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Some results on parallel alternating methods

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Abstract—In this paper, we investigate two parallel alternating methods for solving the system of linear equations $Ax = b$ and give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix. Furthermore, we give one example to show our results.

Keywords—nonsingular H-matrix, parallel alternating method, convergence.

I. Introduction

FOR the large system of linear equations

$$Ax = b, \tag{1}$$

where A is a nonsingular square matrix of order n, $x, b \in R^n$. Benzi and Szyld [1] analyzed the following alternating method:

Given an initial vector $x^{(0)}$, for $k = 0, 1, 2, \cdots$,

$$x^{(k+\frac{1}{2})} = M^{-1}Nx^{(k)} + M^{-1}b,$$
$$x^{(k+1)} = P^{-1}Qx^{(k+\frac{1}{2})} + P^{-1}b,$$

where $A = M - N = P - Q$ are two splittings of A. They proved its convergence under certain conditions when the coefficient matrix A is a monotone matrix or a symmetric positive definite matrix.

In paper [2], Climent and Perea introduced two parallel alternating iterative methods.

Assume that

$$A = M_l - N_l = P_l - Q_l, \quad l = 1, 2, \cdots, p, \tag{2}$$

where M_l and P_l nonsingular matrices; E_l satisfy $\sum_{l=1}^{p} E_l = I$ (I is an identity matrix), where E_l are diagonal and $E_l \geq 0$.

Method 1: Let $x^{(0)}$ be a starting vector, $\varepsilon > 0$ is a given precision. For $k = 1, 2, \cdots$,

$$x_l^{(k+\frac{1}{2})} = (M_l^{-1}N_l)^{\mu(k,l)}x^{(k)} + \sum_{i=0}^{\mu(k,l)-1} (M_l^{-1}N_l)^i M_l^{-1}b,$$

$$x_l^{(k+1)} = (P_l^{-1}Q_l)^{\nu(k,l)}x_l^{(k+\frac{1}{2})} + \sum_{i=0}^{\nu(k,l)-1} (P_l^{-1}Q_l)^i P_l^{-1}b,$$

$$x^{(k+1)} = \sum_{l=1}^{p} E_l x_l^{(k+1)},$$

If $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$, then quit.

It is easy to notice that the iterative matrix of Method 1 is

$$T = \sum_{l=1}^{p} E_l (P_l^{-1}Q_l)^{\nu(k,l)} (M_l^{-1}N_l)^{\mu(k,l)}.$$

Method 2: Let $x^{(0)}$ be a starting vector, $\varepsilon > 0$ is a given precision. For $k = 1, 2, \cdots$,

$$\begin{aligned} x^{(k+\frac{1}{2})} = & \sum_{l=1}^{p} E_l (M_l^{-1}N_l)^{\mu(k,l)} x^{(k)} \\ & + \sum_{l=1}^{p} E_l [\sum_{i=0}^{\mu(k,l)-1} (M_l^{-1}N_l)^i M_l^{-1}]b, \\ x^{(k+1)} = & \sum_{l=1}^{p} F_l (P_l^{-1}Q_l)^{\nu(k,l)} x_l^{(k+\frac{1}{2})} \\ & + \sum_{l=1}^{p} F_l [\sum_{i=0}^{\nu(k,l)-1} (P_l^{-1}Q_l)^i P_l^{-1}]b. \end{aligned}$$

If $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$, then quit.

It is easy to notice that the iterative matrix of Method 2 is

$$S = [\sum_{l=1}^{p} F_l (P_l^{-1}Q_l)^{\nu(k,l)}][\sum_{l=1}^{p} E_l (M_l^{-1}N_l)^{\mu(k,l)}].$$

In this paper, we give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix.

II. Preliminaries

Let $A \in R^{n\times n}$. We denote by $A \geq 0$ a nonnegative matrix, $|A|$ the absolute value of matrix A, and $\rho(A)$ the spectral radius of A.

Definition 2.1 Let $A = B - C$ be a splitting of A. If $B^{-1} \geq 0, B^{-1}C \geq 0$, then $A = B - C$ is a weak regular splitting[3]. If $B^{-1} \geq 0, C \geq 0$, then $A = B - C$ is a regular splitting[4]. If B is an M-matrix and $C \geq 0$, then $A = B - C$ is an M-splitting[5].

In paper [2], a weak regular splitting is also called a weak nonnegative splitting of the first type.

