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Neighbors of Indefinite Binary Quadratic Forms

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***Abstract*—**In this paper, we derive some algebraic identities on right and left neighbors $R(F)$ and $L(F)$ of an indefinite binary quadratic form $F = F(x, y) = ax^2 + bxy + cy^2$ of discriminant $\Delta = b^2 - 4ac$. We prove that the proper cycle of F can be given by using its consecutive left neighbors. Also we construct a connection between right and left neighbors of F.

***Keywords*—**Quadratic form, indefinite form, cycle, proper cycle, right neighbor, left neighbor.

I. PRELIMINARIES.

A real binary quadratic form F is a polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2 \quad (1)$$

with real coefficients a, b, c. We denote it by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. F is an integral form if and only if $a, b, c \in Z$, and is called indefinite if and only if $\Delta(F) > 0$. An indefinite form $F = (a, b, c)$ of discriminant Δ is said to be reduced if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \quad (2)$$

Most properties of quadratic forms can be giving by the aid of extended modular group $\overline{\Gamma}$ (see [5]). Gauss (1777-1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$\begin{aligned} gF(x, y) &= \left(ar^2 + brs + cs^2\right)x^2 \\ &\quad + (2art + bru + bts + 2csu)\,xy \\ &\quad + \left(at^2 + btu + cu^2\right)y^2 \end{aligned} \quad (3)$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$. Hence two forms F and G are called equivalent if and only if there exists a $g \in \overline{\Gamma}$ such that $gF = G$. If $\det g = 1$, then F and G are called properly equivalent, and if $\det g = -1$, then F and G are called improperly equivalent. If a form F is improperly equivalent to itself, then it called ambiguous.

Let $\rho(F)$ denotes the normalization (it means that replacing F by its normalization) of $(c, -b, a)$. To be more explicit, we set

$$\rho^i(F) = (c, -b + 2cr_i, cr_i^2 - br_i + a), \quad (4)$$

where

$$r_i = r_i(F) = \begin{cases} sign(c)\left\lfloor \frac{b}{2|c|} \right\rfloor & for\ |c| \geq \sqrt{\Delta} \\ sign(c)\left\lfloor \frac{b+\sqrt{\Delta}}{2|c|} \right\rfloor & for\ |c| < \sqrt{\Delta} \end{cases} \quad (5)$$

for $i \geq 0$. Then the number r_i is called the reducing number and the form $\rho^i(F)$ is called the reduction of F. Further, if F is reduced, then so is $\rho^i(F)$ by (2). In fact, ρ^i is a permutation of the set of all reduced indefinite forms.

Now consider the following transformations

$$\chi(F) = \chi(a, b, c) = (-c, b, -a)$$
$$\tau(F) = \tau(a, b, c) = (-a, b, -c).$$

If $\chi(F) = F$, that is, $F = (a, b, -a)$, then F is called symmetric. The cycle of F is the sequence $((\tau\rho)^i(G))$ for $i \in \mathbf{Z}$, where $G = (A, B, C)$ is a reduced form with $A > 0$ which is equivalent to F. The cycle and proper cycle of F can be given by the following theorem.

Theorem 1.1: Let $F = (a, b, c)$ be a reduced indefinite quadratic form of discriminant Δ. Then the cycle of F is a sequence $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ of length l, where $F_0 = F = (a_0, b_0, c_0)$,

$$s_i = |s(F_i)| = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor$$

and

$$\begin{aligned} F_{i+1} &= (a_{i+1}, b_{i+1}, c_{i+1}) \\ &= \left(|c_i|,\ -b_i + 2s_i|c_i|,\ -(a_i + b_is_i + c_is_i^2)\right) \end{aligned}$$

for $1 \leq i \leq l - 2$. If l is odd, then the proper cycle of F is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length $2l$ and if l is even, then the proper cycle of F is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length l. In this case the equivalence class of F is the disjoint union of the proper equivalence class of F and the proper equivalence class of $\tau(F)$. [1], [4]

The right neighbor of $F = (a, b, c)$ is denoted by $R(F)$ is the form (A, B, C) determined by $A = c$, $b + B \equiv 0(mod\,2A)$, $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$ and $B^2 - 4AC = \Delta$. It is clear from definition that

$$R(F) = \begin{pmatrix} 0 & -1 \\ 1 & -\delta \end{pmatrix}(a, b, c), \quad (6)$$

where $b + B = 2c\delta$. The left neighbor is hence

$$L(F) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(c, b, a) = \chi\tau(R(c, b, a)). \quad (7)$$

So F is properly equivalent to its right and left neighbors (for further details on binary quadratic forms see [1], [2], [3], [4]).

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II. Neighbors of Indefinite Quadratic Forms.

In this section, we will derive some properties of neighbors of indefinite quadratic forms. In [6], we proved the following theorem.

Theorem 2.1: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ be the cycle of F of length l and let $R^i(F_0)$ be the consecutive right neighbors of $F = F_0$ for $i \geq 0$.

1) If l is odd, then the proper cycle of F is

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$$

of length $2l$.

2) If l is even, then the proper cycle of F is

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0)$$

of length l.

Also we proved that if l is odd, then $R^{\frac{l-1}{2}}(F_0)$ and $R^{\frac{3l-1}{2}}(F_0)$ are the symmetric right neighbors of F. Further we proved the following corollary and two theorems in [6].

