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Numerical Study of a Class of Nonlinear Partial Differential Equations

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***Abstract*—**In this work, we derive two numerical schemes for solving a class of nonlinear partial differential equations. The first method is of second order accuracy in space and time directions, the scheme is unconditionally stable using Von Neumann stability analysis, the scheme produced a nonlinear block system where Newton's method is used to solve it. The second method is of fourth order accuracy in space and second order in time. The method is unconditionally stable and Newton's method is used to solve the nonlinear block system obtained. The exact single soliton solution and the conserved quantities are used to assess the accuracy and to show the robustness of the schemes. The interaction of two solitary waves for different parameters are also discussed.

***Keywords*—**Crank-Nicolson Scheme, Douglas Scheme, Partial Differential Equations

I. Introduction

In this work, we aim to solve numerically the class of nonlinear partial differential equations [3]

$$\left.\begin{aligned} & i\,\frac{\partial \psi_1}{\partial t} + p\left(\frac{\partial^2 \psi_1}{\partial x^2} + A_1 \frac{\partial^2 \psi_1}{\partial y^2}\right) + B_1 |\psi_1|^2 \psi_1 + C_1 \psi_1 \psi_2 = 0, \\ & A_2 \frac{\partial^2 \psi_2}{\partial t^2} + \left(\frac{\partial^2 \psi_2}{\partial x^2} - B_2 \frac{\partial^2 \psi_2}{\partial y^2}\right) + C_2 \left(|\psi_1|^2\right)_{xx} = 0, \\ & -\infty < x < \infty,\ -\infty < y < \infty,\ t \ge 0, \end{aligned}\right\} \quad (1)$$

where $\psi_1(x,y,t)$ is a complex valued function of the spatial coordinates x, y and the time t, $\psi_2(x,y,t)$ is a real valued function. And $p, A_j, B_j, C_j\ (j=1,2)$ are real constants which prove that: $p \neq 0, B_1 \neq 0, C_1 \neq 0, C_2 \neq 0$.

The exact solution of (1) is

$$\begin{aligned} \psi_1(x,y,t) &= e^{i\eta} f(\xi) \\ &= \pm e^{i\eta} \sqrt{\frac{(\alpha_1^2 + A_1\alpha_2^2)\,(\alpha_1^2 - B_2\alpha_2^2 + A_2\beta^2)}{C_1C_2\alpha_1^2 - B_1(\alpha_1^2 - B_2\alpha_2^2 + A_2\beta^2)} 2p}\ \tanh \xi, \end{aligned} \quad (2)$$

$$\begin{aligned} \psi_2(x,y,t) = g(\xi) &= \frac{C}{\alpha_1^2 - B_2\alpha_2^2 + A_2\beta^2} \\ &\quad - \frac{C_2\alpha_1^2\,(\alpha_1^2 + A_1\alpha_2^2)}{C_1C_2\alpha_1^2 - B_1(\alpha_1^2 - B_2\alpha_2^2 + A_2\beta^2)} 2p\ \tanh^2 \xi, \end{aligned} \quad (3)$$

where

$$\eta = k_1 x + k_2 y + \omega t + \eta_0,\ \xi = \alpha_1 x + \alpha_2 y + \beta t + \xi_0,$$

$$\omega = -p\left[2\left(\alpha_1^2 + A_1\alpha_2^2\right) + \left(k_1^2 + A_1 k_2^2\right)\right] + \frac{C_1 C}{\alpha_1^2 - B_2\alpha_2^2 + A_2\beta^2},$$

$$\beta = -2p\left(k_1\alpha_1 + A_1 k_2 \alpha_2\right),$$

C is an integration constant, $\alpha_1, \alpha_2, k_1, k_2, \omega, \beta, \xi_0, \eta_0$ are real constants.

The class of nonlinear partial differential equations (1) has the conserved quantities

$$\left.\begin{aligned} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_1|^2\, dx\, dy = constant, \\ & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_2|^2\, dx\, dy = constant, \end{aligned}\right\} \quad (4)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\left[\overline{\psi_1}\,\frac{\partial^2 \psi_1}{\partial x \partial y} - \psi_1 \frac{\partial^2 \overline{\psi_1}}{\partial x \partial y}\right] dx\, dy = constant. \quad (5)$$

To avoid complex computation, we assume

$$\left.\begin{aligned} & \psi_1(x,y,t) = u_1(x,y,t) + i\,u_2(x,y,t), \\ & \psi_2(x,y,t) = u_3(x,y,t), \\ & \frac{\partial u_3}{\partial t} = u_4(x,y,t), \end{aligned}\right\} \quad (6)$$

where $\left\{u_j(x,y,t)\right\}_{j=1}^{4}$ are real functions.

