Momentum and heat transfer in the flow of a viscoelastic fluid past a porous flat plate subject to suction or blowing

Motahar Reza and Anadi Sankar Gupta

Abstract—An analysis is made of the flow of an incompressible viscoelastic fluid (of small memory) over a porous plate subject to suction or blowing. It is found that velocity at a point increases with increase in the elasticity in the fluid. It is also shown that wall shear stress depends only on suction and is also independent of the material of fluids. No steady solution for velocity distribution exists when there is blowing at the plate. Temperature distribution in the boundary layer is determined and it is found that temperature at a point decreases with increase in the elasticity in the fluid.

Keywords—Viscoelastic fluid, Flow past a porous plate, Heat transfer

I. Introduction

IT is known that liquids respond like elastic solids to impulses, which are very rapid compared to the time, it takes for the molecular order associated with short range forces in the liquid to relax. After this, all liquids behave like viscous fluids with signals propagating by diffusion rather than by waves. For liquids with small molecules this time of relaxation is estimated as 10^{-13} or 10^{-10} seconds depending on the fluids. Waves associated with such liquids move with speeds of $10^5 cm/s$, or even faster [1]. However, there are liquids, which are known to have much longer times of relaxation. Polymers mixed in Newtonian solvents, polymer melts like molten plastics or high viscosity silicone oils are examples. These fluids are known as viscoelastic fluids. The longest times of relaxation for these fluids are of practical interest; times we can read on clock, of the order of milliseconds to minutes. Such fluids have become important industrially. Specifically in polymer processing applications as well as in chemical industry, one deals with flow of viscoelastic fluids. Kaloni[2] investigated the fluctuating flow of a viscoelastic fluid past an infinite porous plate subject to inform suction. The steady flow of an incompressible second grade fluid past an infinite porous plate subject to suction or blowing was investigated by Rajagopal and Gupta[3]. This fluid shows normal stress differences in shear flow and akin to a viscoelastic fluid. To the best of our knowledge, heat transfer in the flow of a viscoelastic fluid past a porous surface subject to suction or blowing has not been studied in the literature, although this problem is important in polymer processing applications.

In this paper we investigate momentum and heat transfer in the steady flow of a viscoelastic fluid (obeying Walters' liquid B$'$ model[4]) past an infinite porous flat plate subject to suction or blowing. It is shown that steady solution for velocity distribution exists only for suction (and not blowing) at the plate. Steady temperature distribution in the boundary layer is found in the case when the porous plate subject to suction is held at constant temperature.

The motivation and the implication of this study is to explore the influence of suction or blowing on the control of separation as well as heat transfer in flow of viscoelastic fluids.

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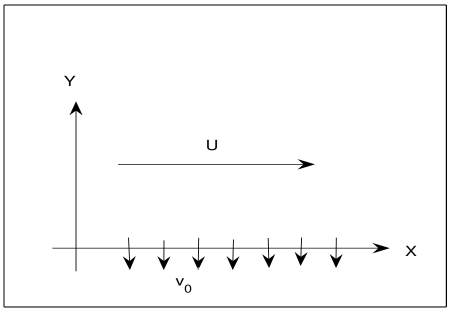

Fig. 1. A sketch of the physical problem.

II. Flow analysis

Consider the steady flow of an incompressible viscoelastic fluid past an infinite porous flat plate subject to uniform suction or blowing. The flow configuration is shown in Fig. 1. The constitutive equation for an incompressible viscoelastic fluid obeying Walters' liquid B$'$ model[4] is

$$S_{ik} = -p\, g_{ik} + S'_{ik} \quad (1)$$

$$S'_{ik} = 2\int_{-\infty}^{t} \Psi(t-t') \frac{\partial x^i}{\partial x'^m} . \frac{\partial x^k}{\partial x'^r} . e^{(1)mr}(x', t')dt', \quad (2)$$

where covariant suffixes are written below, contravariant suffixes above, and the usual summation convention for repeated suffixes is assumed. Further

$$\Psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau} e^{-(t-t')/\tau} d\tau \quad (3)$$

where $N(\tau)$ is the distribution function of relaxation times τ. In these equations, S_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} is the metric tensor of a convected coordinate system $x^i, x^{'i}\left(= x^{'i}(x,t,t^{'})\right)$ the position at the time $t^{'}$ of the element that is instantaneously at the point x^i at time t, and $e_{jk}^{(i)}$ is the rate-of-strain tensor.

