World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:7, No:4, 2013

Exact Pfaffian and N-Soliton Solutions to a (3+1)-Dimensional Generalized Integrable Nonlinear Partial Differential Equations

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***Abstract*—The objective of this paper is to use the Pfaffian technique to construct different classes of exact Pfaffian solutions and N-soliton solutions to some of the generalized integrable nonlinear partial differential equations in (3+1) dimensions. In this paper, I will show that the Pfaffian solutions to the nonlinear PDEs are nothing but Pfaffian identities. Solitons are among the most beneficial solutions for science and technology, from ocean waves to transmission of information through optical fibers or energy transport along protein molecules. The existence of multi-solitons, especially three-soliton solutions, is essential for information technology: it makes possible undisturbed simultaneous propagation of many pulses in both directions.

***Keywords*—Bilinear operator, G-BKP equation, Integrable nonlinear PDEs, Jimbo-Miwa equation, Ma-Fan equation, N-soliton solutions, Pfaffian solutions.

I. Introduction

THE aim of this work is to use the Pfaffian technique, along with the Hirota bilinear method to construct different classes of exact solutions to various of generalized integrable nonlinear partial differential equations. The analysis of traveling wave solutions to integrable nonlinear partial differential equations plays a pivotal role in the study of nonlinear physical phenomena. Solitons [1], [2], [3], are among the most beneficial wave solutions for science and technology.

The derivation and solutions of integrable nonlinear partial differential equations in two spatial dimensions have been the holy Grail in the field of nonlinear science since the late 1970s. The prestigious Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations, as well as the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations are prototypical examples of integrable nonlinear partial differential equations in (1+1) and (2+1) dimensions respectively. However, one question remains: Do there exist Pfaffian and soliton solutions to generalized integrable nonlinear partial differential equations in (3+1) dimensions?

Generally, it is a difficult task to find exact solutions of nonlinear partial differential equations. Moreover, even if one manages to find a strategy for solving one particular nonlinear partial differential equation, in general, such a strategy may not be applicable to other nonlinear partial differential equations.

In this paper, I obtained a set of explicit exact Pfaffian and N-soliton solutions to the (3+1)-dimensional generalized integrable nonlinear partial differential equations, including a generalized B-type KP equation, soliton equations of Jimbo-Miwa type and the nonlinear Ma-Fan equation, A set of sufficient conditions consisting of systems of linear partial differential equations involving free parameters and continuous functions is generated to guarantee that the Pfaffian solves these generalized equations.

Examples of the Pfaffian and N-soliton solutions are explicitly computed. The numerical simulations of the obtained solutions are illustrated and plotted for different parameters involved in the solutions (see [1]).

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Manuscript received March 29, 2013; revised Apirl 5, 2013.

A. Bilinear forms

The Leibniz rule for normal derivatives is given by

$$\begin{aligned}&\frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n}\alpha(x,t)\beta(x,t)\\ =\ &\frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n}\alpha(x+y,t+s)\beta(x+y,t+s)\mid_{s=0,\ y=0}.\end{aligned}$$

Similarly, the usual Hirota derivatives (or D-operators) are defined by [2]:

$$\begin{aligned}&D_t^m D_x^n\alpha(x,t)\cdot\beta(x,t)\\ =\ &\frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n}\alpha(x+y,t+s)\beta(x-y,t-s)\mid_{s=0,\ y=0},\end{aligned}$$

or equivelently, by

$$\begin{aligned}&D_t^m D_x^n\alpha(t,x)\cdot\beta(t,x)\\ =\ &(\partial_t-\partial_{t'})^m(\partial_x-\partial_{x'})^n\alpha(t,x)\beta(t',x')\mid_{t'=t,\ x'=x}.\end{aligned}$$

Writing out the last equation for the case of one variable, we can obtain Hirota derivatives:

$$D_x^n\alpha(x)\cdot\beta(x)=\frac{\partial^n}{\partial y^n}\alpha(x+y)\beta(x-y)\mid_{y=0}.$$

Further, we get a nice property of D-operators that normal derivatives don't have:

$$D_x^n\varphi\cdot\varphi=0\quad\text{for } n \text{ is odd.}$$

The following properties are easily seen from the definition:

- $$D_\xi^m\varphi\cdot 1=\frac{\partial^m}{\partial\xi^m}\varphi.$$
- $$D_\xi^m\varphi\cdot\psi=(-1)^m D_\xi^m\psi\cdot\varphi.$$

- $$D_\xi^m \varphi \cdot \varphi = 0, \text{ for odd.} m.$$
- $$D_\xi^m \varphi \cdot \psi = D_\xi^{m-1}(\varphi_\xi \psi - \varphi \psi_\xi).$$
- $$D_\xi^m \varphi \cdot \varphi = 2D_\xi^{m-1}(\varphi_\xi \cdot \varphi), \text{ for odd.} m.$$
- $$D_\xi D_\eta \varphi \cdot \varphi = 2D_\xi(\varphi_\eta \cdot \varphi) = 2D_\eta(\varphi_\xi \cdot \varphi).$$
- $$D_\xi(\varphi\psi \cdot \omega) = \frac{\partial \varphi}{\partial \xi}\psi\omega + \varphi D_\xi(\psi \cdot \omega).$$
- $$\exp(\epsilon D_\xi)(\varphi \cdot \psi) = \varphi(x+\epsilon)\psi(x-\epsilon).$$

The relation between the normal derivative and the Hirota D-operators can be found in Ref. [2], here we list some of them:

$$2\frac{\partial^2}{\partial x^2}\log\varphi = \frac{D_x^2\varphi\cdot\varphi}{\varphi^2}, \tag{1}$$

$$2\frac{\partial^2}{\partial x \partial t}\log\varphi = \frac{D_x D_t\varphi\cdot\varphi}{\varphi^2}, \tag{2}$$

$$2\frac{\partial^4}{\partial x^4}\log\varphi = \frac{D_x^4\varphi\cdot\varphi}{\varphi^2} - 3\left(\frac{D_x^2\varphi\cdot\varphi}{\varphi^2}\right)^2. \tag{3}$$

