World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:6, No:8, 2012

Adaptive Fourier Decomposition Based Signal Instantaneous Frequency Computation Approach

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Abstract—There have been different approaches to compute the analytic instantaneous frequency with a variety of background reasoning and applicability in practice, as well as restrictions. This paper presents an adaptive Fourier decomposition and (α-counting) based instantaneous frequency computation approach. The adaptive Fourier decomposition is a recently proposed new signal decomposition approach. The instantaneous frequency can be computed through the so called mono-components decomposed by it. Due to the fast energy convergency, the highest frequency of the signal will be discarded by the adaptive Fourier decomposition, which represents the noise of the signal in most of the situation. A new instantaneous frequency definition for a large class of so-called simple waves is also proposed in this paper. Simple wave contains a wide range of signals for which the concept instantaneous frequency has a perfect physical sense. The α-counting instantaneous frequency can be used to compute the highest frequency for a signal. Combination of these two approaches one can obtain the IFs of the whole signal. An experiment is demonstrated the computation procedure with promising results.

Keywords—Adaptive Fourier decomposition, Fourier series, signal processing, instantaneous frequency

I. Introduction

THE the time-frequency literature, frequency is defined as the derivative of the phase of the signal. The physical meaning of the frequency at a time moment t is the averaging vibrating times of a certain vibration during the 2π-length interval. It is a non-negative quantity varying along with time. Based on this idea people found the Fourier type, including window Fourier transforms; the wavelet type; and other types of frequencies. Some of these mentioned types of frequencies cannot give the time-varying amplitude frequency representation of signals. A sophisticated but delicate way of defining time-varying amplitude-phase representation and therefore instantaneous frequency is through an application of Hilbert transformation. Due to connection of Hilbert transformation and complex analytic functions, such defined instantaneous frequency (IF) is called *analytic IF* [8], [9].

In Fourier analysis, each exponential has a precise frequency. The frequencies and the spectra of those frequencies can be easily calculated. This provides a broad overview of the characteristics of the signal, which is important for theoretical considerations. However, Fourier analysis suffers from the absence of time localization, especially the time-varying frequencies, of the given signal. In this situation, IF is interpreted as the average frequency of the component signal at each time. To make the representation more local, one started with window Fourier transforms. Lately wavelet transforms were studied, where frequencies correspond with the dilation parameters [14]. However, these mentioned types of frequencies cannot give the time-varying frequency representations or instantaneous characteristics of signals. Another restriction of the Fourier decomposition is that the decomposition may lead to slow convergence due to the fact that the principal sine and cosine components contributing significant shares of the total energy of the original signal may arrive late.

A novel signal decomposition approach – Adaptive Fourier Decomposition (AFD) is proposed recently [11], [12] with proven mathematical foundations. The AFD is designed to treat the above mentioned two restrictions of Fourier transform. The decomposing components which make up the signal are adaptively selected under the principle of fast convergence in energy (Maximal Projection Principle). Such decompositions, therefore, can represent certain characteristic properties of the given signal.

The mathematical foundation of AFD is built up based on two important concepts. One is analytic function, the other is mono-component. Analytic function is a well known concept with good properties [4], [7]. The *mono-component* is defined as follows [9], [10]. A function (or signal), $s(t)$, no matter being complex- or real-valued, is said to be a mono-component, if $s(t)+iHs(t)$ is the boundary value of a function in the Hardy space H^2, where H is the Hilbert transform of s; and, furthermore, with the amplitude-phase representation $s(t)+iHs(t)=\rho(t)e^{i\theta(t)}$ there holds $\theta^{'}(t)\geq 0$. The *analytic phase derivative* $\theta^{'}(t)$ is called the *IF* if and only if the requirement $\theta^{'}(t)\geq 0$ is met, or, equivalently, s is a mono-component.

The Hilbert transformation involved in the definition is a crucial subject [5], [13]: a function $s=u+iv$, where u, v are real-valued with certain integrability property, is the boundary value of an analytic function in the corresponding Hardy space if and only if $v=Hu$ [11].