Corollary 2.2: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ be the cycle of F of length l.

1) If l is odd, then

$$R^i(F_0) = \begin{cases} F_i & i \ is \ even \\ \tau(F_i) & i \ is \ odd \end{cases}$$

for $1 \leq i \leq l-1$ and

$$R^i(F_0) = \begin{cases} F_{i-l} & i \ is \ even \\ \tau(F_{i-l}) & i \ is \ odd \end{cases}$$

for $l \leq i \leq 2l-1$.

2) If l is even, then

$$R^i(F_0) = \begin{cases} F_i & i \ is \ even \\ \tau(F_i) & i \ is \ odd \end{cases}$$

for $1 \leq i \leq l-1$.

Theorem 2.3: If l is odd, then F has $2l-1$ right neighbors and if l is even, then F has $l-1$ right neighbors.

Theorem 2.4: If l is odd, then

1) $R^i(F_0) = \chi\tau(R^{2l-1-i}(F_0))$ for $1 \leq i \leq 2l-2$ and $R^{2l-1}(F_0) = \chi\tau(F_0)$.

2) $R^i(F_0) = \tau(R^{i+l}(F_0))$, $R^l(F_0) = \tau(F_0)$ for $l \leq i \leq l-1$ and $R^i(F_0) = \tau(R^{i-l}(F_0))$ for $l+1 \leq i \leq 2l-1$.

In [7], we also derived some algebraic identities on proper cycles and right neighbors of F. Now we can return our problem. Then we can give the following theorems.

Theorem 2.5: If l is odd, then in the proper cycle of F, we have

1) $R^i(F_0) = \tau(F_{i-l})$ for $l \leq i \leq 2l-1$.

2) $\chi\tau(R^i(F_0)) = R^{2l-1-i}(F_0)$ for $0 \leq i \leq l-1$.

Proof: 1) Let $F_0 = F = (a_0, b_0, c_0)$. Then applying (6), we get

$$\begin{aligned}
F_0 &= (a_0,\, b_0,\, c_0) \\
R^1(F_0) &= (a_1, b_1, c_1) \\
R^2(F_0) &= (a_2,\, b_2,\, c_2) \\
&\cdots \\
R^{\frac{l-3}{2}}(F_0) &= \left(a_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, c_{\frac{l-3}{2}}\right) \\
R^{\frac{l-1}{2}}(F_0) &= \left(a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, c_{\frac{l-1}{2}}\right) \\
R^{\frac{l+1}{2}}(F_0) &= \left(-c_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, -a_{\frac{l-3}{2}}\right) \\
&\cdots \\
R^{l-3}(F_0) &= (-c_2, b_2, -a_2) \\
R^{l-2}(F_0) &= (-c_1, b_1, -a_1) \\
R^{l-1}(F_0) &= (-c_0, b_0, -a_0) \\
R^{l}(F_0) &= (-a_0,\, b_0, -\,c_0) \\
R^{l+1}(F_0) &= (-a_1, b_1, -c_1) \\
R^{l+2}(F_0) &= (-a_2,\, b_2,\, -c_2) \\
&\cdots \\
R^{\frac{3l-3}{2}}(F_0) &= \left(-a_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, -c_{\frac{l-3}{2}}\right) \\
R^{\frac{3l-1}{2}}(F_0) &= \left(-a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, -c_{\frac{l-1}{2}}\right) \\
R^{\frac{3l+1}{2}}(F_0) &= \left(c_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, a_{\frac{l-3}{2}}\right) \\
&\cdots \\
R^{2l-3}(F_0) &= (c_2, b_2, a_2) \\
R^{2l-2}(F_0) &= (c_1, b_1, a_1) \\
R^{2l-1}(F_0) &= (c_0, b_0, a_0).
\end{aligned}$$

Hence it is clear that

$$\begin{aligned}
R^{l}(F_0) &= \tau(F_0) \\
R^{l+1}(F_0) &= \tau(F_1) \\
R^{l+2}(F_0) &= \tau(F_2) \\
&\cdots \\
R^{\frac{3l-3}{2}}(F_0) &= \tau(F_{\frac{l-3}{2}}) \\
R^{\frac{3l-1}{2}}(F_0) &= \tau(F_{\frac{l-1}{2}}) \\
R^{\frac{3l+1}{2}}(F_0) &= \tau(F_{\frac{l+1}{2}}) \\
&\cdots \\
R^{2l-3}(F_0) &= \tau(F_{l-3}) \\
R^{2l-2}(F_0) &= \tau(F_{l-2}) \\
R^{2l-1}(F_0) &= \tau(F_{l-1}).
\end{aligned}$$

So $R^i(F_0) = \tau(F_{i-l})$ for $l \leq i \leq 2l-1$.