We would like to express integrable nonlinear partial differential equations in the Hirota Bilinear form as

$$P(D)\varphi\cdot\varphi = 0, \tag{4}$$

where $P(D)$ is a polynomial in D, φ is a new dependent variable for the bilinear form of the given partial differential equation. So, the first step of our work is to develop appropriate transformations of nonlinear partial differential and difference equations which require that the differential equations are in quadratic form in dependent variables. For example, let us consider the KdV equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{5}$$

Introducing the following dependent variable transformation:

$$u = 2(\log\varphi)_{xx}, \tag{6}$$

Eq. (5) becomes

$$2(\log\varphi)_{xt} + 3\left[2(\log\varphi)_{xx}\right]^2 + 2(\log\varphi)_{4x} = 0. \tag{7}$$

The second step in our work is to use Hirota D-operators to find the bilinear form of the considered equation by a polynomial of Hirota D-operator. For example, we proceed to bilinearise the KdV Eq. (5). Using Eqs.(1), (2) and (3) in Eq. (7), the KdV Eq. (5) becomes

$$\frac{D_x D_t\varphi\cdot\varphi}{\varphi^2} + 3\left(\frac{D_x^2\varphi\cdot\varphi}{\varphi^2}\right)^2 + \frac{D_x^4\varphi\cdot\varphi}{\varphi^2} - 3\left(\frac{D_x^2\varphi\cdot\varphi}{\varphi^2}\right)^2 = 0. \tag{8}$$

Cancelling the second and fourth terms, this is simplified to

$$(D_x D_t + D_x^4)\varphi\cdot\varphi = 0, \tag{9}$$

or equivalently

$$D_x(D_t + D_x^3)\varphi\cdot\varphi = 0. \tag{10}$$

Eq. (10) is the Hirota bilinear form of the KdV equation.

B. Pfaffians

In this work, we will use the Pfaffian identities to search for exact solutions of the generalized B-type Kadomtsev–Petviashvili equation. In what follows, we will introduce three useful lemmas about the Pfaffian expansions and derivatives formulation. Let us recall some basics about the Pfaffian. Let $\Delta = \det(\delta_{i,j})_{1\le i,j\le 2N}$ be the determinant of an $2N \times 2N$ skew-symmetric matrix, then the Pfaffian associated with Δ is denoted conventionally by [9], [4]:

$$\begin{aligned}\text{Pf }(\delta_{i,j})_{1\le i,j\le 2N} &= (\alpha_1, \alpha_2, ..., \alpha_{2N}) \\ &= \left|\begin{array}{cccc} \delta_{1,2} & \delta_{1,3} & \cdots & \delta_{1,2N} \\ & \delta_{2,3} & \cdots & \delta_{2,2N} \\ & & \ddots & \vdots \\ & & & \delta_{2N-1,2N} \end{array}\right|.\end{aligned}$$

When $N = 1, 2$, the Pfaffian read

$$\begin{aligned}(\alpha_1, \alpha_2) &= \delta_{1,2}, \\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \delta_{1,2}\delta_{3,4} - \delta_{1,3}\delta_{2,4} + \delta_{1,4}\delta_{2,3}.\end{aligned}$$

Lemma 1: Let A be the determinant of an $m \times m$ skew-symmetric matrix,

$$A = \det(a_{i,j}) \quad (1 \le i, j \le m = 2n), \tag{11}$$

then the Pfaffian of order n, can be obtained from the above determinant, and is denoted

$$\text{Pf }(a_{i,j})_{1\le i,j\le 2n} = (1, 2, ..., 2n), \tag{12}$$

and the right-hand side of (12), can be expanded as

$$(1, 2, ..., 2n) = \sum\nolimits_\sigma sgn(\sigma) \prod_{i=1}^{n} (\sigma(2i-1), \sigma(2i)), \tag{13}$$

where the summation is taken over all permutations

$$\sigma = \begin{pmatrix} 1 & 2 & ... & 2n \\ i_1 & i_2 & ... & i_{2n} \end{pmatrix}$$

with

$$i_1 < i_2,\ i_3 < i_4,\ ...,\ i_{2n-1} < i_{2n},\ i_1 < i_3 < \ ... < i_{2n-1},$$

and $sgn(\sigma) = (-1)^{inv(\sigma)}$.

We have several expansion theorems on Pfaffian. Here we describe two of them which are relevant to the present paper.

Lemma 2: Let n be a positive integer, then

$$\begin{aligned}&(a_1, a_2, 1, 2, ..., 2n) \\ =\ & \sum_{j=2}^{2n}(-1)^j (a_1, a_2, 1, j) \left[(2, 3, ..., \hat{j}, ...2n)\right. \\ & \left. + (1, j)(a_1, a_2, 2, 3, ..., \hat{j}, ...2n)\right] \\ & - (a_1, a_2)(a_1, a_2, 1, 2, ..., 2n),\end{aligned} \tag{14}$$

and

$$\begin{aligned}&(b_1, b_2, c_1, c_2, 1, 2, ..., 2n)\\ =& \sum_{j=1}^{2n}\sum_{k=j+1}^{2n}(-1)^{j+k-1}(b_1, b_2, j, k)\\ &\times(c_1, c_2, 1, 2, ..., \hat{j}, ..., \hat{k}, ...2n),\end{aligned} \tag{15}$$

provided that

$$(b_j, c_k) = 0 \text{ , for } j, k = 1, 2.$$

Proof: see [11]. ■

We shall use the equation (14) and the equation (15) to express the derivatives of the Pfaffain by the Pfaffians of lower order. Identities for determinants and Pfaffians are of great interest in many branches of mathematics. In the next lemma we present two Pfaffian identities which correspond to the Jacobi's determinant identity.