Fourier transform is well accepted because it has mathematical roots in analytic functional theory, representing complicated signals into the physically realizable basic signals of meaningful IFs. Physically realizable signals are identical with those being boundary values of analytic functions; while meaningful IF amounts to say that in their polar coordinate representations their phase derivatives are non-negative functions. Different from Fourier transform which decomposes a given signal into fixed sin and cosine functions, AFD decomposes a given signal into mono-components, which are boundary values of analytic functions. Sin and cosine functions are particular examples of mono-components. In this situation, Fourier transform is a special case of mono-component decompositions. There is a large pool of mono-components. To different signals, the

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decomposed mono-components by AFD are not fixed ones like Fourier decomposition. The decomposing mono-components are selected based on the given signal. That is the reason that the decomposition is said to be *adaptive*.

A qualified IF should satisfy the following three conditions. The first is the time varying property; the second is non-negativity; and the third is that it should reflect the vibrating frequency. The last requirement amounts to require that the definition of the IF should give rise to n if it is applied to $\cos nt$. The analytic signal method to define (analytic) IF satisfies the three required conditions with the reservation that not every signal has analytic IF [3], [8], [10]. In this paper we also define for a large class of signals called simple waves (SWs) the α-Counting Instantaneous Frequency (α-CIF). This definition is straight forward and easy to apply. The α-CIF satisfies the three mentioned conditions. For each mono-component decomposed by AFD, we can obtain its IF. In some situation, a signal cannot be fully approximated by the reconstruction of AFD due to fast convergency, the α-CIF is used to compute the IF of the residue. Some experiment results are illustrated in this paper. The results show that the IF of any given signal can be computed through AFD and α-CIF combination.

This paper is organized as follow. The mathematical foundation of the AFD based IF computation algorithm and α-CIF computation algorithm principles is introduced in section II. AFD based IF computation algorithm is presented in section III. The experiment results are shown in section IV. The conclusions are drawn in section V.

II. Mathematical Foundation

A. Mathematical Foundation Of The AFD Based IF Computation

Let $G(z)$ be a function in H^2, or, equivalently, an analytic signal of finite energy. Assume that B_n are of the form

$$B_n(z) = \frac{\sqrt{1-\mid a_n \mid^2}}{1-\overline{a_n}z}\prod_{p=1}^{n-1}\frac{z-a_p}{1-\bar{a}_p z}, \qquad n = 1, 2, \ldots, \quad (1)$$

where a_n are complex numbers inside the unit circle, $a_n = x+iy$, which are the parameters to be adaptively chosen in the algorithm. The system $\{B_n\}$ is called a Takenaka-Malmquist (TM) system or a rational orthogonal system [1], [2]. Since the parameters a_n will be adaptively chosen, the study of the TM system does not fall in the traditional one [11].

Note that in the expression

$$B_n(z) = O_n I_n, \quad (2)$$

where

$$O_n(z) = \frac{\sqrt{1-\mid a_n \mid^2}}{1-\overline{a_n}z},$$

and

$$I_n = \prod_{p=1}^{n-1}\frac{z-a_p}{1-\bar{a}_p z},$$

the part I_n as a Blaschke product, is always a mono-component [10]. But the part O_n can be in some cases, a pre-mono-component. A complex-valued signal is called a *pre-mono-component* if there exists a positive number M such that $e^{iMt}s(t)$ is a mono-component. In the signal processing language, it means that riding on a carrier frequency e^{iMt}, or after a phase modulation by e^{iMt}, $M > 0$, the signal becomes a mono-component. Obviously, every mono-component is a pre-mono-component [10]. Therefore, their product, i.e. B_n, is always a pre-mono-component, and sometimes a mono-component.

It can be easily verified that if $a_{n_0} = 0$, then all B_n, $n \geq n_0$, are mono-components. The Maximal Projection (matching pursuit) Principle is as follows. Denote

$$e_{\{a\}} = B_1(z) = \frac{\sqrt{1-\mid a \mid^2}}{1-\bar{a}z}.$$

the reproducing kernel of the Hardy space. Then for any $G \in H^2$, there exists a_1 in the open unit disc $\mathbf{D}$ such that

$$\left|< G(z), e_{\{a_1\}} >\right| = max\{\left|< G, e_{\{a\}} >\right| : a \in \mathbf{D}\} \quad (3)$$

The AFD decomposition stands for the expansion

$$G(z) = \sum_{p=1}^{\infty} c_p B_p(z), \quad (4)$$

where c_p are the coefficients

$$c_p =< G, B_p >= \int_0^{2\pi} G(e^{it})\overline{B_p(e^{it})})dt \quad (5)$$

where the parameters $a_1, \ldots, a_p$ are consecutively determined, by means of the Maximal Projection Principles [11].