2) Similarly we find that

$$\begin{aligned}
\chi\tau(F_0) &= R^{2l-1}(F_0) \\
\chi\tau(R^1(F_0)) &= R^{2l-2}(F_0) \\
\chi\tau(R^2(F_0)) &= R^{2l-3}(F_0) \\
&\cdots \\
\chi\tau(R^{\frac{l-3}{2}}(F_0)) &= R^{\frac{3l+1}{2}}(F_0)
\end{aligned}$$

$$\begin{aligned}
\chi\tau(R^{\frac{l-1}{2}}(F_0)) &= R^{\frac{3l-1}{2}}(F_0)\\
\chi\tau(R^{\frac{l+1}{2}}(F_0)) &= R^{\frac{3l-3}{2}}(F_0)\\
&\cdots\\
\chi\tau(R^{l-3}(F_0)) &= R^{l+2}(F_0)\\
\chi\tau(R^{l-2}(F_0)) &= R^{l+1}(F_0)\\
\chi\tau(R^{l-1}(F_0)) &= R^{l}(F_0).
\end{aligned}$$

So $\chi\tau(R^i(F_0)) = R^{2l-1-i}(F_0)$ for $0 \leq i \leq l-1$. ■

Now we consider the left neighbors of F. Recall that the left neighbor of F is defined to be

$$L(F) = L(a,b,c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(c,b,a).$$

Then we can give the following theorem.

Theorem 2.6: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ denote the cycle of F. If l is odd, then

1)

$$L^i(F_0) = \begin{cases} \tau(F_{l-i}) & i \text{ is odd} \\ F_{l-i} & i \text{ is even} \end{cases}$$

for $1 \leq i \leq l$ and

$$L^i(F_0) = \begin{cases} \tau(F_{2l-i}) & i \text{ is odd} \\ F_{2l-i} & i \text{ is even} \end{cases}$$

for $l+1 \leq i \leq 2l$.

2)

$$\tau(L^i(F_0)) = \begin{cases} F_{l-i} & i \text{ is odd} \\ \tau(F_{l-i}) & i \text{ is even} \end{cases}$$

for $1 \leq i \leq l$ and

$$\tau(L^i(F_0)) = \begin{cases} F_{2l-i} & i \text{ is odd} \\ \tau(F_{2l-i}) & i \text{ is even} \end{cases}$$

for $l+1 \leq i \leq 2l$.

3)

$$\chi(L^i(F_0)) = \begin{cases} \tau(F_{i-1}) & i \text{ is odd} \\ F_{i-1} & i \text{ is even} \end{cases}$$

for $1 \leq i \leq l$ and

$$\chi(L^i(F_0)) = \begin{cases} \tau(F_{i-l-1}) & i \text{ is odd} \\ F_{i-l-1} & i \text{ is even} \end{cases}$$

for $l+1 \leq i \leq 2l$.

Proof: 1) Applying (7), we get

$$\begin{aligned}
L^1(F_0) &= (c_0, b_0, a_0) = \tau(F_{l-1})\\
L^2(F_0) &= (-c_1, b_1, -a_1) = F_{l-2}\\
L^3(F_0) &= (c_2, b_2, a_2) = \tau(F_{l-3})\\
&\cdots\\
L^l(F_0) &= (-a_0,\, b_0,\, -c_0) = \tau(F_0)\\
L^{l+1}(F_0) &= (-c_0, b_0, -a_0) = F_{l-2}\\
&\cdots\\
L^{2l-1}(F_0) &= (-a_1, b_1, -c_1) = \tau(F_1)\\
L^{2l}(F_0) &= (a_0, b_0, c_0) = F_0.
\end{aligned}$$

So the result is clear. The others can be proved similarly. ■

Note that we proved in Theorem 2.1 that the proper cycle of F can be given by using its consecutive right neighbors. Similarly we can give the following theorem.

Theorem 2.7: Let $L^i(F)$ denote the consecutive left neighbors of F.

1) If l is odd, then the proper cycle of $F = F_0$ is

$$F_0 \sim L^{2l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0)$$

of length $2l$.

2) If l is even, then the proper cycle of $F = F_0$ is

$$F_0 \sim L^{l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0)$$

of length l.

Proof: 1) Let l be odd. Then by Theorem 1.1 the proper cycle of F is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length $2l$. We also see Theorem 2.6 that

$$L^i(F_0) = \begin{cases} \tau(F_{l-i}) & i \text{ is odd} \\ F_{l-i} & i \text{ is even} \end{cases}$$

for $1 \leq i \leq l$ and

$$L^i(F_0) = \begin{cases} \tau(F_{2l-i}) & i \text{ is odd} \\ F_{2l-i} & i \text{ is even} \end{cases}$$

for $l+1 \leq i \leq 2l$. So the proper cycle of F is $F_0 \sim L^{2l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0)$.

Similarly it can be shown that if l is even, then the proper cycle of F is $F_0 \sim L^{l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0)$. ■

Example 2.1: **1)** The cycle of $F = (1, 5, -4)$ is $F_0 = (1, 5, -4) \sim F_1 = (4, 3, -2) \sim F_2 = (2, 5, -2) \sim F_3 = (2, 3, -4) \sim F_4 = (4, 5, -1)$ of length 5. So its proper cycle is hence

$$F_0 = (1,5,-4) \sim F_1 = (-4,3,2) \sim F_2 = (2,5,-2) \sim F_3 = (-2,3,4) \sim F_4 = (4,5,-1) \sim F_5 = (-1,5,4) \sim F_6 = (4,3,-2) \sim F_7 = (-2,5,2) \sim F_8 = (2,3,-4) \sim F_9 = (-4,5,1)$$

of length 10. The consecutive left neighbors of F are

$$\begin{gathered}
L^1(F) = (-4,5,1), L^2(F) = (2,3,-4),\\
L^3(F) = (-2,5,2), L^4(F) = (4,3,-2),\\
L^5(F) = (-1,5,4), L^6(F) = (4,5,-1),\\
L^7(F) = (-2,3,4), L^8(F) = (2,5,-2),\\
L^9(F) = (-4,3,2), L^{10}(F) = F.
\end{gathered}$$