Lemma 3: Let m and n be positive integers, then

$$\begin{aligned}&(a_1, a_2, ..., a_{2m}, 1, 2, ..., 2n)(1, 2, ..., 2n)\\ =& \sum_{s=2}^{2m}(-1)^s(a_1, a_s, 1, 2, ..., 2n)\times\\ &(a_1, a_2, ..., \hat{a}_s, ..., a_{2m}, 1, 2, ..., 2n),\end{aligned} \tag{16}$$

and

$$\begin{aligned}&(a_1, a_2, ..., a_{2m-1}, 1, 2, 3, ..., 2n-1)(1, 2, ..., 2n)\\ =& \sum_{s=1}^{2m-1}(-1)^{s-1}(a_s, 1, ..., 2n-1)\times\\ &(a_1, a_2, ..., \hat{a}_s, ..., a_{2m-1}, 1, ..., 2n).\end{aligned} \tag{17}$$

Proof: see [11]. ■

We shall use the equation (16) with $m = 2$ to get the desired identity.

II. Pfaffian solutions

A. The (3+1)-D BKP Eq

It is known that one of the most interesting problems in the Sato description of the KP hierarchy or its extensions, is that of describing lower-dimensional integrable systems whose solutions form a subset of the solution space for the KP hierarchy [8].

The process by means of which such systems are singled out is known as a reduction of the KP hierarchy, or of one of its extensions. The spectrum of such reductions ranges from (2+1)-dimensional systems, through (1+1)-dimensional ones, all the way to integrable ordinary differential equations. Interesting integrable systems arise at many different stages throughout the reduction process.

The hierarchy of (2+1)-dimensional systems whose solutions only make up a subset of the KP solution space, but which still appear as an integrable hierarchy in its own right.

The BKP Hierarchy. A well-known example of such a subhierarchy is the so-called BKP hierarchy [7]. Its name derives from the fact that, whereas $\mathfrak{gl}(\infty)$ can be identified with the infinite-rank Kac-Moody algebra A_∞, the Lie algebra that underlies the BKP hierarchy is of B-type, (B_∞) [10].

In this work, we investigate a generalized B-type KP equation [1]:

$$\frac{\partial^2 u}{\partial y\partial t} - \frac{\partial^4 u}{\partial y\partial x^3} - 3\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + 3\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial z^2} = 0,$$

which can be written in terms of the Hirota bilinear operator.

When $z = x$, the above equation possesses the same nonlinearity as the Sawada-Kotera equation:

$$\frac{\partial u}{\partial t} + 15\frac{\partial}{\partial x}(u^3 + u\frac{\partial^2 u}{\partial x^2}) + \frac{\partial^5 u}{\partial x^5} = 0,$$

and the model equation for shallow water waves:

$$\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t\partial x^2} - 3u\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x}\int_x^\infty \frac{\partial u}{\partial t}dx' + \frac{\partial u}{\partial x} = 0.$$

In fact, the Sawada-Kotera equation and the model equation for shallow water waves belong to a class of the B-type KP equations. Moreover, the B-type KP hierarchy is obtained from the standard KP hierarchy by imposing an extra condition between the Lax operator and its adjoint. A well known standard reduction of this hierarchy is the Sawada-Kotera equation.

We consider the following (3+1)-dimensional non-linear equation [1]:

$$\frac{\partial^2 u}{\partial y\partial t} - \frac{\partial^4 u}{\partial y\partial x^3} - 3\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + 3\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial z^2} = 0. \tag{18}$$

Under the dependent variable transformation:

$$u = 2\frac{\partial}{\partial x}(\ln\tau), \tag{19}$$

the above equation (18) is mapped into the Hirota bilinear equation:

$$\left(D_t D_y - D_x^3 D_y + 3D_x^2 + 3D_z^2\right)\tau\cdot\tau = 0. \tag{20}$$

We can rewrite the equation (20) in terms of τ as follows

$$\begin{aligned}&\left(\frac{\partial^2\tau}{\partial y\partial t} - \frac{\partial^4\tau}{\partial y\partial x^3} + 3\frac{\partial^2\tau}{\partial x^2} + 3\frac{\partial^2\tau}{\partial z^2}\right)\tau\\ &-\frac{\partial\tau}{\partial t}\frac{\partial\tau}{\partial y} + \frac{\partial^3\tau}{\partial x^3}\frac{\partial\tau}{\partial y} + 3\frac{\partial^3\tau}{\partial y\partial x^2}\frac{\partial\tau}{\partial x}\\ &-3\frac{\partial^2\tau}{\partial x^2}\frac{\partial^2\tau}{\partial y\partial x} - 3\left(\frac{\partial\tau}{\partial x}\right)^2 - 3\left(\frac{\partial\tau}{\partial z}\right)^2 = 0.\end{aligned} \tag{21}$$

We would like to present a sufficient conditions which guarantees that the Pfaffian solves the equation (21).

1) Sufficient conditions: We can introduce now the following Pfaffian:

$$\tau_n = \text{Pf}(a_{ij})_{1\le i,j\le 2n}, \tag{22}$$

$$a_{ij} = C_{ij} + \int_{-\infty}^{x} D_x\ f_i(x)\cdot f_j(x)dx, \quad i, j = 1, 2, ..., 2n, \tag{23}$$

where ($C_{ij} = -C_{ji}$ for $i \neq j$) are constants and all f_i satisfy the linear differential equations:

$$\frac{\partial f_i}{\partial y} = \lambda_1 \frac{\partial f_i}{\partial x_{-1}},\ \frac{\partial f_i}{\partial z} = \lambda_2 \frac{\partial f_i}{\partial x},\ \frac{\partial f_i}{\partial t} = \frac{\partial^3 f_i}{\partial x^3}, \qquad (24)$$

where

$$\lambda_1 = a^2 + 1, \text{ and } \lambda_2 = a, \qquad (25)$$

where a is free parameter and f_i has the boundary condition $f_i(-\infty) = 0$ for $i = 1, 2, ..., 2n$ and $\frac{\partial f_i}{\partial x_{-1}}$ is defined by

$$\frac{\partial f_i}{\partial x_{-1}} =: \int_{-\infty}^{x} f_i(x)dx. \qquad (26)$$

In what follows, as an application of the Pfaffian techniques, we shall construct a class of exact Pfaffian solutions to the (3+1)-dimensional generalized B-type Kadomtsev–Petviashvili equation.