This will reduce (1) to the system

$$\left.\begin{aligned} & \frac{\partial u_1}{\partial t} + p\left[\frac{\partial^2 u_2}{\partial x^2} + A_1 \frac{\partial^2 u_2}{\partial y^2}\right] + u_2\left[B_1\left(u_1^2 + u_2^2\right) + C_1 u_3\right] = 0, \\ & \frac{\partial u_2}{\partial t} - p\left[\frac{\partial^2 u_1}{\partial x^2} + A_1 \frac{\partial^2 u_1}{\partial y^2}\right] - u_1\left[B_1\left(u_1^2 + u_2^2\right) + C_1 u_3\right] = 0, \\ & \frac{\partial u_3}{\partial t} - u_4 = 0, \\ & A_2 \frac{\partial u_4}{\partial t} + \left[\frac{\partial^2 u_3}{\partial x^2} - B_2 \frac{\partial^2 u_3}{\partial y^2}\right] + C_2\left(u_1^2 + u_2^2\right)_{xx} = 0. \end{aligned}\right\} \quad (7)$$

The paper is organized as follows: in Section 2, finite difference method is used to derive two numerical schemes. In section 3, numerical results for single soliton and the interaction of two solitons are given. The error and the conserved quantities are used to assess the efficiency of the proposed methods. Concluding remarks contained in Section 4.

II. Numerical Method

Consider the class of nonlinear partial differential equations (1) in a finite domain [4]-[5]-[6]

$$\left.\begin{aligned} & \frac{\partial u_1}{\partial t} + p\left[\frac{\partial^2 u_2}{\partial x^2} + A_1 \frac{\partial^2 u_2}{\partial y^2}\right] + u_2\left[B_1\left(u_1^2 + u_2^2\right) + C_1 u_3\right] = 0, \\ & \frac{\partial u_2}{\partial t} - p\left[\frac{\partial^2 u_1}{\partial x^2} + A_1 \frac{\partial^2 u_1}{\partial y^2}\right] - u_1\left[B_1\left(u_1^2 + u_2^2\right) + C_1 u_3\right] = 0, \\ & \frac{\partial u_3}{\partial t} - u_4 = 0, \\ & A_2 \frac{\partial u_4}{\partial t} + \left[\frac{\partial^2 u_3}{\partial x^2} - B_2 \frac{\partial^2 u_3}{\partial y^2}\right] + C_2\left(u_1^2 + u_2^2\right)_{xx} = 0. \end{aligned}\right\} \quad (8)$$

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In the region $R = [x_R \le x \le x_L,\ y_R \le y \le y_L] \times [t \ge 0]$ with the initial conditions

$$\left.\begin{aligned}\psi_1(x,y,0) &= g_1(x,y),\\ \psi_2(x,y,0) &= g_2(x,y),\end{aligned}\right\} \ -\infty < x < \infty, -\infty < y < \infty, \qquad (9)$$

and boundary conditions

$$\left.\begin{aligned}\frac{\partial \psi_1}{\partial x} &= \frac{\partial \psi_2}{\partial x} = 0 \quad at \quad x = x_L, x_R, \quad t \ge 0,\\ \frac{\partial \psi_1}{\partial y} &= \frac{\partial \psi_2}{\partial y} = 0 \quad at \quad y = y_L, y_R, \quad t \ge 0.\end{aligned}\right\} \qquad (10)$$

A. Crank-Nicolson Scheme

We will adopt in Crank-Nicolson type replacement, which is of second order accurate in time and it work well with longer time steps because of its stability properties. So the full discretization of (10) is

$$\begin{aligned}&A\left(U_{l,m}^{n+1} - U_{l,m}^{n}\right)\\ &+ r_1\left(U_{l+1,m}^{n+1} - 2U_{l,m}^{n+1} + U_{l-1,m}^{n+1} + U_{l+1,m}^{n} - 2U_{l,m}^{n} + U_{l-1,m}^{n}\right)\\ &+ r_2\left(U_{l,m+1}^{n+1} - 2U_{l,m}^{n+1} + U_{l,m-1}^{n+1} + U_{l,m+1}^{n} - 2U_{l,m}^{n} + U_{l,m-1}^{n}\right)\\ &+ kD\left(U_{l,m}^{*}\right) = 0,\end{aligned} \qquad (11)$$

for $l,m = 1,2,\ldots,N$, $n = 0,1,\ldots,NT$,

where $r_1 = \dfrac{kB}{2h^2}$, $r_2 = \dfrac{kC}{2h^2}$, $U_{l,m}^{*} = \dfrac{U_{l,m}^{n+1} + U_{l,m}^{n}}{2}$.