So it is easily seen that the proper cycle of F is

$$F \sim L^9(F) \sim L^8(F) \sim L^7(F) \sim L^6(F) \sim L^5(F) \sim L^4(F) \sim L^3(F) \sim L^2(F) \sim L^1(F).$$

2) The cycle of $F = (1, 8, -5)$ is $F_0 = (1, 8, -5) \sim F_1 = (5, 2, -4) \sim F_2 = (4, 6, -3) \sim F_3 = (3, 6, -4) \sim F_4 = (4, 2, -5) \sim F_5 = (5, 8, -1)$ of length 6. So its proper cycle is

$$F_0 = (1, 8, -5) \sim F_1 = (-5, 2, 4) \sim F_2 = (4, 6, -3) \sim F_3 = (-3, 6, 4) \sim F_4 = (4, 2, -5) \sim F_5 = (-5, 8, 1).$$

The left neighbors of F are

$$L^1(F) = (-5, 8, 1), L^2(F) = (4, 2, -5), L^3(F) = (-3, 6, 4), L^4(F) = (4, 6, -3), L^5(F) = (-5, 2, 4), L^6(F) = F.$$

So its proper cycle is $F \sim L^5(F) \sim L^4(F) \sim L^3(F) \sim L^2(F) \sim L^1(F)$.

From above theorem, we can give the following result.

Theorem 2.8: If l is odd, then F has $2l - 1$ left neighbors and if l is even it has $l - 1$ left neighbors.

Proof: Let l be odd. Then we get

$$\begin{aligned}
F_0 &= (a_0,\, b_0,\, c_0)\\
F_1 &= (a_1, b_1, c_1)\\
F_2 &= (a_2,\, b_2,\, c_2)\\
F_3 &= (a_3, b_3, c_3)\\
&\cdots\\
F_{\frac{l-3}{2}} &= \left(a_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, c_{\frac{l-3}{2}}\right)\\
F_{\frac{l-1}{2}} &= \left(a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, -a_{\frac{l-1}{2}}\right)\\
F_{\frac{l+1}{2}} &= \left(-c_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, -a_{\frac{l-3}{2}}\right)\\
&\cdots\\
F_{l-3} &= (-c_2, b_2, -a_2)\\
F_{l-2} &= (-c_1, b_1, -a_1)\\
F_{l-1} &= (-c_0, b_0, -a_0).
\end{aligned}$$

The first left neighbor of $F = F_0$ is

$$\begin{aligned}
L^1(F_0) &= (a_1, b_1, c_1)\\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(c_0, b_0, a_0)\\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (a_0, -b_0 + 2a_0\delta_0, c_0 - \delta_0 b_0 + a_0\delta_0^2)\\
&= (c_0 - \delta_0 b_0 + a_0\delta_0^2, -b_0 + 2a_0\delta_0, a_0)\\
&= (c_0, b_0, a_0).
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
L^2(F_0) &= (-c_1, b_1, -a_1)\\
L^3(F_0) &= (c_2, b_2, a_2)\\
L^4(F_0) &= (-c_3, b_3, -a_3)\\
&\cdots\\
L^l(F_0) &= (-a_0,\, b_0, -c_0)\\
L^{l+1}(F_0) &= (-c_0, b_0, -a_0)\\
&\cdots\\
L^{2l-1}(F_0) &= (-a_1, b_1, -c_1)\\
L^{2l}(F_0) &= (a_0, b_0, c_0) = F_0.
\end{aligned}$$

So F has $2l - 1$ left neighbors. Similarly it can be shown that F has $l - 1$ left neighbors if l is even. ■

Theorem 2.9: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ be the cycle of F of length l. If l is odd, then

1) $L(F_i) = \tau(F_{i-1})$ for $1 \leq i \leq l - 1$ and $L(F_0) = \tau(F_{l-1})$.

2) $L(F_i) = \chi\tau(F_{l-i})$ for $1 \leq i \leq l - 1$ and $L(F_0) = \chi\tau(F_0)$.