If compulsorily selecting all $a_n = 0$, then (4) becomes the Fourier series decomposition. Thus AFD is an improvement of the Fourier decomposition. The essence of the algorithm is to select the parameters according to the given signal to be decomposed. The Maximal Projection Principle is in spirit of the so called, and developed, *matching pursuit* or *greedy algorithm* [6]. But it is a realizable variation of the latter [11].

In practice the analytic signal to be decomposed is given by a set of discrete data on the boundary. Denote

$$G(e^{it_k}) = x_k, k = 1, 2, \cdots, M, \quad (6)$$

where

$$t_k = 2\pi\Delta k, \quad \Delta = \frac{1}{M}, \quad t_k \in [0, 2\pi]. \quad (7)$$

First we need calculate the energy of the given signal in terms of the data. Denote the energy of signal G by $\|G\|^2$, that is given by the norm of G in the Hilbert space. The discretization of the integral formula for $\|G\|^2$ is

$$\|G\|^2 \approx 2\pi\Delta\sum_{k=1}^{M}|u_k|^2. \quad (8)$$

Through discretization of the integral formula (4), we have

$$|c_1|^2 = \left|\Delta\sqrt{2\pi(1-|a_1|)^2}\sum_{k=1}^{M}u_k\frac{1}{1-a_1 e^{-i2\pi\Delta k}}\right|^2. \quad (9)$$

According to the Maximal Projection Principle, an a_1 exists in $\mathbf{D}$ that gives rise to the maximal value of $|c_1|^2$. To find a_1

maximizing $|c_1|^2$ is to solve a global extreme problem of a differentiable function in the unit disc. This can be done by the standard method in calculus. It can be done by matlab. Denote such $|c_1|^2$ by $\max|c_1|^2$, which is the energy of the first decomposition component G_1, where

$$G_1 = c_1 B_1 \tag{10}$$

$$B_1 = \frac{\sqrt{1-|a_1|^2}}{\sqrt{2\pi}} \frac{1}{1-\bar{a}_1 e^{it}}, \tag{11}$$

and $c_1 = \langle G, B_1 \rangle$.

Comparing the energy between the given signal and the output signal, $\|G\|^2 - \max|c_1|^2$, we can judge how close it is from the first partial sum to the original given signal. Then repeat this process.

We have the following general relations.

$$|c_p|^2 = |\Delta\sqrt{2\pi(1-|a_p|^2)}\sum_{k=1}^{M}\frac{u_k}{1-a_p e^{-i2\pi\Delta k}} \prod_{q=1}^{p-1}\frac{1-\bar{a}_q e^{e^{i2\pi\Delta k}}}{e^{i2\pi\Delta k}-a_q}| \tag{12}$$

$$G_n = \sum_{p=1}^{n} c_p B_p, \tag{13}$$

where

$$c_p = \langle G, B_p \rangle. \tag{14}$$

The energy difference is calculated by the equation

$$\|G\|^2 - \sum_{p=1}^{n} \max|c_p|^2. \tag{15}$$

ADF for non-analytic, and in particular, real-valued signals, is based on the relation $s = 2Re\{s^+\} - c_0$, where s^+ is the $\frac{1}{2}$ multiple of the analytic signal associated with viz. $s^+ = \frac{1}{2}(s + iHs)$. The algorithm is subject to some changes. Now the data (7) is given for the non-analytic signal s, but the formula (8) that is suitable for data of $G = s^+$ now is not suitable. The energy formula should be data of s not s^+. We first deduce [10]

$$\|G\|^2 \approx \pi\Delta\sum_{k=1}^{M}|u_k|^2 + \pi\Delta\left|\sum_{k=1}^{M}u_k\right|^2. \tag{16}$$

In terms of the original data u_k the formula (14), and therefore (13), will remain to be valid for computing $\langle G, B\rangle$, for the inner product automatically eliminates the role of $s^- = s - s^+$. That is, owing to the orthogonality property between the inner and outer Hardy spaces,

$$\begin{aligned} < S, B_n > &= < S^+, B_n > + < S^-, B_n > \\ &= < S^+, B_n > \qquad (17) \\ &= < G, B_n > . \qquad (18) \end{aligned}$$

Once we have the decomposition for s^+ we have that for s.