The scheme form a nonlinear block system can be solved by using Newton's method.

1. *Accuracy of the Scheme*

Truncation error which is given by

$$\begin{aligned}T_{l,m}^{n} = \Bigg[& A\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{k^2}{4}\frac{\partial^2}{\partial t^2}\left\{B\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} + D(u)\right\}\\ & + \frac{h^2}{12}\left\{B\frac{\partial^4 u}{\partial x^4} + C\frac{\partial^4 u}{\partial y^4}\right\} + \frac{kh^2}{24}\frac{\partial}{\partial t}\left\{B\frac{\partial^4 u}{\partial x^4} + C\frac{\partial^4 u}{\partial y^4}\right\} + \ldots\Bigg]_{l,m}^{n}.\end{aligned} \qquad (12)$$

This means the scheme in (12) is second order accuracy in space and time. And it is consistent, since the principal part of the truncation error will be vanish as $h,k \to 0$.

2. *Stability of the Scheme*

$$\left.\begin{aligned}U_{l,m}^{n} &= G^n e^{i\beta l h} e^{i\gamma m h},\\ \delta_x^2 U_{l,m}^{n} &= -4\left(\sin^2\frac{\beta h}{2}\right) G^n e^{i\beta l h} e^{i\gamma m h},\\ \delta_y^2 U_{l,m}^{n} &= -4\left(\sin^2\frac{\gamma h}{2}\right) G^n e^{i\beta l h} e^{i\gamma m h},\end{aligned}\right\} \qquad (13)$$

where $i = \sqrt{-1}$, $\beta,\gamma \in \Re$, $G \in \Re^{4\times 4}$, by substituting in (11) and after some manipulation we get

$$\left.\begin{aligned}\lambda_j &= \frac{1+i\xi}{1-i\xi} \Rightarrow |\lambda_j| = \sqrt{\frac{1+\xi^2}{1+\xi^2}} \Rightarrow |\lambda_j| = 1,\ j = 1,2,3.\\ \lambda_4 &= \frac{A_2 + i\xi}{A_2 - i\xi} \Rightarrow |\lambda_4| = \sqrt{\frac{A_2^2+\xi^2}{A_2^2+\xi^2}} \Rightarrow |\lambda_4| = 1.\end{aligned}\right\} \qquad (14)$$

This means that the suggested scheme is unconditionally stable according to Von Neumann stability analysis in the linearized sense, which means that no restriction on the grid sizes of h and k [1]-[2].

B. Douglas Scheme

Douglas scheme is fourth order in space and second order in time for the system in can be obtained

$$\begin{aligned}G\left(U_{l,m}^{*}\right) = k\Bigg[& \frac{25}{36}D\left(U_{l,m}^{*}\right) + \frac{5}{72}\Big\{D\left(U_{l+1,m}^{*}\right) + D\left(U_{l-1,m}^{*}\right)\\ & + D\left(U_{l,m+1}^{*}\right) + D\left(U_{l,m-1}^{*}\right)\Big\} + \frac{1}{144}\Big\{D\left(U_{l+1,m+1}^{*}\right)\\ & + D\left(U_{l+1,m-1}^{*}\right) + D\left(U_{l-1,m+1}^{*}\right) + D\left(U_{l-1,m-1}^{*}\right)\Big\}\Bigg],\end{aligned} \qquad (15)$$

for $l,m = 1,2,\ldots,N$, $n = 0,1,\ldots,NT$,

Where $r_1 = \dfrac{kB}{2h^2}$, $r_2 = \dfrac{kC}{2h^2}$, $U_{l,m}^{*} = \dfrac{U_{l,m}^{n+1} + U_{l,m}^{n}}{2}$.

The scheme form a nonlinear block system can be solved by using Newton's method.