Proof: 1) Let $F = F_0 = (a_0, b_0, c_0)$. Then

$$\begin{aligned}
F_1 &= (a_1, b_1, c_1)\\
&= \left(|c_0|,\, -b_0 + 2s_0|c_0|,\, -(a_0 + b_0 s_0 + c_0 s_0^2)\right)\\
&= \left(-c_0,\, -b_0 - 2s_0 c_0,\, -a_0 - b_0 s_0 - c_0 s_0^2\right). \qquad (8)
\end{aligned}$$

Now we try to determine the first left neighbor of F_1. Applying its definition, we get

$$\begin{aligned}
L(F_1) &= L\left(-c_0,\, -b_0 - 2s_0 c_0,\, -a_0 - b_0 s_0 - c_0 s_0^2\right)\\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R\left(-a_0 - b_0 s_0 - c_0 s_0^2, -b_0 - 2s_0 c_0, -c_0\right). \qquad (9)
\end{aligned}$$

So we have to find out the right neighbor of $(-a_0 - b_0 s_0 - c_0 s_0^2, -b_0 - 2s_0 c_0, -c_0)$. To get this we make the change of variables $x \to y$ and $y \to -x - \delta_0 y$. Then we get

$$\begin{aligned}
& R\left(-a_0 - b_0 s_0 - c_0 s_0^2, -b_0 - 2s_0 c_0, -c_0\right)\\
={}& (-a_0 - b_0 s_0 - c_0 s_0^2)y^2 + (-b_0 - 2s_0 c_0)y(-x - \delta_0 y)\\
& +(-c_0)(-x - \delta_0 y)^2\\
={}& -c_0 x^2 + (b_0 + 2c_0 s_0 - 2c_0\delta_0)xy \qquad (10)\\
& +(-a_0 - b_0 s_0 - c_0 s_0^2 + b_0\delta_0 + 2s_0 c_0\delta_0 - c_0\delta_0^2)y^2.
\end{aligned}$$

Also for $i = 0$, we get $s_0 = -\delta_0$. So (10) becomes

$$\begin{aligned}
& R\left(-a_0 - b_0 s_0 - c_0 s_0^2, -b_0 - 2s_0 c_0, -c_0\right)\\
={}& -c_0 x^2 + (b_0 - 2c_0\delta_0 - 2c_0\delta_0)xy\\
& +(-a_0 + b_0\delta_0 - c_0\delta_0^2 + b_0\delta_0 - 2\delta_0^2 c_0 - c_0\delta_0^2)y^2. \qquad (11)
\end{aligned}$$

Since $s_0 = -\delta_0 = 0$, (11) becomes

$$\begin{aligned}
& R\left(-a_0 - b_0 s_0 - c_0 s_0^2, -b_0 - 2s_0 c_0, -c_0\right)\\
&= -c_0 x^2 + b_0 xy - a_0 y^2. \qquad (12)
\end{aligned}$$

So applying (9) and (12), we get

$$\begin{aligned}
L(F_1) &= L\left(-c_0,\, -b_0 - 2s_0 c_0,\, -a_0 - b_0 s_0 - c_0 s_0^2\right)\\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R\left(-a_0 - b_0 s_0 - c_0 s_0^2, -b_0 - 2s_0 c_0, -c_0\right)\\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (-c_0, b_0, -a_0)\\
&= (-a_0, b_0, -c_0)\\
&= \tau(F_0).
\end{aligned}$$

Similarly we find that $L(F_2) = \tau(F_1)$, $L(F_3) = \tau(F_2)$, $\cdots, L(F_{l-1}) = \tau(F_{l-2})$ and $L(F_0) = \tau(F_{l-1})$. The other case can be proved similarly. ■

Example 2.2: The cycle of $F = (1, 7, -6)$ is

$$F_0 = (1,7,-6) \sim F_1 = (6,5,-2) \sim F_2 = (2,7,-3) \sim$$
$$F_3 = (3,5,-4) \sim F_4 = (4,3,-4) \sim F_5 = (4,5,-3) \sim$$
$$F_6 = (3,7,-2) \sim F_7 = (2,5,-6) \sim F_8 = (6,7,-1).$$

Then

$$\begin{aligned}
L(F_0) &= L(1,7,-6) = (-6,7,1) = \tau(F_8) = \chi\tau(F_0)\\
L(F_1) &= L(6,5,-2) = (-1,7,6) = \tau(F_0) = \chi\tau(F_8)\\
L(F_2) &= L(2,7,-3) = (-6,5,2) = \tau(F_1) = \chi\tau(F_7)\\
L(F_3) &= L(3,5,-4) = (-2,7,3) = \tau(F_2) = \chi\tau(F_6)\\
L(F_4) &= L(4,3,-4) = (-3,5,4) = \tau(F_3) = \chi\tau(F_5)\\
L(F_5) &= L(4,5,-3) = (-4,3,4) = \tau(F_4) = \chi\tau(F_4)\\
L(F_6) &= L(3,7,-2) = (-4,5,3) = \tau(F_5) = \chi\tau(F_3)\\
L(F_7) &= L(2,5,-6) = (-3,7,2) = \tau(F_6) = \chi\tau(F_2)\\
L(F_8) &= L(6,7,-1) = (-2,6,5) = \tau(F_7) = \chi\tau(F_1)
\end{aligned}$$

as we wanted.

From above theorem, we can give the following corollary.

Corollary 2.10: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ be the cycle of F of length l. If l is odd, then

1) $\tau(L^i(F_0)) = L^{i+l}(F_0)$ for $1 \leq i \leq l$.

2) $\chi(L^i(F_0)) = L^{l+1-i}(F_0)$ for $1 \leq i \leq l$ and $\chi(L^i(F_0)) = L^{3l+1-i}(F_0)$ for $l+1 \leq i \leq 2l$.

Theorem 2.11: Let $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ be the cycle of F of length l. If l is odd, then $L^{\frac{l+1}{2}}(F_0)$ and $L^{\frac{3l+1}{2}}(F_0)$ are the symmetric left neighbors of F.