In the present paper, we study the flow of liquid B$^{'}$ using boundary layer approximations. Since boundary layer flows are most likely to develop in viscoelastic fluid that are mobile and not highly elastic, we restrict the discussion to liquids with short memories (i.e. small relaxation times). The constitute equation (2) can then be written in the simplified form as

$$S^{'ik} = 2\mu e^{(1)ik} - 2k_0 \frac{D}{Dt} e^{(1)ik} \tag{4}$$

where $\mu\left(= \int_0^\infty N(\tau)d\tau\right)$ is the limiting viscosity at small rates of shear, $k_0 = \int_0^\infty \tau N(\tau)d\tau$, and terms involving $\int_0^\infty \tau^n N(\tau)d\tau$ $(n \geq 2)$ have been neglected. Further D/Dt denotes convected differentiation of a tensor quantity in relation to the material in motion as defined by Oldroyd [5]. For a contravariant tensor b^{ik},

$$\frac{Db^{ik}}{Dt} = \frac{\partial b^{ik}}{\partial t} + v^m \frac{\partial b^{ik}}{\partial x^m} - b^{im}\frac{\partial v^k}{\partial x^m} - b^{mk}\frac{\partial v^i}{\partial x^m}, \tag{5}$$

where v^j is the velocity vector. In (4), k_0 is the elastic constant of the fluid and is a measure of the relaxation time of the fluid.

We take x-axis along the plate, y-axis being normal to it. Since the plate is infinite, in the steady state the physical variables except pressure depend on y only. We seek a velocity field of the form

$$u = v^{(1)} = u(y), \;\; v = v^{(2)} = v(y), \;\; v^{(3)} = 0. \tag{6}$$

The equation of continuity then gives

$$v = -v_0, \tag{7}$$

where v_0 is the constant velocity at the plate with $v_0 > 0$ for suction and $v_0 < 0$ for blowing. Using (1), (4), (5), (6) and (7), the steady boundary layer equations for the flow are given by

$$-v_0\frac{du}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{k_0}{\rho}v_0\frac{d^3u}{dy^3} + \frac{\mu}{\rho}\frac{d^2u}{dy^2}, \tag{8}$$

$$\frac{\partial p}{\partial y} = O(\delta), \tag{9}$$

where p and ρ denote pressure and density, respectively and δ stands for the boundary layer thickness. In driving these equations, it is tacitly assumed that in addition to the usual boundary layer approximations, the viscoelastic and the inertial terms in the equations of motion are of the same order of magnitude. Hence it is necessary that both $\frac{\mu}{\rho}$ and $\frac{k_0}{\rho}$ are of $O(\delta^2)$.

The boundary conditions are

$$u(0) = 0, \;\; u \to U \;\; \text{as}\; y \to \infty. \tag{10}$$

Equation (9) shows that p is a function of x only since the boundary layer thickness is very small. Then equation (8) shows that $\frac{\partial p}{\partial x}$ is at most a constant. Since far away from the plate the free stream velocity is uniform, it follows from (8) that $\frac{\partial p}{\partial x} = 0$. Hence p is constant throughout the flow.

Introducing the dimensionless variables

$$\eta = \frac{\rho U y}{\mu}, \;\; \overline{u}(\eta) = \frac{u(y)}{U}, \tag{11}$$

we obtain from (8) with $p = constant$

$$k_1\frac{d^3\overline{u}}{d\eta^3} + \frac{1}{s}\frac{d^2\overline{u}}{d\eta^2} + \frac{d\overline{u}}{d\eta} = 0, \tag{12}$$

where the dimensionless elastic parameter k_1 and the dimensionless suction parameter s are given by

$$k_1 = \frac{\rho k_0 U^2}{\mu^2}, \quad s = \frac{v_0}{U}. \tag{13}$$

The boundary conditions follow from (10) as

$$\overline{u}(0) = 0, \;\; \overline{u} \to 1 \;\; \text{as}\; \eta \to \infty. \tag{14}$$

It is noticed that presence of elasticity in the fluid results in a third-order equation, whereas in the viscous case $(k_0 = 0)$, the order of the equation is two. It would thus appear that an additional boundary condition must be imposed to obtained a unique solution. However implicit in the derivation of (12) is the neglect of terms of order k_1^2 since k_1, which is measure (dimensionless) of relaxation time, is very small for a viscoelastic fluid with small memory. This, of course, means that the characteristic time scale of the fluid flow is large compared with the relaxation time of the fluid. Thus following, Ray Mahapatra and Gupta[6], we seek a solution of (12) of the form

$$\overline{u}(\eta) = u_0(\eta) + k_1u_1(\eta) + 0(k_1^2), \tag{15}$$

which is valid for sufficiently small k_1. Substituting (15) in (12) and equating different powers of k_1, we get

$$\frac{d^2u_0}{d\eta^2} + s\frac{du_0}{d\eta} = 0, \tag{16}$$

$$\frac{d^2u_1}{d\eta^2} + s\frac{du_1}{d\eta} = -s\frac{d^3u_0}{d\eta^3}. \tag{17}$$