It's obvious that an M-splitting is a regular splitting and a regular splitting is a weak regular splitting.

Definition 2.2([6]) Let $A \in R^{n\times n}$. $A = M - N (M, N \in R^{n\times n})$ is called as an H-splitting if $< M > -|N|$ is an M-matrix. If $< A >=< M > -|N|$, then $A = M - N$ is called as an H-compatible splitting.

III. Convergence Theorems

In this section, we give convergence theorems for the parallel alternating methods when the coefficient matrix is a nonsingular H-matrix.

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Lemma 3.1[2] Let $A \in R^{n\times n}$ and $A^{-1} \geq 0$. If $A = M_l - N_l = P_l - Q_l$ $(l = 1, 2, \cdots, p)$ are all weak nonnegative splittings of the first type, then

$$\rho(T) < 1,$$

where

$$T = \sum_{l=1}^{p} E_l (P_l^{-1} Q_l)^{\nu(k,l)} (M_l^{-1} N_l)^{\mu(k,l)}.$$

Lemma 3.2[2] Let $A \in R^{n\times n}$ and $A^{-1} \geq 0$. If $A = M_l - N_l = P_l - Q_l (l = 1, 2, \cdots, p)$ are all weak nonnegative splittings of the first type, then

$$\rho(S) < 1,$$

where

$$S = [\sum_{l=1}^{p} F_l (P_l^{-1} Q_l)^{\nu(k,l)}][\sum_{l=1}^{p} E_l (M_l^{-1} N_l)^{\mu(k,l)}].$$

Lemma 3.3[7] If $A \in R^{n\times n}$is a nonsingular H-matrix, then $|A^{-1}| \leq < A >^{-1}$.

Theorem 3.1 Let $A \in R^{n\times n}$ be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l \ (l = 1, 2, \cdots, p)$$

are H-splittings while $B \in R^{n\times n}$ be a nonsingular M-matrix,

$$B =< M_l > -|N_l| =< P_l > -|Q_l| \ (l = 1, 2, \cdots, p),$$

then Method 1 converges to the unique solution of (**??**) for any starting vector $x^{(0)}$.

Proof: We will show that $\rho(T) < 1$.

It is obvious that $\rho(T) < 1$ if $\rho(|T|) < 1$. From

$$< M_l > -|N_l| =< P_l > -|Q_l| = B$$

is a nonsingular M-matrix. So $< M_l >, < P_l >$ $(l = 1, 2, \cdots, p)$ are all M-matrices, and

$$< M_l > -|N_l| =< P_l > -|Q_l| = B \ (l = 1, 2, \cdots, p)$$

are M-splittings of B. So M_l, P_l $(l = 1, 2, \cdots, p)$ are all H-matrices. Moreover,

$$|M_l^{-1}| \leq < M_l >^{-1}, |P_l^{-1}| \leq < P_l >^{-1}.$$

Thus we obtain

$$\begin{aligned} |T| &= |\sum_{l=1}^{p} E_l (P_l^{-1} Q_l)^{\nu(k,l)} (M_l^{-1} N_l)^{\mu(k,l)}| \\ &\leq \sum_{l=1}^{p} E_l (< P_l >^{-1} |Q_l|)^{\nu(k,l)} (< M_l >^{-1} |N_l|)^{\mu(k,l)} \\ &= \bar{T}. \end{aligned}$$

We use Lemma 3.1 to see immediately that $\rho(\bar{T}) < 1$. Therefore, $\rho(|T|) < 1$ and $\rho(T) < 1$, we obtain the conclusion of this theorem.

If $B =< A >$ in Theorem 3.1, then we can obtain the following corollary.

Corollary 3.1 Let $A \in R^{n\times n}$ be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l (l = 1, 2, \cdots, p)$$

are H-compatible splittings, then Method 1 converges to the unique solution of (**??**) for any starting vector $x^{(0)}$.

Theorem 3.2 Let $A \in R^{n\times n}$ be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l \ (l = 1, 2, \cdots, p)$$

are H-splittings while $B \in R^{n\times n}$ be a nonsingular M-matrix,

$$B =< M_l > -|N_l| =< P_l > -|Q_l| \ (l = 1, 2, \cdots, p),$$

then Method 2 converges to the unique solution of (**??**) for any starting vector $x^{(0)}$.

Proof: We will show that $\rho(S) < 1$.