Proof: We know that F has $2l-1$ left neighbors when l is odd. Also

$$\begin{aligned}
L^1(F_0) &= (c_0, b_0, a_0)\\
L^2(F_0) &= (-c_1, b_1, -a_1)\\
L^3(F_0) &= (c_2, b_2, a_2)\\
&\cdots\\
L^{\frac{l-1}{2}}(F_0) &= (-a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, -c_{\frac{l-1}{2}})\\
L^{\frac{l+1}{2}}(F_0) &= (-a_{\frac{l+1}{2}}, b_{\frac{l+1}{2}}, -a_{\frac{l+1}{2}})\\
L^{\frac{l+3}{2}}(F_0) &= (c_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, a_{\frac{l-1}{2}})\\
&\cdots\\
L^{l}(F_0) &= (-a_0, b_0, -c_0)\\
L^{l+1}(F_0) &= (-c_0, b_0, -a_0)\\
&\cdots\\
L^{\frac{3l-1}{2}}(F_0) &= (a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, c_{\frac{l-1}{2}})\\
L^{\frac{3l+1}{2}}(F_0) &= (a_{\frac{l+1}{2}}, b_{\frac{l+1}{2}}, -a_{\frac{l+1}{2}})\\
L^{\frac{3l+3}{2}}(F_0) &= (-c_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, -a_{\frac{l-1}{2}})\\
&\cdots\\
L^{2l-1}(F_0) &= (-a_1, b_1, -c_1)\\
L^{2l}(F_0) &= (a_0, b_0, c_0).
\end{aligned}$$

So $L^{\frac{l+1}{2}}(F_0)$ and $L^{\frac{3l+1}{2}}(F_0)$ are symmetric left neighbors. ■

Theorem 2.12: If l is odd, then in the proper cycle of F, we have

1) $L^i(F_0) = F_{2l-i}$ for $1 \leq i \leq 2l$.

2) $L^i(F_0) = \tau(F_{l-i})$ for $1 \leq i \leq l$ and $L^i(F_0) = \tau(F_{3l-i})$ for $l+1 \leq i \leq 2l$.

3) $L^i(F_0) = \chi(F_{l-1+i})$ for $1 \leq i \leq l$ and $L^i(F_0) = \chi(F_{i-l-1})$ for $l+1 \leq i \leq 2l$.

4) $L^i(F_0) = \chi\tau(F_{i-1})$ for $1 \leq i \leq 2l$.

Proof: 1) Before starting our proof, we try to determine the cycle and proper cycle of F. To get this let $F = F_0 = (a_0, b_0, c_0)$. Then the cycle of F is $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-2} \sim F_{l-1}$, where

$$\begin{aligned}
F_0 &= (a_0, b_0, c_0)\\
F_1 &= (a_1, b_1, c_1)\\
F_2 &= (a_2, b_2, c_2)\\
F_3 &= (a_3, b_3, c_3)\\
&\cdots\\
F_{\frac{l-3}{2}} &= \left(a_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, c_{\frac{l-3}{2}}\right)\\
F_{\frac{l-1}{2}} &= \left(a_{\frac{l-1}{2}}, b_{\frac{l-1}{2}}, -a_{\frac{l-1}{2}}\right)\\
F_{\frac{l+1}{2}} &= \left(-c_{\frac{l-3}{2}}, b_{\frac{l-3}{2}}, -a_{\frac{l-3}{2}}\right)\\
&\cdots\\
F_{l-3} &= (-c_2, b_2, -a_2)\\
F_{l-2} &= (-c_1, b_1, -a_1)\\
F_{l-1} &= (-c_0, b_0, -a_0).
\end{aligned}$$

So the proper cycle of F is hence $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1} \sim F_l \sim F_{l+1} \sim F_{l+2} \sim \cdots \sim F_{2l-2} \sim F_{2l-1}$, where

$$\begin{aligned}
F_0 &= (a_0, b_0, c_0)\\
F_1 &= (-a_1, b_1, -c_1)\\
F_2 &= (a_2, b_2, c_2)\\
F_3 &= (-a_3, b_3, -c_3)\\
&\cdots\\
F_{l-2} &= (c_1, b_1, a_1)\\
F_{l-1} &= (-c_0, b_0, -a_0)\\
F_l &= (-a_0, b_0, -c_0)\\
F_{l+1} &= (a_1, b_1, c_1)\\
&\cdots\\
F_{2l-2} &= (-c_1, b_1, -a_1)\\
F_{2l-1} &= (c_0, b_0, a_0).
\end{aligned}$$

Now we determine the left neighbors of $F = F_0$. Then applying (7), we get

$$\begin{aligned}
L^1(F_0) &= (c_0, b_0, a_0) = F_{2l-1}\\
L^2(F_0) &= (-c_1, b_1, -a_1) = F_{2l-2}\\
&\cdots\\
L^l(F_0) &= (-a_0, b_0, -c_0) = F_l\\
L^{l+1}(F_0) &= (-c_0, b_0, -a_0) = F_{l-1}
\end{aligned}$$

$$\begin{aligned} &\cdots \\ L^{2l-1}(F_0) &= (-a_1, b_1, -c_1) = F_1 \\ L^{2l}(F_0) &= (a_0, b_0, c_0) = F_0. \end{aligned}$$

So $L^i(F_0) = F_{2l-i}$ for $1 \leq i \leq 2l$.