2) Pfaffian solutions:

Theorem 4: (Sufficient condition) Let $f_i,\ i = 1, 2, ..., 2n$, satisfy (24), then the Pfaffian defined by (22) solves the Hirota bilinear equation (20) and the function $u = 2\ln(\tau_n)_x$ solves the (3+1)-dimensional generalized B-type Kadomtsev–Petviashvili equation (18).

Proof: See Appendix A. ■

3) N-soliton solutions: The system (24) has solution in the form

$$\begin{aligned} f_i &= \sum_{j=1}^{p} d_{ij}\exp(\xi_{ij}), && (27)\\ \xi_{ij} &= k_{ij}x + \lambda_1 k_{ij}^{-1} y + \lambda_2 k_{ij} z + k_{ij} t + \xi_{ij}^0, && (28)\end{aligned}$$

where $d_{ij},\ k_{ij}$, and ξ_{ij}^0 are free parameters and p is arbitrary natural number. In particular we have the following specific solutions

$$\begin{aligned} f_i &= \exp(\xi_i), && (29)\\ \xi_i &= k_i x + \lambda_1 k_i^{-1} y + \lambda_2 k_i z + k_i t + \xi_i^0, && (30)\end{aligned}$$

where k_i and ξ_i^0 are free parameters and $\lambda_1,\ \lambda_2$ are given in the equation (25). In order to investigate the solutions of (20), we choose special values for $(C_{ij})_{n\times n}$ and the functions f_i. For example, let

$$\begin{aligned} f_i &= \exp(\xi_i), && (31)\\ \xi_i &= k_i x + \lambda_1 k_i^{-1} y + \lambda_2 k_i z + k_i t + \xi_i^0, && (32)\end{aligned}$$

we obtain

$$(i, j) = C_{ij} + \frac{k_i - k_j}{k_i + k_j} f_i f_j. \qquad (33)$$

Let us consider the two-soliton and three-soliton expression of the equation (20). For the two-soliton solution we may choose $C_{12} = C_{34} = 1,\ C_{13} = C_{14} = C_{23} = C_{24} = 0$. Therefore,

$$\tau_2 = (1\ 2)(3\ 4) - (1\ 3)(2\ 4) + (1\ 4)(2\ 3),$$

$$\begin{aligned}\tau_2 = {} & 1 + \frac{k_1 - k_2}{k_1 + k_2}\exp(\xi_1 + \xi_2)\\ & + \frac{k_3 - k_4}{k_3 + k_4}\exp(\xi_3 + \xi_4)\\ & + \frac{(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)}{(k_1 + k_2)(k_1 + k_3)(k_1 + k_4)}\\ & \times \frac{(k_2 - k_3)(k_2 - k_4)(k_3 - k_4)}{(k_2 + k_3)(k_2 + k_4)(k_3 + k_4)}\exp(\sum_{i=1}^{4}\xi_i).\end{aligned}$$

Putting

$$\eta_i = \xi_i + \xi_{i+1} + \delta_i, \text{ where } \exp(\delta_i) = \frac{k_i - k_{i+1}}{k_i - k_{i+1}}, \qquad (34)$$

we may rewrite τ_2 as

$$\tau_2 = 1 + \exp(\eta_1) + \exp(\eta_3) + k_{12}^{34}\exp(\eta_1 + \eta_3), \qquad (35)$$

where

$$k_{ij}^{lm} =: \frac{(k_i - k_l)(k_i - k_m)(k_j - k_l)(k_j - k_m)}{(k_i + k_l)(k_i + k_m)(k_j + k_l)(k_j + k_m)}. \qquad (36)$$

In a similar way we can obtain the three-soliton expression for equation (20), we may choose $C_{12} = C_{34} = C_{56} = 1$, otherwise $C_{ij} = 0$, the

$$\begin{aligned}\tau_3 = {} & [(1\ 6)(2\ 3)(4\ 5) - (1\ 6)(2\ 4)(3\ 5) + (1\ 6)(2\ 5)(3\ 4)]\\ & -[(1\ 3)(2\ 6)(4\ 5) - (1\ 4)(2\ 6)(3\ 5) + (1\ 5)(2\ 6)(3\ 4)]\\ & +[(1\ 2)(3\ 6)(4\ 5) - (1\ 4)(2\ 5)(3\ 6) + (1\ 5)(2\ 4)(3\ 6)]\\ & -[(1\ 2)(3\ 5)(4\ 6) - (1\ 3)(2\ 5)(4\ 6) + (1\ 5)(2\ 3)(4\ 6)]\\ & +[(1\ 2)(3\ 4)(5\ 6) - (1\ 3)(2\ 4)(5\ 6) + (1\ 4)(2\ 3)(5\ 6)],\end{aligned}$$

we may rewrite τ_3 as

$$\begin{aligned}\tau_3 = {} & 1 + \exp(\eta_1) + \exp(\eta_3)\\ & + \exp(\eta_5) + k_{12}^{34}\exp(\eta_1 + \eta_3)\\ & + k_{12}^{56}\exp(\eta_1 + \eta_5) + k_{34}^{56}\exp(\eta_3 + \eta_5)\\ & + k_{123}^{456}\exp(\eta_1 + \eta_3 + \eta_5), \qquad (37)\end{aligned}$$

where

$$k_{ijp}^{lmn} =: k_{ij}^{pl} k_{ij}^{mn} k_{pl}^{mn}. \qquad (38)$$

Therefore, if we put $k_{ij}^{lm} =: \exp(K_{ij}^{lm})$, the N-soliton solution of the equation (20) might be expressed as:

$$\tau_N = \sum \exp(\sum_{i=1}^{N} \mu_{2i-1}\eta_{2i-1} + \sum_{i<j<l<m}^{(2N)} K_{ij}^{lm}\mu_i\mu_l), \qquad (39)$$

where $\sum$ denotes the summation over all possible combinations of $\mu_1 = 0, 1,\ \mu_2 = 0, 1, ...,\ \mu_{2N} = 0, 1$, and $\sum_{i<j<l<m}^{(2N)}$ is the sum over all $i,\ j,\ l,\ m\ (i < j < l < m)$ chosen from $\{1, 2, ..., 2N\}$. Furthermore, the equation (18) has the solution $u = 2\frac{\partial}{\partial x}(\ln \tau_N)$.