An alternative algorithm strategy is to reduce the inner product (14) between the given analytic function G, or the Hardy space projection $G = s^+$ of the given non-analytic signal s, and the entries B_k to one between some recursively induced analytic signals g_k and the evaluator $e_{\{a_k\}}$. The two strategies have their respective merits of which we refer to the fundamental literature [10].

The IF could be computed by the following formula.

$$(\varphi_n)'(t) = \sum_{k=1}^{n-1}\frac{1-|a_l|^2}{1-2|a_l|xos(t-\theta_l)+|a_l|^2}. \tag{19}$$

where $a_l = |a_l|e^{i\theta_l}, l = 1, 2, ..., n$.

B. Mathematical Foundation Of The α-CIF Computation Algorithm

Definition II.1. *We call a function on a finite or an infinite interval a simple wave (SW) if it is a continuous function with finitely many strict local extrema.*

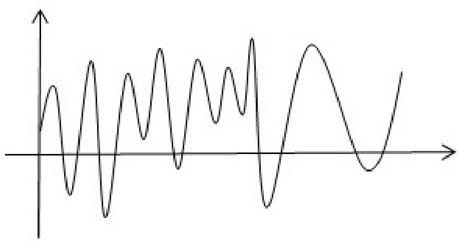

Fig. 1. The example of SW

The example of SW is shown in Fig. 1. We say that f has a *strict local maximum* at t_0 if

$$f(t_0 + \Delta t) < f(t_0)$$

for all sufficiently small $\Delta t > 0$. *Strict local minimum* is defined similarly. By saying that f is *monotone* in an interval we mean

$$f(t+\Delta t) \geq f(t) \quad \text{for all} \quad t \quad \text{and} \quad \Delta t > 0$$

whenever both t and $t + \Delta t$ are in the interval; or

$$f(t+\Delta t) \leq f(t) \quad \text{for all} \quad t \quad \text{and} \quad \Delta t > 0$$

whenever both t and $t+\Delta t$ are in the interval. The above two cases are said to be, respectively, *monotonously increasing* or, *monotonously decreasing*, or, in brief, *increasing* or *decreasing*.

For any $\alpha > 0$, the quantity

$$\begin{aligned} \frac{1}{2}[&\max\{2k : t-\alpha \leq t_{2k} \leq t+\alpha\} \\ &-\min\{2k : t-\alpha \leq t_{2k} \leq t+\alpha\}] \end{aligned} \tag{20}$$

represents the (integer) number of the vibrations that f has in the interval $[t-\alpha, t+\alpha]$. Then the average of the above quantity over the interval of length 2α multiplied by 2π

represents the number of vibrations over the interval of length 2π. The last quantity is given by

$$\frac{\pi}{2\alpha}[max\{2k : t-\alpha \leq t_{2k} \leq t+\alpha\} \\ -\min\{2k : t-\alpha \leq t_{2k} \leq t+\alpha\}] \quad (21)$$

that is defined the α-counting frequency of f at the moment t. Note that it is technical to choose $\alpha > 0$ for practical problems. If α is chosen too large, then what we are about average of the frequency, that dose not the time-varying property; and if α is chosen too small, say, smaller than the time needed to have a full vibration, then we get $\theta'_\alpha(t) = 0$.

Assumption the defined α-CIF satisfies the first two of the required three conditions for IF, viz. the time-varying and non-negativity conditions. Now we show that it also satisfies the third condition. Simple computation shows that α-CIF $\theta'_\alpha(t) = n$ for all t in $cosnt$.

III. AFD Based IF Computation Algorithm

Based on the described principles in section II, the AFD based IF computation method is proposed. As the convergency of AFD is based on the energy. For fast convergency, we only use the first fewer mono-components to reconstruct the original signal. The most high frequency residue is discarded. In most of the situation, the discarded part is usually the noise of the signal. If the user really would like to compute the IF of the residue, α-CIF is used as a supplementary for the AFD. The flowchart of the AFD based IF computation algorithm is shown in Algorithm 1. In the next section, we will give the experimental results based on this procedure.

Algorithm 1 AFD Based IF Computation Method

Input: Original signal S.
Output: IFs.