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$$\begin{aligned}T_{l,m}^{n} = \Bigg[& A\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{k^2}{4}\left\{\frac{\partial^2}{\partial t^2}\left(B\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} + D(u)\right)\right\}\\ & + \frac{h^4}{144}\left\{A\frac{\partial}{\partial t}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial x^2 \partial y^2}\right) + \left(B\frac{\partial^6 u}{\partial x^4 \partial y^2} + C\frac{\partial^6 u}{\partial x^2 \partial y^4}\right)\right\}\\ & + \left(\frac{\partial^4 D(u)}{\partial x^4} + \frac{\partial^4 D(u)}{\partial x^2 \partial y^2} + \frac{\partial^4 D(u)}{\partial y^4}\right)\Bigg\} + \cdots\Bigg]_{l,m}^{n}.\end{aligned} \qquad (16)$$

This means the scheme in (16) is fourth order accuracy in space and second order time. And it is consistent, since the principal part of the truncation error will be vanish as $h,k \to 0$

2. Stability of the Scheme

By substituting in (15) and after some manipulation we get

$$\left.\begin{aligned}\lambda_j &= \frac{\eta - i\xi}{\eta + i\xi} \Rightarrow |\lambda_j| = \sqrt{\frac{\eta^2+\xi^2}{\eta^2+\xi^2}} \Rightarrow |\lambda_j| = 1,\ j = 1,2,3.\\ \lambda_4 &= \frac{A_2\eta - i\xi}{A_2\eta + i\xi} \Rightarrow |\lambda_4| = \sqrt{\frac{(A_2\eta)^2+\xi^2}{(A_2\eta)^2+\xi^2}} \Rightarrow |\lambda_4| = 1.\end{aligned}\right\} \qquad (17)$$

This means that the suggested scheme is unconditionally stable according to Von Neumann stability analysis in the linearized sense, which means that no restriction on the grid sizes of h and k.

III. Numerical Results

A. Single Soliton

In this test we choose the parameters [1]

$$\begin{gathered}A_1 = 1, B_1 = \frac{1}{2}, C_1 = -1, A_2 = 0, B_2 = -1,\\ C_2 = 1, \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{2}, k_1 = \frac{1}{2}, k_2 = \frac{1}{4},\\ \eta_0 = \frac{1}{4}, \xi_0 = \frac{1}{4}, p = \frac{1}{2}, C = 0, k = 0.01,\\ h = 0.1,\ tol = 10^{-6}, t = 0,1,\ldots,25,\\ -20 \le x \le 30, -20 \le y \le 30.\end{gathered}$$

The results of $L_\infty(\psi_1), L_\infty(\psi_2)$ and conserved quantities in the tow schemes are given in following Tables.

TABLE I
SINGLE SOLITON BY CRANK-NICOLSON SCHEME

t	$L_\infty(\psi_1)$	$L_\infty(\psi_2)$	$con\,1$	$con\,2$
0.00	0.000000	0.000000	1.843851	-0.657426
5.00	0.000263	0.000372	1.843851	-0.657425
10.00	0.001305	0.000417	1.843854	-0.657424
15.00	0.001861	0.000763	1.843816	-0.657425
20.00	0.001614	0.000542	1.843870	-0.657426

cons1= mass conservation, con2= momentum conservation

TABLE II
SINGLE SOLITON BY DOUGLAS SCHEME

t	$L_\infty(\psi_1)$	$L_\infty(\psi_2)$	$con\,1$	$con\,2$
0.00	0.000000	0.000000	1.843851	-0.657426
5.00	0.000109	0.000113	1.843851	-0.657425
10.00	0.000654	0.000127	1.843852	-0.657426
15.00	0.000671	0.000409	1.843843	-0.657426
20.00	0.000720	0.000198	1.843865	-0.657426

cons1= mass conservation, con2= momentum conservation

We notice that both schemes are given almost the same results regarding the conserved quantities but the error in Douglas scheme is high accurate than the error in Crank-Nicolson scheme.

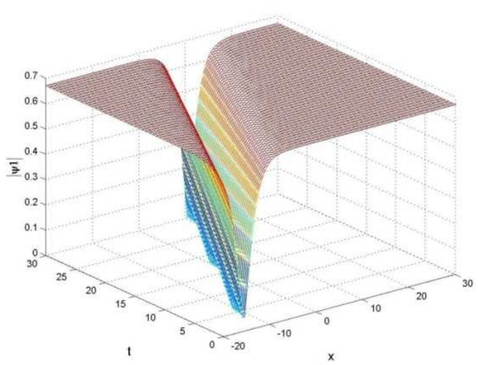

Fig. 1 (a) Single soliton with $k = 0.01, h = 0.1, \eta_0 = \xi_0 = \frac{1}{4}$.