B. The (3+1)-D nonlinear Ma-Fan Eq

In this subsection, I would like to discuss the existence of the Pfaffian solutions to the following (3+1)-dimensional nonlinear Ma-Fan equation [12]:

$$\frac{\partial^2 u}{\partial z \partial t} - \frac{\partial^4 u}{\partial x^3 \partial y} - 3\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + 3\frac{\partial^2 u}{\partial x^2} = 0, \qquad (40)$$

which belongs to a class of generalized BKP equations, and can be written in terms of the Hirota bilinear operator.

When $z = y$, this equation reduces to the B-type Kadomtsev–Petviashvili equation [2], and so it is also a generalized BKP equation.

Under the dependent variable transformation

$$u = 2\frac{\partial}{\partial x}(\ln \phi), \qquad (41)$$

the above (3+1)-dimensional nonlinear Ma-Fan equation is mapped into the Hirota bilinear equation [12]:

$$\left(D_t D_z - D_x^3 D_y + 3D_x^2\right)\phi \cdot \phi = 0. \qquad (42)$$

1) Sufficient conditions: In this subsections, I will use the Pfaffian identities to search for exact solutions to the (3+1) dimensional nonlinear Ma-Fan equation (40). Let us introduce the following Pfaffian

$$\phi_n = \mathrm{Pf}\ (a_{i,j})_{1\le i,j\le 2n}, \qquad (43)$$

$$a_{ij} = C_{ij} + \int_{-\infty}^{x} D_x\ \psi_i \cdot \psi_j dx \quad i,j = 1,2,...,2n, \qquad (44)$$

where $C_{ij} = (-C_{ji}$ for $i \neq j)$ are constants and all ψ_i, $1 \le i \le 2n$, satisfy the linear differential equations:

$$\psi_{i,y} = \int_{-\infty}^{x} \psi_i(x)dx,\ \psi_{i,z} = \alpha \int_{-\infty}^{x} \psi_i(x)dx,$$
$$\psi_{i,t} = \beta\frac{\partial^3 \psi_i}{\partial x^3}, \qquad (45)$$

where

$$\beta = \frac{1}{\alpha}, \qquad (46)$$

with α being an arbitrary constant, and all ψ_i satisfying the boundary condition $\psi_i(-\infty) = 0$ for $i = 1,2,...,2n$.

2) Pfaffian solutions:

Theorem 5: (Sufficient condition) If $\psi_i(x,y,z,t)$, $1 \le i \le 2n$, satisfy (45), then the Pfaffian defined by (43) solve the Hirota bilinear equation (42) and the function $u = 2(\ln\phi)_x$ solve the (3+1)-dimensional Ma-Fan equation (40).

Proof: See [5]. ■

3) N-soliton solutions: The system (45) has a solution in the form:

$$\psi_i = \sum_{j=1}^{p} d_{ij}\exp(\xi_{ij}),$$
$$\xi_{ij} = k_{ij}x + k_{ij}^{-1}y + \alpha k_{ij}^{-1}z + \beta k_{ij}^3 t + \xi_{ij}^0, \qquad (47)$$

where $d_{ij},\ k_{ij}$, and ξ_{ij}^0 are free parameters and p is arbitrary natural number. In particular, we have the following solutions

$$\psi_i = \exp(\xi_i),$$
$$\xi_i = k_i x + k_i^{-1}y + \alpha k_i^{-1}z + \beta k_i^3 t + \xi_i^0, \qquad (48)$$

where k_i and ξ_i^0 are free parameter, and α, β are satisfy the equation (46). In order to explore solutions of (42), we choose special values for $(C_{ij})_{n\times n}$ and the functions ψ_i. For example, introduce

$$\psi_i = \exp(\xi_i),$$
$$\xi_i = k_i x + k_i^{-1}y + \alpha k_i^{-1}z + \beta k_i^3 t + \xi_i^0, \qquad (49)$$

and then we obtain

$$(i,j) = C_{ij} + \frac{k_i - k_j}{k_i + k_j}\psi_i\psi_j. \qquad (50)$$

Let us consider two-soliton and three-soliton solutions of the equation (42). For a two-soliton solution. We may choose $C_{12} = C_{34} = 1,\ C_{13} = C_{14} = C_{23} = C_{24} = 0$. Therefore,

$$\phi_2 = (1\ 2)(3\ 4) - (1\ 3)(2\ 4) + (1\ 4)(2\ 3),$$

Putting

$$\eta_i =: \xi_i + \xi_{i+1} + \delta_i,\ \text{where}\ \exp(\delta_i) =: \frac{k_i - k_{i+1}}{k_i - k_{i+1}}, \qquad (51)$$

equivalently, we have

$$\phi_2 = 1 + e^{\eta_1} + e^{\eta_3} + k_{12}^{34}e^{\eta_1+\eta_3}, \qquad (52)$$

where

$$k_{ij}^{lm} =: \frac{(k_i - k_l)(k_i - k_m)(k_j - k_l)(k_j - k_m)}{(k_i + k_l)(k_i + k_m)(k_j + k_l)(k_j + k_m)}. \qquad (53)$$