The boundary conditions for $u_0(\eta)$ and $u_1(\eta)$ are obtained from (14) as

$$u_0(0) = 0, \;\; u_0 \to 1 \;\; \text{as} \;\; \eta \to \infty, \tag{18}$$

$$u_1(0) = 0, \;\; u_1 \to 0 \;\; \text{as} \;\; \eta \to \infty. \tag{19}$$

The solution of (16) satisfying (18) is

$$u_0(\eta) = 1 - e^{-s\eta}. \tag{20}$$

Substituting for $u_0(\eta)$ from (20) in (17) and solving the resulting equation using the boundary conditions (19), we obtain

$$u_1 = s^3\eta e^{-s\eta}. \tag{21}$$

Thus, to neglect $O(k_1^2)$, the velocity distribution is given by

$$\overline{u}(\eta) = 1 - e^{-s\eta} + k_1s^3\eta e^{-s\eta}. \tag{22}$$

It follows from (22) that for a given value of the suction s, the velocity at a point increases with increase in the elastic parameter k_1.

Fig. 2 shows the velocity distribution for several values of s with $k_1 = 0.01$. It is observed that at a given point, the velocity increases with increase in suction. It can be readily shown that no steady solution for velocity distribution exists when there is blowing at the plate. This follows from the fact that in the case of blowing, $s < 0$ and there is no solution of (16) satisfying the boundary conditions (18).

Shear stress at the plate is given by

$$(\tau_{xy})_{y=0} = \left(\mu \frac{du}{dy} + k_0 v_0 \frac{d^2 y}{dy^2}\right)_{y=0}. \quad (23)$$

Subnutrition from (11), (13) and (22) in (23) gives the dimensionless shear stress at the plate as

$$(\tau_{xy})_{y=0} / \rho U^2 = s. \quad (24)$$

This shows that wall shear stress depends only on suction and is also independent of the material of fluids.

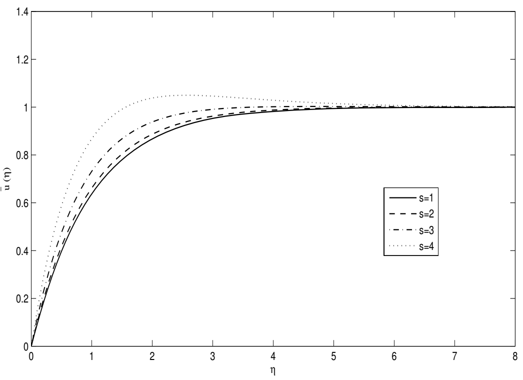

Fig. 2. Variation of velocity $\overline{u}(\eta)$ for several values of the suction parameter s.

III. Heat transfer

Let us now consider the heat transfer equation in the flow of a viscoelastic fluid to determine the temperature distribution in the flow. In this context it is necessary to establish the energy balance for a fluid element in motion and to consider it in conjunction with the equation of motion. It is to be noted that during the motion of a viscoelastic fluid, a certain amount of energy is stored up in the fluid as strain energy, while some energy is lost due to viscous dissipation. Thus for an incompressible viscoelastic fluid the energy balance is determined by the internal energy, the conduction of heat, the convection of heat flow with the flow, the generation of heat through viscous dissipation and the strain (or deformation) energy stored in the fluid due to its elastic properties.

The transfer of heat in the steady flow of the viscoelastic fluid past a porous plate subject to suction can be expressed in the form of the energy equation given by

$$-v_0 \frac{dT}{dy} = \frac{\lambda}{\rho c_p} \frac{d^2 T}{dy^2} + \frac{\mu}{\rho c_p} \left(\frac{du}{dy}\right)^2 + \frac{k_0 v_0}{\rho c_p} \frac{du}{dy} \frac{d^2 u}{dy^2}, \quad (25)$$

where T, λ and c_p denote the temperature, thermal conductivity and the specific heat of the fluid, respectively. Note that the second and third term on the right hand side of (25) denote the terms due to viscous dissipation and strain energy, respectively. The boundary conditions are

$$T = T_w \text{ at } y = 0, \;\; T \to T_\infty \text{ as } y \to \infty, \quad (26)$$

where T_w and T_∞ are constants with $T_w > T_\infty$.