1: Decompose S with AFD.
2: The number of the iterations is N.
3: **for** $i = 1 : N$ **do**
4:     Implementing the $i-$th decomposition;
5:     Obtaining $i-$th mono-component B_i;
6:     Getting the IF for each mono-component IF_i.
7: **end for**
8: Reconstructing the signal with reverse AFD;
9: Obtaining the reconstructed signal S_r;
10: Obtaining the residue R of the signal through $R = S - S_r$;
11: Computing the α-CIF IF_α for the signal R;
12: The IFs of the signal are $IF_i + IF_\alpha$.

IV. Experimental Results and Analysis

In this section, we demonstrate the procedure and effectiveness of the proposed method on signal IF computation. The experiment data is selected from closing price of Hong Kong's Hang Seng Index for the period from 2nd January, 2008 to 9th March, 2012. There were total 1056 stock market trading days during the selected period. The original signal is shown in Fig. 2. The closing price of Hong Kong's Hang Seng Index is selected as the experiment data due to the higher vibration of the signal.

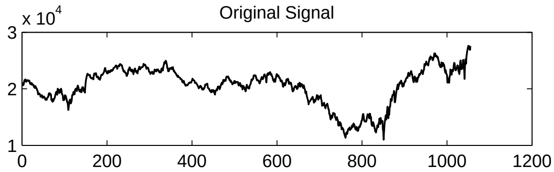

Fig. 2. The original signal

A. IF Computation Through AFD Decomposed Mono-components

If we choose the energy difference between the original signal and the reconstructed signal as 0.0005, 21 mono-components are obtained through AFD decomposition. The reconstruction signal is illustrated in Fig. 3. Some selected decomposing mono-components are shown in Fig. 4. The corresponding IFs of the mono-components are illustrated in Fig. 5. All IFs of 21 decomposed mono-components are illustrated together in Fig. 6. The spectrum of the signal through AFD is shown in Fig. 7.

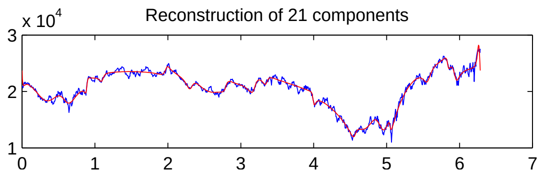

Fig. 3. The reconstruction signal

B. IF Computation Through α-CIF Algorithm

The residue of AFD decomposition can be obtained by subtracting the reconstructed signal from the original signal. It is illustrated in Fig. 8. From the reconstructed signal in Fig. 3, we can see that the 21 decompositions can approximate the original signal quite well. In most of the situation, the residue is the noise of the signal and could be discarded. In some specific situation, if people really want to investigate the IF of the residue, α-CIF algorithm can be used to compute it. After applying the α-CIF algorithm, the IF of residue is obtained as seen in Fig. 9.

C. The IFs of The Original Signal

Put together the IFs obtained through AFD and IF through α-CIF Algorithm to obtain the IFs of the original signal. It is illustrated in Fig. 10. As the IF of the residue is highest frequency in the original image, the magnitude of the IF is much more greater than the IFs computed through AFD, which normally represent the comparatively lower frequencies. It is hardly to see them all in the same graph due to the large difference between the highest frequency and lower ones.

V. Conclusion

In this paper, two different types of IFs computation approaches are presented. The defined AFD IF and α-CIF satisfy the required three conditions for IF. The IFs computed through

AFD represent the normal lower frequencies, in which in most of the applications it could be enough for use. The IF computed through α-CIF represents the highest frequency of the signal. Combination of the two types of the IFs, one can obtain the whole IFs of the original signal. The effectiveness of the proposed method is demonstrated by a practical signal with high vibrations. Please note that the α-CIF is sensitive to the signal noise. Denoising algorithm can be applied first to smooth the signal in order to make the method more effective.

ACKNOWLEDGMENT

This work was supported by University of Macau research grant MYRG144(Y1-L2)-FST11-ZLM.