In a similar way, we can obtain a three-soliton solution for the equation (42). We may choose $C_{12} = C_{34} = C_{56} = 1$, otherwise $C_{ij} = 0$, and the ϕ function now is

$$\begin{aligned}\phi_3 &= 1 + \exp(\eta_1) + \exp(\eta_3) + \exp(\eta_5)\\ &+k_{12}^{34}\exp(\eta_1+\eta_3) + k_{12}^{56}\exp(\eta_1+\eta_5)\\ &+k_{34}^{56}\exp(\eta_3+\eta_5) +\ k_{123}^{456}\exp(\eta_1+\eta_3+\eta_5),\end{aligned} \qquad (54)$$

where

$$k_{ijp}^{lmn} =: k_{ij}^{pl}k_{ij}^{mn}k_{pl}^{mn}. \qquad (55)$$

Therefore, if we put $k_{ij}^{lm} =: \exp(K_{ij}^{lm})$, the N-soliton solution of the equation (42) can be expressed as

$$\phi_N = \sum \exp\left(\sum_{i=1}^{N}\mu_{2i-1}\eta_{2i-1} + \sum_{i<j<l<m}^{(2N)} K_{ij}^{lm}\mu_i\mu_l\right), \qquad (56)$$

where $\sum$ denotes the summation over all possible combinations of $\mu_1 = 0,1,\ \mu_2 = 0,1,...,\ \mu_{2N} = 0,1$, and $\sum_{i<j<l<m}^{(2N)}$ is the sum over all $i,\ j,\ l,\ m\ (i<j<l<m)$ chosen from $\{1,2,...,2N\}$. Furthermore, the nonlinear Ma-Fan equation (40) has the N-soliton solution $u = 2\frac{\partial}{\partial x}(\ln\phi_N)$.

C. The (3+1)-D soliton Eq of Jimbo-Miwa type

The Jimbo-Miwa equation is the second equation in the well known KP hierarchy of integrable systems, which is used to describe certain interesting (3+1)-dimensional waves in physics but not pass any of the conventional integrability tests [13], [14]. The equation arose in physics in connection with the nonlinear weaves with a weak dispersion.

In this subsection, I would like to discuss the non-linear soliton equations [12]:

$$2v_{yt} + v_{xxxy} + 3v_{xx}v_y + 3v_x v_{yx} - 3v_{zz} = 0, \tag{57}$$

which can be written in terms of the Hirota bilinear operator. In fact, the above soliton equations belong to a class of 3+1 dimensional soliton equations of Jimbo-Miwa type presented in [12].

Under the dependent variable Cole-Hopf transformations [12]

$$v = 2(\ln \omega)_x, \tag{58}$$

the above (3+1)-dimensional nonlinear Jimbo-Miwa type equation is mapped into two Hirota bilinear equations:

$$\left(2D_t D_y + D_x^3 D_y - 3D_z^2\right)\omega \cdot \omega = 0, \tag{59}$$

1) Sufficient conditions: In what follows we would like to discuss Pfaffian solutions to the (3+1)-dimensional soliton equations of Jimbo-Miwa type *(57)*. Let us take the following Pfaffian

$$\omega_n = \text{Pf }(\mu_{i,j})_{1\le i,j\le 2n}, \tag{60}$$

$$\mu_{ij} = C_{ij} + \int_{-\infty}^{x} D_x\ \zeta_i \cdot \zeta_j dx \quad i,j = 1,2,...,2n, \tag{61}$$

where $C_{ij} = (-C_{ji}$ for $i \neq j)$ are constants and all ζ_i, $1 \le i \le 2n$, satisfy the linear integro differential equations:

$$\zeta_{i,y} = 2\alpha^2 \int_{-\infty}^{x} \zeta_i(x)dx, \quad \zeta_{i,z} = \sqrt{2}\alpha\zeta_{i,x},$$
$$\zeta_{i,t} = -\frac{1}{2}\frac{\partial^3 \zeta_i}{\partial x^3}, \tag{62}$$

where α being an arbitrary nonzero parameter, and all ζ_i satisfying the boundary condition $\zeta_i(-\infty) = 0$ for $i = 1,2,...,2n$.

2) Pfaffian solutions:

Theorem 6: If $\zeta_i(x,y,z,t)$, $1 \le i \le 2n$, satisfy (62), then the Pfaffian defined by (60) solves the Hirota bilinear equation (59) and the function $v = 2(\ln \omega_n)_x$ solves the (3+1)-dimensional soliton equation of *Jimbo-Miwa* type (57).

Proof: See [6]. ■

3) N-soliton solutions: The system (62) has the solution in the form

$$\begin{aligned} \zeta_i &= \sum_{j=1}^{p} \rho_{ij} e^{\vartheta_{ij}}, \\ \vartheta_{ij} &= l_{ij}x + 2\alpha^2 l_{ij}^{-1} y + \sqrt{2}\alpha l_{ij} z - \frac{1}{2} l_{ij}^3 t + \vartheta_{ij}^0, \end{aligned} \tag{63}$$

where ρ_{ij}, l_{ij}, and ϑ_{ij}^0 are free parameters and p is an arbitrary natural number. In particular, we have the following specific solutions

$$\begin{aligned} \zeta_i &= e^{\vartheta_i}, \\ \vartheta_i &= l_i x + 2\alpha^2 l_i^{-1} y + \sqrt{2}\alpha l_i z - \frac{1}{2} l_i^3 t + \vartheta_i^0, \end{aligned} \tag{64}$$

where l_i and ϑ_i^0 are free parameter, and α is an arbitrary constant. In order to investigate those solutions of (59), we choose special values for $(C_{ij})_{n\times n}$ and the functions ζ_i. For example, letting

$$\zeta_i = e^{\vartheta_i},$$
$$\vartheta_i = l_i x + 2\alpha^2 l_i^{-1} y + \sqrt{2}\alpha l_i z - \frac{1}{2} l_i^3 t + \vartheta_i^0, \tag{65}$$

we obtain

$$(i,j) = C_{ij} + \frac{l_i - l_j}{l_i + l_j}\zeta_i\zeta_j. \tag{66}$$

Let us consider two-soliton and three-soliton solutions for the Eq. (59). For a two-soliton solution, we may choose $C_{12} = C_{34} = 1,\ C_{13} = C_{14} = C_{23} = C_{24} = 0$. Then

$$\omega_2 = (1\ 2)(3\ 4) - (1\ 3)(2\ 4) + (1\ 4)(2\ 3),$$

Putting

$$\theta_i = \vartheta_i + \vartheta_{i+1} + \delta_i, \text{ where } e^{\delta_i} = \frac{k_i - k_{i+1}}{k_i - k_{i+1}}, \tag{67}$$

we may rewrite ω_2 as

$$\omega_2 = 1 + e^{\theta_1} + e^{\theta_3} + l_{12}^{34} e^{\theta_1+\theta_3}, \tag{68}$$

where

$$l_{ij}^{lm} = \frac{(l_i - l_l)(l_i - l_m)(l_j - l_l)(l_j - l_m)}{(l_i + l_l)(l_i + l_m)(l_j + l_l)(l_j + l_m)}. \tag{69}$$