Introducing the dimensionless temperature θ as

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad (27)$$

we find from (25) upon using (11) and (13)

$$\frac{d^2\theta}{d\eta^2} + sPr \frac{d\theta}{d\eta} = -EPr \left[\left(\frac{d\overline{u}}{d\eta}\right)^2 + k_1 s \frac{d\overline{u}}{d\eta} \frac{d^2 \overline{u}}{d\eta^2} \right], \quad (28)$$

where

$$Pr = \frac{\mu c_p}{\lambda}, \;\; E = \frac{U^2}{c_p (T_w - T_\infty)}. \quad (29)$$

It is clear from above that the temperature distribution $\theta(\eta)$ depends on four dimensionless parameters : (i) the elastic parameter k_1, (ii) the Prandtl number Pr, (iii) the Eckert number E (which characterizes viscous dissipation in the flow) and (iv) the suction parameter s. The boundary conditions for $\theta(\eta)$ are obtained from (26) as

$$\theta(0) = 1, \;\; \theta(\infty) = 0. \quad (30)$$

Substitution for $\overline{u}(\eta)$ from (22) in (28) and integration of the resulting equation subject to the boundary conditions (30) gives

$$\theta(\eta) = (1+M_1) e^{-sPr\eta} - (M_1 + M_2 \eta + M_3 \eta^2) e^{-2s\eta} \quad \text{for } Pr \neq 2, \quad (31)$$

and for $Pr = 2$,

$$\theta(\eta) = \left[1 + \tfrac{2EPr}{8s^3} \left\{(2s^2 A_1 + sB_1 + C_1)\eta\right\}\right] e^{-2s\eta} + \left[\tfrac{2EPr}{8s^3} \left\{(2s^2 B_1 + sC_1)\eta^2 + \tfrac{2}{3} s^2 C_1 \eta^3\right\}\right] e^{-2s\eta} \quad (32)$$

where

$$M_1 = -E \left[\tfrac{(2s^2 A_1 + sB_1 + C_1)}{4s^4} + \tfrac{(N_1^2 A_1 - N_1 B_1 + 2C_1)}{sN_1^3} \right],$$

$$M_2 = -E \left[\tfrac{(s^2 B_1 + sC_1)}{2s^4} + \tfrac{(N_1^2 B_1 - 2N_1 C_1)}{sN_1^3} \right],$$

$$M_3 = -E \left[\tfrac{C_1}{2s} + \tfrac{C_1}{sN_1} \right], \; N_1 = s(Pr - 2),$$

$$A_1 = s^2 \left(1 + k_1 s^2\right)\left(1 - 2k_1^2 s^4\right), \;\; B_1 = k_1 s^5 \left(3k_1^2 s^4 - 2\right)$$

$$C_1 = k_1^2 s^8 \left(1 - k_1 s^2\right).$$

Fig. 3 shows the temperature distribution for various values of the elastic parameter k_1 with $Pr = 5, E = 4$ and $s = 2$. It is observed that the temperature at a point decreases with increase in k_1. Fig. 4 displays the temperature distribution for various values of the Eckert number E with $Pr = 5, s = 2$, and $k_1 = 0.005$. As expected, temperature at a point increases with increasing E and overshoot in temperature $(T > T_w)$ occurs near the plate for $E > 2$. It is observed from Fig. 5 that for fixed values of Pr, E and k_1, temperature near the plate increases with increase in the suction parameter s but further away from the plate, temperature at a point decreases with increase in suction.

Finally we point out that no steady distribution of temperature exists when there is uniform blowing at the plate. In fact, Equation (28) governing the temperature distribution

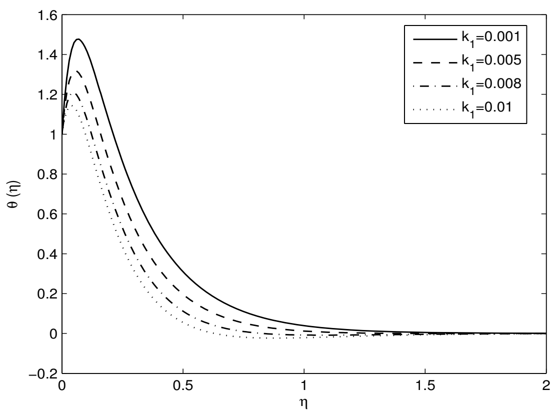

Fig. 3. Variation of θ for several values of k_1 with $s = 2$, $Pr = 10$ & $E = 4$.

does not have any solution satisfying the boundary conditions (30) in the case of blowing. This is plausible on physical grounds since the fluid at large distance from the plate gets continually heated due to convection of heat away from the plate by blowing.