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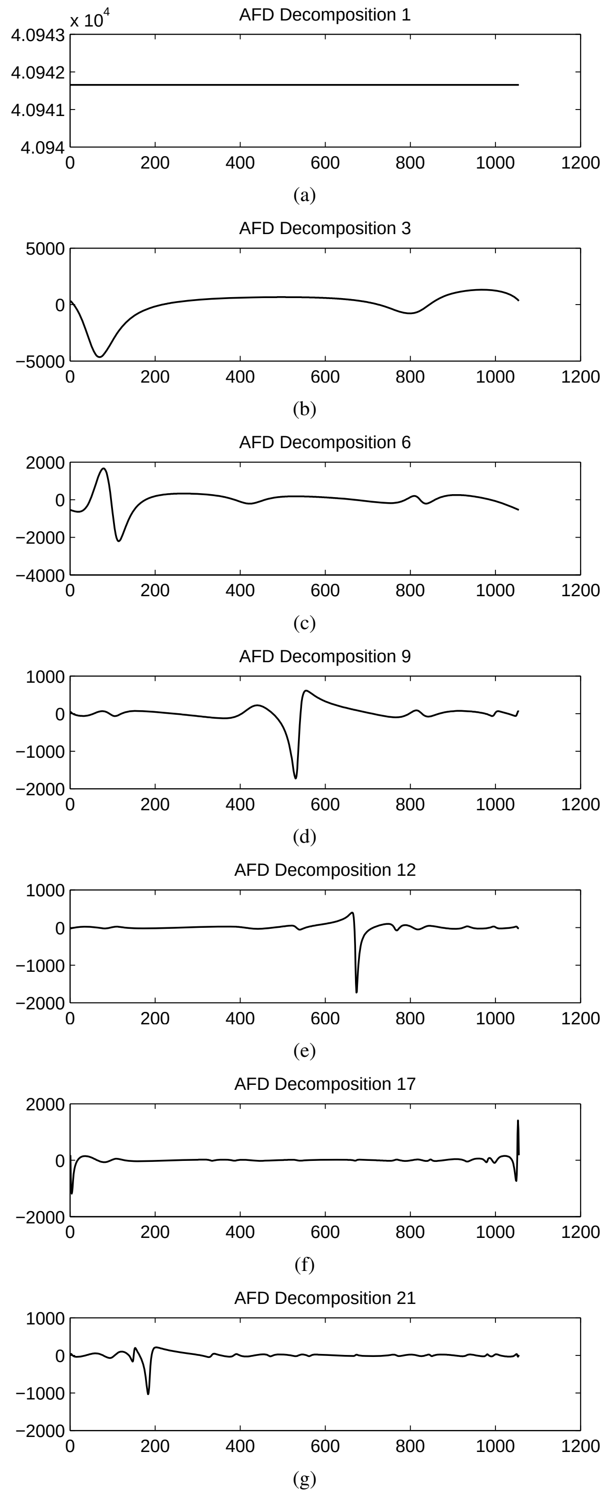
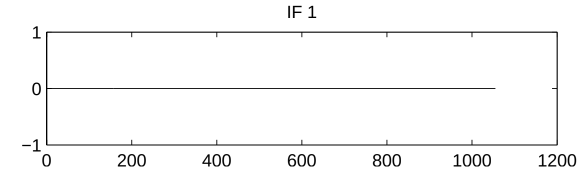
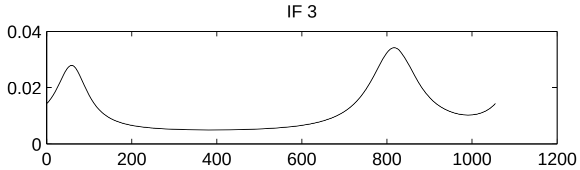


Fig. 4. (a) The 1st mono-component (b) The 3rd mono-component (c) The 6th mono-component (d) The 9th mono-component (e) The 12th mono-component (f) The 17th mono-component (g) The 21th mono-component



(a)



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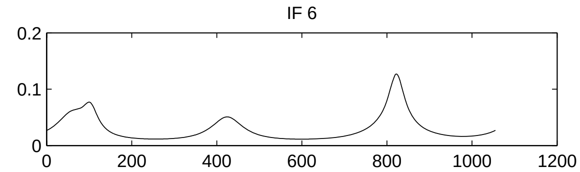

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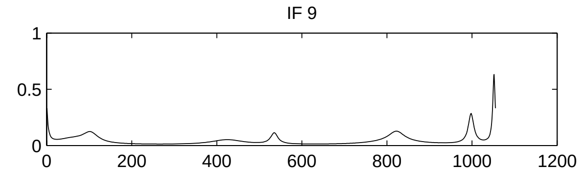