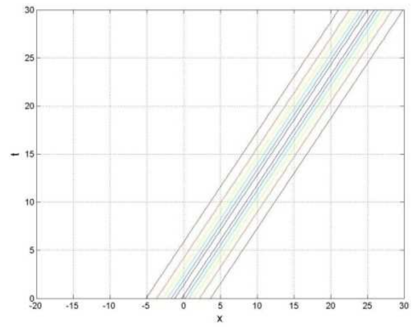

Fig. 1 (b) Single soliton with $k = 0.01, h = 0.1, \eta_0 = \xi_0 = \frac{1}{4}$.

B. Two Solitons Interaction

In this test we choose the parameters

$$A_1 = 1, B_1 = \frac{1}{2}, C_1 = -1, A_2 = 0, B_2 = -1,$$
$$C_2 = 1, \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{2}, k_1 = \frac{1}{2}, k_2 = \frac{1}{4},$$
$$\eta_{0_1} = \frac{1}{4}, \eta_{0_2} = \frac{2}{3}, \ \xi_{0_1} = \frac{1}{4}, \ \xi_{0_2} = \frac{2}{5}, p = \frac{1}{2},$$
$$C = 0, \ tol = 10^{-6}, \ k = 0.001, \ h = 0.025,$$
$$x_1 = 10, x_2 = 30, y_1 = 10, y_2 = 30, t = 0, \dots, 60.$$

TABLE III
TWO SOLUTIONS BY CRANK-NICOLSON SCHEME

t	$con\,1$	$con\,2$
0.00	4.378413	-1.236993
15.00	4.378502	-1.236851
25.00	4.378314	-1.236971
39.00	4.378252	-1.236879
43.00	4.378306	-1.236642

cons1= mass conservation, con2= momentum conservation

TABLE IV
TWO SOLUTIONS BY DOUGLAS SCHEME

t	$con\,1$	$con\,2$
0.00	4.378413	-1.236993
15.00	4.378422	-1.236904
25.00	4.378435	-1.236942
39.00	4.378400	-1.236750
43.00	4.378417	-1.236801

cons1= mass conservation, con2= momentum conservation

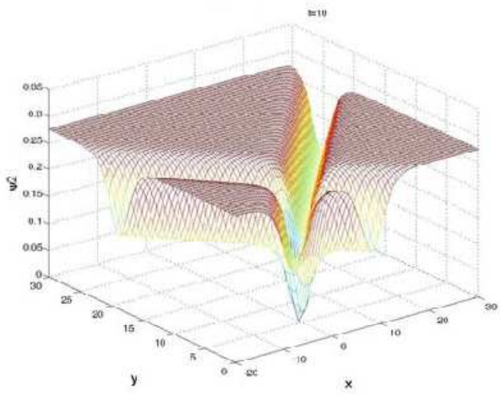

Fig. 2 (a) Interaction of two solitons with
$$k = 0.001, h = 0.025, \xi_{0_1} = \frac{1}{4}, \xi_{0_2} = \frac{2}{5}, \eta_{0_1} = \frac{1}{4}, \eta_{0_2} = \frac{2}{3}.$$

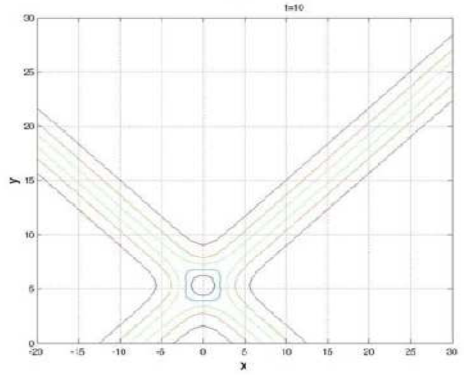

Fig. 2 (b) Interaction of two solitons with
$$k = 0.001, h = 0.025, \xi_{0_1} = \frac{1}{4}, \xi_{0_2} = \frac{2}{5}, \eta_{0_1} = \frac{1}{4}, \eta_{0_2} = \frac{2}{3}.$$

IV. Conclusion

In this work we have solved a class of nonlinear partial differential equations using two difference schemes. In Crank-Nicolson Scheme, we got a nonlinear block system where Newton's method is used to solve it. In Douglas scheme we present a nonlinear block system which can be solved by Newton's method. Single soliton and the interaction of two solitons are used to assess the performance of these methods. We show that both methods simulate the solution in a very nice way and keep the conserved quantities are almost constants. As a conclusion we can say Crank-Nicolson Scheme is faster than Douglas scheme.

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