limit as $\beta \to 1$, equation (19) becomes

$$S_2(\lambda,t) = -\frac{1}{1+\lambda t}\begin{bmatrix}-(\lambda t)^2\log\lambda t + \lambda t\log\lambda t\\ -(1+\lambda t)^2\log(1+\lambda t) + \lambda t(1+\lambda t)\log\lambda t\\ -(1+\lambda t)\log(1+\lambda t) + \lambda t(1+\lambda t)\log(1+\lambda t)\end{bmatrix}$$

$$= \frac{2\left\{(1+\lambda t)\log(1+\lambda t) - \lambda t \log \lambda t\right\}}{1+\lambda t} \quad (20)$$

Differentiating equation (18) w.r.t. λt , and taking limit as $\alpha \to \beta$, and then finally taking limit as $\beta \to 1$, we get the following result:

$$\frac{d}{d(\lambda t)} S_2(\lambda, t) = -\frac{2\log \lambda t}{(1+\lambda t)^2}$$

Thus, uncertainty increases if $\lambda t > 1$ and decreases if $\lambda t \le 1$.

Also from equation (20), we have

Max $S_2(\lambda, t) = 2\log 2$, when $\lambda t = 1$.

Further, when $t = 0$, the uncertainty is zero and when $t \to \infty$, we have the following expression from equation (20):

$$\underset{t\to\infty}{Lt}\, S_2(\lambda, t) = 0$$

Thus, in this case also, the uncertainty starts with zero value at $t = 0$ and ends with zero value as $t \to \infty$, and in between, it attains the maximum value at $\lambda t = 1$, that is, at t = $\frac{1}{\lambda}$.

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