2) Similarly we obtain

$$\begin{aligned} L^1(F_0) &= (c_0, b_0, a_0) = \tau(F_{l-1}) \\ L^2(F_0) &= (-c_1, b_1, -a_1) = \tau(F_{l-2}) \\ &\cdots \\ L^{l-2}(F_0) &= (-a_2, b_2, -c_2) = \tau(F_2) \\ L^{l-1}(F_0) &= (a_1, b_1, c_1) = \tau(F_1) \\ L^{l}(F_0) &= (-a_0, b_0, -c_0) = \tau(F_0) \\ L^{l+1}(F_0) &= (-c_0, b_0, -a_0) = \tau(F_{2l-1}) \\ L^{l+2}(F_0) &= (c_1, b_1, a_1) = \tau(F_{2l-2}) \\ &\cdots \\ L^{2l-1}(F_0) &= (-a_1, b_1, -c_1) = \tau(F_{l+1}) \\ L^{2l}(F_0) &= (a_0, b_0, c_0) = \tau(F_l). \end{aligned}$$

So $L^i(F_0) = \tau(F_{l-i})$ for $l \leq i \leq l$ and $L^i(F_0) = \tau(F_{3l-i})$ for $l+1 \leq i \leq 2l$.

The others are proved similarly. ■

Example 2.3: The cycle of $F = (1, 7, -6)$ is

$$\begin{aligned} &F_0 = (1,7,-6) \sim F_1 = (6,5,-2) \sim F_2 = (2,7,-3) \sim \\ &F_3 = (3,5,-4) \sim F_4 = (4,3,-4) \sim F_5 = (4,5,-3) \sim \\ &F_6 = (3,7,-2) \sim F_7 = (2,5,-6) \sim F_8 = (6,7,-1) \end{aligned}$$

and hence the proper cycle of is

$$\begin{aligned} &F_0 = (1,7,-6) \sim F_1 = (-6,5,2) \sim F_2 = (2,7,-3) \sim \\ &F_3 = (-3,5,4) \sim F_4 = (4,3,-4) \sim F_5 = (-4,5,3) \sim \\ &F_6 = (3,7,-2) \sim F_7 = (-2,5,6) \sim F_8 = (6,7,-1) \sim \\ &F_9 = (-1,7,6) \sim F_{10} = (6,5,-2) \sim F_{11} = (-2,7,3) \sim \\ &F_{12} = (3,5,-4) \sim F_{13} = (-4,3,4) \sim F_{14} = (4,5,-3) \sim \\ &F_{15} = (-3,7,2) \sim F_{16} = (2,5,-6) \sim F_{17} = (-6,7,1). \end{aligned}$$

The left neighbors of F are

$$\begin{aligned} &L^1(F_0) = (-6,7,1) = F_{17},\ L^2(F_0) = (2,5,-6) = F_{16}, \\ &L^3(F_0) = (-3,7,2) = F_{15}, L^4(F_0) = (4,5,-3) = F_{14}, \\ &L^5(F_0) = (-4,3,4) = F_{13}, L^6(F_0) = (3,5,-4) = F_{12} \\ &L^7(F_0) = (-2,7,3) = F_{11},\ L^8(F_0) = (6,5,-2) = F_{10}, \\ &L^9(F_0) = (-1,7,6) = F_9, L^{10}(F_0) = (6,7,-1) = F_8, \\ &L^{11}(F_0) = (-2,5,6) = F_7, L^{12}(F_0) = (3,7,-2) = F_6 \\ &L^{13}(F_0) = (-4,5,3) = F_5,\ L^{14}(F_0) = (4,3,-4) = F_4, \\ &L^{15}(F_0) = (-3,5,4) = F_3, L^{16}(F_0) = (2,7,-3) = F_2, \\ &L^{17}(F_0) = (-6,5,2) = F_1,\ L^{18}(F_0) = (1,7,-6) = F_0. \end{aligned}$$

Here, $L^5(F_0)$ and $L^{14}(F_0)$ are symmetric left neighbors of F by Theorem 2.11.

Now we give the connection between right and left neighbors of F. To get this we can give the following theorem.

Theorem 2.13: Let $R^i(F_0)$ and $L^i(F_0)$ be denote the right and left neighbors of F, respectively.

1) If l is odd, then $L^i(F_0) = R^{2l-i}(F_0)$ for $1 \leq i \leq 2l-1$.

2) If l is even, then $L^i(F_0) = R^{l-i}(F_0)$ for $1 \leq i \leq l-1$.