It is obvious that $\rho(S) < 1$ if $\rho(|S|) < 1$. From

$$< M_l > -|N_l| =< P_l > -|Q_l| = B$$

is a nonsingular M-matrix. So $< M_l >, < P_l >$ $(l = 1, 2, \cdots, p)$ are all M-matrices, and

$$< M_l > -|N_l| =< P_l > -|Q_l| = B \ (l = 1, 2, \cdots, p)$$

are M-splittings of B. So M_l, P_l $(l = 1, 2, \cdots, p)$ are all H-matrices. Moreover,

$$|M_l^{-1}| \leq < M_l >^{-1}, |P_l^{-1}| \leq < P_l >^{-1}.$$

Thus we obtain

$$\begin{aligned} |S| &= |[\sum_{l=1}^{p} F_l (P_l^{-1} Q_l)^{\nu(k,l)}][\sum_{l=1}^{p} E_l (M_l^{-1} N_l)^{\mu(k,l)}]| \\ &\leq \sum_{l=1}^{p} F_l (< P_l >^{-1} |Q_l|)^{\nu(k,l)} \sum_{l=1}^{p} E_l (< M_l >^{-1} |N_l|)^{\mu(k,l)} \\ &= \bar{S}. \end{aligned}$$

We use Lemma 3.2 to see immediately that $\rho(\bar{S}) < 1$. Therefore, $\rho(|S|) < 1$ and $\rho(S) < 1$, we obtain the conclusion of this theorem.

If $B =< A >$ in Theorem 3.2, then we can obtain the following corollary.

Corollary 3.2 Let $A \in R^{n\times n}$ be a nonsingular H-matrix,

$$A = M_l - N_l = P_l - Q_l \ (l = 1, 2, \cdots, p)$$

are H-compatible splittings, then Method 2 converges to the unique solution of (**??**) for any starting vector $x^{(0)}$.

Example

$$A = \begin{bmatrix} 10 & 3 & 8 \\ 4 & 10 & 0 \\ 7 & 3 & 12 \end{bmatrix}$$

is a nonsingular H-matrix. Let

$$M_1 = \begin{bmatrix} 10 & 3 & 5 \\ 5 & 10 & 0 \\ 6 & 5 & 12 \end{bmatrix} \quad M_2 = \begin{bmatrix} 10 & 3 & 6 \\ 5 & 10 & 0 \\ 6 & 5 & 12 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 10 & 3 & 4 \\ 5 & 10 & 0 \\ 6 & 5 & 12 \end{bmatrix} \quad P_2 = \begin{bmatrix} 10 & 3 & 4 \\ 5 & 10 & 0 \\ 5 & 5 & 12 \end{bmatrix}$$

$N_l = M_l - A$, $Q_l = P_l - A(l = 1, 2), \mu(k, l) = \nu(k, l) = 2.$

It's easy to test that

$$< M_1 > -|N_1| =< M_2 > -|N_2| =< P_1 > -|Q_1|$$

$$=< P_2 > -|Q_2| = \begin{bmatrix} 10 & -3 & -8 \\ -6 & 10 & 0 \\ -7 & -7 & 12 \end{bmatrix}$$

is a nonsingular M-matrix, but

$$\begin{bmatrix} 10 & -3 & -8 \\ -6 & 10 & 0 \\ -7 & -7 & 12 \end{bmatrix} \neq < A >,$$

so

$$A = M_l - N_l = P_l - Q_l \ (l = 1, 2)$$

are H-splittings.

Case 1: We choose

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$T = \begin{bmatrix} \frac{48}{7033} & -\frac{89}{5135} & -\frac{237}{10805} \\ -\frac{31}{12518} & \frac{123}{21529} & \frac{40}{9861} \\ -\frac{52}{17127} & \frac{212}{15969} & \frac{150}{9149} \end{bmatrix}$$

$$\rho(T) = \frac{103}{3963} < 1.$$

Case 2: We choose $E_1 = I/3$, $E_2 = 2I/3$, $F_1 = 3I/4$, $F_2 = I/4$, $l = 1, 2$, then

$$S = \begin{bmatrix} \frac{191}{31989} & -\frac{119}{7022} & -\frac{231}{12751} \\ -\frac{64}{28429} & \frac{19}{3855} & \frac{25}{6782} \\ -\frac{30}{11587} & \frac{75}{6682} & \frac{133}{8366} \end{bmatrix}$$

$$\rho(S) = \frac{163}{6917} < 1.$$

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