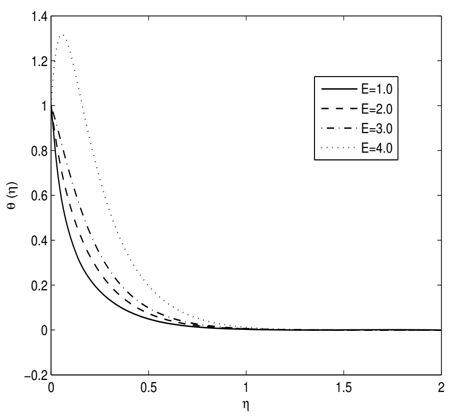

Fig. 4. Variation of θ for several values of E with $s = 2$, $Pr = 10$ & $k_1 = 0.005$.

IV. CONCLUSION

The analysis of flow of a viscoelastic fluid past an infinite porous plate shows that steady solution for velocity distribution exists only when there is suction at the plate. It is observed that the velocity at a point increases with increase in the elastic parameter k_1. Shear stress at the wall is found to decrease with increase in k_1. Solution of the heat transfer equation shows that steady temperature distribution exists for flow of a viscoelastic fluid past a porous plate only when there is suction (and not blowing) at the plate. It is observed that temperature at a point decreases with increase in the elastic parameter. It is further shown that temperature at a point increases with increase in the suction parameter s near the plate but further away from the plate, temperature at a point decreases with increase in s.

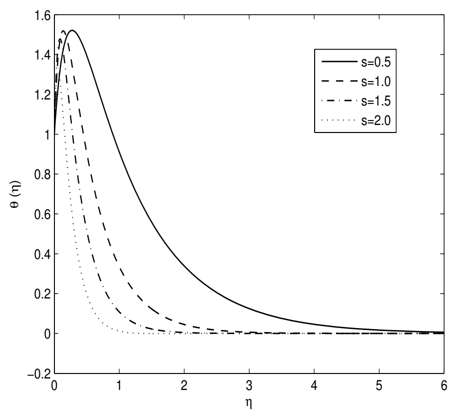

Fig. 5. Variation of θ for several values of s with $E = 4$, $Pr = 10$ & $k_1 = 0.005$.

In this context, we say a few words about the significance of these results. For viscoelastic fluids (e.g., polymers and their solutions), the viscous heating effect might be appreciable in all viscometers, including capillary tubes (Brodkey[7]). Since such effects are to be avoided as far as possible, one should know when they are liable to be important. From our foregoing analysis of temperature distribution, we find that for a viscoelastic fluid (small memory), temperature at a point decreases with increase in the elastic parameter. Thus elasticity of the fluid mitigates to a large extent the undesirable effects of viscous heating in not only viscometric flows but also in flows in other geometries such as the one we have studied here.

REFERENCES

[1] D. D. Joseph, Fluid dynamics of Viscoelastic liquids. *Springer-Verlag, New York*, 1990.
[2] P. N. Kaloni, "Fluctuating flow of an elastico-viscous fluid past a porous flat plate", *Physics of Fluids* **10**, pp. 1344–1346, 1966.
[3] K. R. Rajagopal and A. S. Gupta, "An exact solution for the flow of a non-Newtonian fluid past an infinite porous plate", *Meccanica* **19**, pp.158–160, 1984.
[4] D. W Beard and K. Walters, "Elastico-viscous boundary layer flows", *Proc. Cambridge Philos. Soc. Maths. Phys. Sci.* **60**, pp.667-674, , 1964.
[5] J. G. Oldroyd, "On the formulation of rheological equations of state", *Proc. Royal Soc. Londoon ser A* **200**, 523-541, 1950.
[6] T. Ray Mahapatra and , A. S. Gupta, "Stagnation-point flow of a viscoelastic fluid towards a stretching surface", *Int. Journal of Non-linear Mechanics* **39**, pp.811-820,2004.
[7] R. S. Brodkey, The Phenomena of Fluid Motions. *Dover Publications, INC, New York*, 1967.