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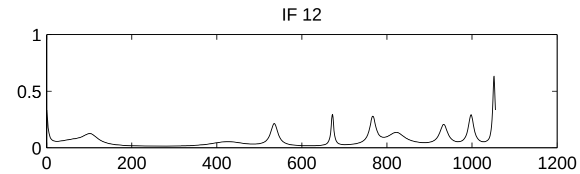
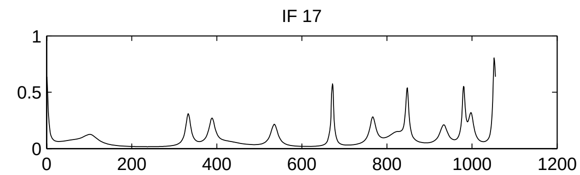
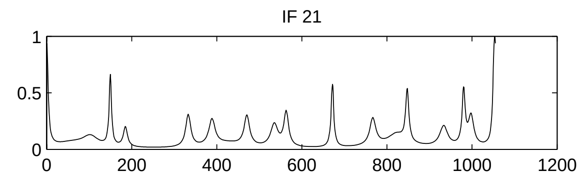


(e)



(f)



(g)

Fig. 5. (a) The IF of 1st mono-component (b) The IF of 3rd mono-component (c) The IF of 6th mono-component (d) The IF of 9th mono-component (e) The IF of 12th mono-component (f) The IF of 17th mono-component (g) The IF of 21th mono-component

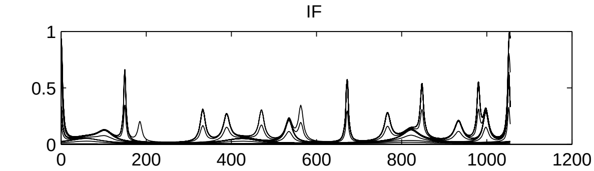


Fig. 6. The IFs computed through AFD

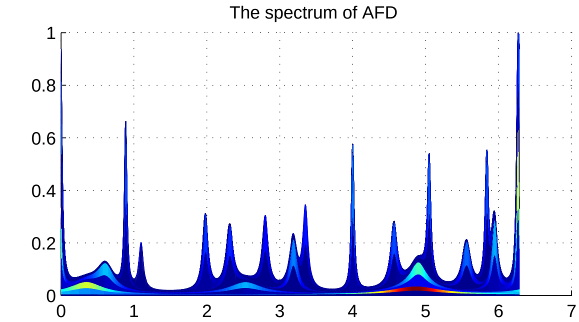


Fig. 7. The spectrum of the signal through AFD

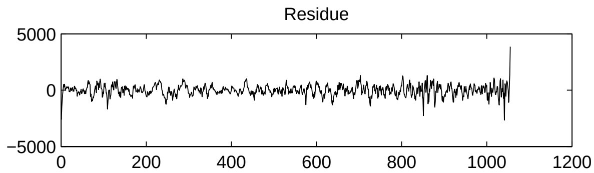


Fig. 8. The residue signal

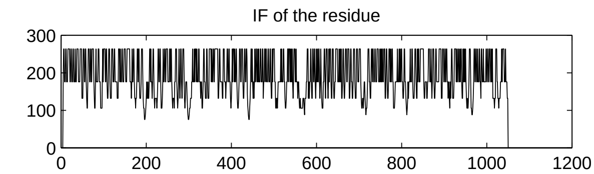


Fig. 9. The IF of the residue signal

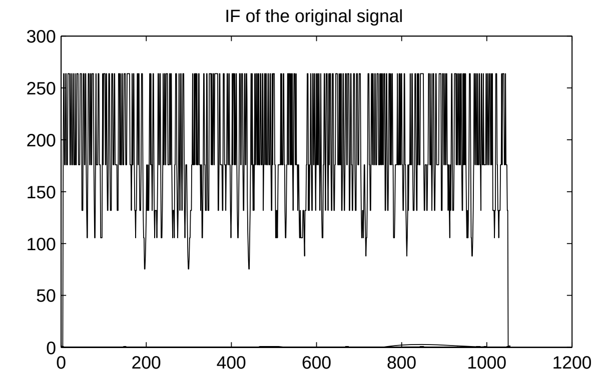


Fig. 10. The IFs of the original signal