In a similar way, we can obtain a three-soliton solution for the equation (59). We may choose $C_{12} = C_{34} = C_{56} = 1$, otherwise $C_{ij} = 0$, and then we may rewrite ω_3 as

$$\begin{aligned} \omega_3 = \ & 1 + e^{\theta_1} + e^{\theta_3} + e^{\theta_5} + \\ & l_{12}^{34} e^{\theta_1+\theta_3} + l_{12}^{56} e^{\theta_1+\theta_5} \\ & + l_{34}^{56} e^{\theta_3+\theta_5} + l_{123}^{456} e^{\theta_1+\theta_3+\theta_5}, \end{aligned} \tag{70}$$

where

$$l_{ijp}^{lmn} = l_{ij}^{pl} l_{ij}^{mn} l_{pl}^{mn}. \tag{71}$$

Therefore, if we put $l_{ij}^{lm} = e^{L_{ij}^{lm}}$, then the N-soliton solution of the equation (59) is expressed as

$$\omega_N = \sum \exp\left(\sum_{i=1}^{N} \beta_{2i-1}\beta_{2i-1} + \sum_{i<j<l<m}^{(2N)} L_{ij}^{lm}\beta_i\beta_l\right), \tag{72}$$

where $\sum$ denotes the summation over all possible combinations of $\beta_1 = 0,1,\ \beta_2 = 0,1,...,\ \beta_{2N} = 0,1$, and $\sum_{i<j<l<m}^{(2N)}$ is the sum over all $i,\ j,\ l,\ m\ (i < j < l < m)$ chosen from $\{1,2,...,2N\}$. Furthermore, the equation (57) has the N-soliton solution $v = 2\frac{\partial}{\partial x}(\ln \omega_N)$.

III. Conclusion

In Subsection II-A2, I have built an Pfaffian formulation for the (3+1)-dimensional generalized B-type Kadomtsev–Petviashvili equation:

$$u_{ty} - u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + 3u_{xx} + 3u_{zz} = 0.$$

The facts used in our construction are the Pfaffian identities. Theorem 4, give the main results on Pfaffian solutions, which say that

$$u = 2\frac{\partial}{\partial x}(\ln \tau_n),\ \tau_n = \text{Pf }(a_{i,j})_{1\le i,j\le 2n},$$

where the elements of τ_n are defined by $a_{ij} = C_{ij} + \int_{-\infty}^{x} D_x\ f_i(x)\cdot f_j(x)dx,\ C_{ij}$ =constant, $i,j = 1,2,...,2n$, with f_i satisfying

$$\frac{\partial f_i}{\partial y} = \lambda_1 \frac{\partial f_i}{\partial x_{-1}},\ \frac{\partial f_i}{\partial z} = \lambda_2 \frac{\partial f_i}{\partial x},\ \frac{\partial f_i}{\partial t} = \frac{\partial^3 f_i}{\partial x^3},$$

where λ_1 and λ_2 are free parameters defined in the equation (25), solves the above (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation. Examples of Pfaffian solutions made, along with a few plots [1]. In Theorem 4, we considered only a specific sufficient conditions: (24), though there is a free parameters λ_1 and λ_2 in the conditions. It would be great to look for more general conditions involving combined equations for Pfaffian solutions.

Moreover, based on the theory of the Bordered determinants and the relation between a Pfaffian and a determinant. We would like to discuss the relation between the generalized (3+1)-dimensional B-*type* KP equation ($GBKP$) and the generalized (3+1)-dimensional A-type KP equation (GKP). Using integration by parts, each Pfaffian entry (i,j) is

$$\begin{aligned} a_{ij} &= C_{ij} + \int_{-\infty}^{x} D_x\ f_i(x)\cdot f_j(x)dx \\ &= C_{ij} + 2\int_{-\infty}^{x} \frac{\partial f_i}{\partial x} f_j dx - f_i f_j. \end{aligned}$$

Therefore, the square of the N-soliton solution τ_N can be written as the determinants

$$\tau_N^2 = \left| C_{ij} + 2\int_{-\infty}^{x} \frac{\partial f_i}{\partial x} f_j dx \right|_{1\le i,j\le 2N}.$$

This determinant is nothing but the Grammian solution of the GKP equation, τ_{GKP}. Hence, we have

$$\tau_{GKP} = \tau_{GBKP}^2.$$

Where GKP stand for the generalized A-type Kadomtsev-Petviashvili equation. We choose a lower limit of the above integrals to be $x = -\infty$, but this is not an essential restriction, the result is the same for any other choice of the lower limit.

In Subsections II-B2 and II-C2, I have built an Pfaffian formulation for the (3+1)-dimensional nonlinear Ma-Fan equation:

$$u_{tz} - u_{xxxy} - 3\left(u_x u_y\right)_x + 3u_{xx} = 0,$$

and the (3+1) dimensional soliton equations of Jimbo-Miwa type:

$$2v_{yt} + v_{xxxy} + 3v_{xx}v_y + 3v_x v_{yx} - 3v_{zz} = 0.$$

The facts used in our construction are the Pfaffian identities and the same technique used in the Subsection II-A2.