Proof: 1) Let l be odd. Then the proper cycle of F can be given by using its consecutive right neighbors, that is, $F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$ by Theorem 2.1. Also by considering the proper cycle $F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1})$ of F, we get

$$R^i(F_0) = \begin{cases} F_i & i\ is\ even \\ \tau(F_i) & i\ is\ odd \end{cases}$$

for $1 \leq i \leq l-1$ and

$$R^i(F_0) = \begin{cases} F_{i-l} & i\ is\ even \\ \tau(F_{i-l}) & i\ is\ odd \end{cases}$$

for $l \leq i \leq 2l-1$ by Corollary 2.2. Also

$$L^i(F_0) = \begin{cases} \tau(F_{l-i}) & i\ is\ odd \\ F_{l-i} & i\ is\ even \end{cases}$$

for $1 \leq i \leq l$ and

$$L^i(F_0) = \begin{cases} \tau(F_{2l-i}) & i\ is\ odd \\ F_{2l-i} & i\ is\ even \end{cases}$$

for $l+1 \leq i \leq 2l$. On the other hand, since the proper cycle of F is $L^{2l}(F_0) \sim L^{2l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0)$, we conclude that $L^i(F_0) = R^{2l-i}(F_0)$ for $1 \leq i \leq 2l-1$.

Similarly if l is even, then $L^i(F_0) = R^{l-i}(F_0)$ for $1 \leq i \leq l-1$. ■

Example 2.4: **1)** The cycle of $F = (1, 5, -4)$ is $F_0 = (1, 5, -4) \sim F_1 = (4, 3, -2) \sim F_2 = (2, 5, -2) \sim F_3 = (2, 3, -4) \sim F_4 = (4, 5, -1)$. The consecutive left and right neighbors of F are

$$\begin{aligned} L^1(F) &= (-4,5,1) = R^9(F) \\ L^2(F) &= (2,3,-4) = R^8(F) \\ L^3(F) &= (-2,5,2) = R^7(F) \\ L^4(F) &= (4,3,-2) = R^6(F) \\ L^5(F) &= (-1,5,4) = R^5(F) \\ L^6(F) &= (4,5,-1) = R^4(F) \\ L^7(F) &= (-2,3,4) = R^3(F) \\ L^8(F) &= (2,5,-2) = R^2(F) \\ L^9(F) &= (-4,3,2) = R^1(F). \end{aligned}$$

2) The cycle of $F = (1, 8, -5)$ is $F_0 = (1, 8, -5) \sim F_1 = (5, 2, -4) \sim F_2 = (4, 6, -3) \sim F_3 = (3, 6, -4) \sim F_4 = (4, 2, -5) \sim F_5 = (5, 8, -1)$. The consecutive left and right neighbors of F are

$$\begin{aligned}L^1(F) = (-5, 8, 1) = R^5(F)\\ L^2(F) = (4, 2, -5) = R^4(F)\\ L^3(F) = (-3, 6, 4) = R^3(F)\\ L^4(F) = (4, 6, -3) = R^2(F)\\ L^5(F) = (-5, 2, 4) = R^1(F).\end{aligned}$$

From above theorem, we can give the following result.

Corollary 2.14: Let $R^i(F_0)$ and $L^i(F_0)$ denote the right and left neighbors of F_0, respectively. If l is odd, then

1) $L^i(F_0) = \tau(R^{l-i}(F_0))$ for $1 \leq i \leq l$ and $L^i(F_0) = \tau(R^{3l-i}(F_0))$ for $l+1 \leq i \leq 2l$.

2) $L^i(F_0) = \chi(R^{i+l-1}(F_0))$ for $1 \leq i \leq l$ and $L^i(F_0) = \chi(R^{i-l-1}(F_0))$ for $l+1 \leq i \leq 2l$.

If l is even, then $L^i(F_0) = \chi\tau(R^{i-1}(F_0))$ for $1 \leq i \leq l-1$.

Finally, we can give the following theorem.

Theorem 2.15: $R(F_0)$ and $L(F_0)$ denote the right and left neighbors of F_0, respectively. Then

$$R(L(F_0)) = L(R(F_0)) = F_0.$$

Proof: Recall that the right neighbor of $F = (a, b, c)$ is the form $R(F) = (A, B, C)$, where $A = c$, $b + B \equiv 0$ $(mod\ 2A)$, $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$ and $B^2 - 4AC = \Delta$. Also $R(F) = [0; -1; 1; -\delta](a, b, c)$ for $b + B = 2c\delta$ and $L(F) = \chi\tau(R(c, b, a))$. For $F = F_0 = (a_0, b_0, c_0)$, we get

$$L(F_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(c_0, b_0, a_0). \tag{13}$$

Now we try to find $R(c_0, b_0, a_0)$. It is easily seen that

$$R(c_0, b_0, a_0) = (a_0, -b_0 + 2a_0\delta_0, c_0 - b_0\delta_0 + a_0\delta_0^2).$$

So (13) becomes

$$L(F_0) = (c_0 - b_0\delta_0 + a_0\delta_0^2, -b_0 + 2a_0\delta_0, a_0).$$

Note that $-b_0 + 2a_0\delta_0 \equiv -b_0(mod\ 2a)$. Also $\sqrt{\Delta} - 2|a_0| < -b_0 + 2a_0\delta_0 < \sqrt{\Delta}$. So if we take the right neighbor of $L(F_0)$, then we get

$$\begin{aligned} R(L(F_0)) &= R(c_0 - b_0\delta_0 + a_0\delta_0^2, -b_0 + 2a_0\delta_0, a_0) \\ &= (a_0, b_0, c_0) \\ &= F_0. \end{aligned}$$

Similarly it can be proved that $L(R(F_0)) = F_0$. ■

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