Appendix A
Proof of Theorem 4

Proof: Let us express the Pfaffian τ_n by means of Lemma 2:

$$\tau_n = (1,2,...,2n) = (\bullet), \tag{73}$$

where $a_{ij} = (i,j)$ and $(d_m, d_m) = 0$, where m is integer. To compute the derivatives of the entries (i,j) and the Pfaffian τ_n, we introduce new Pfaffian entries

$$(d_n, i) = \frac{\partial^n f_i}{\partial x^n}, \qquad (d_{-n}, i) = \frac{\partial^n f_i}{\partial x_{-1}^n},\ \text{for } n \ge 0, \tag{74}$$

by using the equation (23) and the equation (24) we can get

$$\begin{aligned} \frac{\partial}{\partial x}(i,j) &= f_j \frac{\partial f_i}{\partial x} - f_i \frac{\partial f_j}{\partial x} = \left(d_0, d_1, i, j\right), && (75) \\ \frac{\partial}{\partial y}(i,j) &= \lambda_1 \frac{\partial}{\partial x_{-1}} \int_{-\infty}^{x} \left[f_j \frac{\partial f_i}{\partial x} - f_i \frac{\partial f_j}{\partial x} \right] dx \\ &= \lambda_1 \left[f_i \frac{\partial f_j}{\partial x_{-1}} - f_j \frac{\partial f_i}{\partial x_{-1}} \right] = \\ &\quad \lambda_1 \left(d_{-1}, d_0, i, j\right), && (76) \\ \frac{\partial}{\partial z}(i,j) &= \lambda_2 \left(d_0, d_1, i, j\right), && (77) \\ \frac{\partial}{\partial t}(i,j) &= f_j \frac{\partial^3 f_i}{\partial x^3} - f_i \frac{\partial^3 f_j}{\partial x^3} - 2\left[\frac{\partial f_j}{\partial x}\frac{\partial^2 f_i}{\partial x^2} - \frac{\partial f_i}{\partial x}\frac{\partial^2 f_j}{\partial x^2} \right] \\ &= \left(d_0, d_3, i, j\right) - 2\left(d_1, d_2, i, j\right). && (78) \end{aligned}$$

Therefore, from the above results (75)-(78) we have the following differential formulae for τ_n

$$\frac{\partial \tau_n}{\partial x} = \left(d_0, d_1, \bullet\right), \tag{79}$$

$$\frac{\partial \tau_n}{\partial y} = \lambda_1 \left(d_{-1}, d_0, \bullet\right), \tag{80}$$

$$\frac{\partial \tau_n}{\partial z} = \lambda_2 \left(d_0, d_1, \bullet\right), \tag{81}$$

$$\partial \tau$$

$$\frac{\ \ _n}{\partial t} = (d_0, d_3, \bullet) - 2\,(d_1, d_2, \bullet)\,, \tag{82}$$

$$\frac{\partial^2 \tau_n}{\partial x^2} = (d_0, d_2, \bullet)\,, \tag{83}$$

$$\frac{\partial^2 \tau_n}{\partial z^2} = \lambda_2^2\,(d_0, d_2, \bullet)\,, \tag{84}$$

$$\frac{\partial^3 \tau_n}{\partial x^3} = (d_1, d_2, \bullet) + (d_0, d_3, \bullet)\,, \tag{85}$$

$$\frac{\partial^2 \tau_n}{\partial y \partial x} = \lambda_1\,(d_{-1}, d_1, \bullet)\,, \tag{86}$$

$$\frac{\partial^3 \tau_n}{\partial y \partial x^2} = \lambda_1\,[(d_{-1}, d_2, \bullet) + (d_0, d_1, \bullet)\ ]\,, \tag{87}$$

$$\frac{\partial^2 \tau_n}{\partial y \partial t} - \frac{\partial^4 \tau_N}{\partial y \partial x^3} = -3\lambda_1\,[(d_0, d_2, \bullet) + (d_{-1}, d_0, d_1, d_2, \bullet)]\,, \tag{88}$$

where we have used the abbreviated notation $\bullet = 1, 2, ..., 2n$. We can now compute that

$$\begin{aligned} & \left[\frac{\partial^2 \tau_n}{\partial y \partial t} - \frac{\partial^4 \tau_n}{\partial y \partial x^3} + 3\frac{\partial^2 \tau_n}{\partial z^2} + 3\frac{\partial^2 \tau_n}{\partial x^2}\right]\tau_n \\ = \quad & -3\lambda_1\,(d_{-1}, d_0, d_1, d_2, \bullet)\,(\bullet), \\ & \left[-\frac{\partial \tau_n}{\partial t}\frac{\partial \tau_n}{\partial y} + \frac{\partial^3 \tau_n}{\partial x^3}\frac{\partial \tau_n}{\partial y}\right] = 3\lambda_1\,(d_{-1}, d_0, \bullet)\,(d_1, d_2, \bullet)\,, \\ & \left[3\frac{\partial^3 \tau_n}{\partial y \partial x^2}\frac{\partial \tau_n}{\partial x} - 3\frac{\partial^2 \tau_n}{\partial x^2}\frac{\partial^2 \tau_n}{\partial y \partial x} - 3(\frac{\partial \tau_n}{\partial z})^2 - 3(\frac{\partial \tau_n}{\partial x})^2\right] \\ = \quad & 3\lambda_1\,[(d_0, d_1, \bullet)\,(d_{-1}, d_2, \bullet) - (d_0, d_2, \bullet)\,(d_{-1}, d_1, \bullet)\ ]\,. \end{aligned}$$

Substituting the above derivatives of $\tau\ \ _n$ into the LHS of the equation (21), we arrive at

$$\begin{aligned} & \left[\frac{\partial^2 \tau_n}{\partial y \partial t} - \frac{\partial^4 \tau_n}{\partial y \partial x^3} + 3\frac{\partial^2 \tau_n}{\partial z^2} + 3\frac{\partial^2 \tau_n}{\partial x^2}\right]\tau_n \\ & -\frac{\partial \tau_n}{\partial t}\frac{\partial \tau_n}{\partial y} + \frac{\partial^3 \tau_n}{\partial x^3}\frac{\partial \tau_n}{\partial y} \\ & +3\frac{\partial^3 \tau_n}{\partial y \partial x^2}\frac{\partial \tau_n}{\partial x} - 3\frac{\partial^2 \tau_n}{\partial x^2}\frac{\partial^2 \tau_n}{\partial y \partial x} \\ & -3(\frac{\partial \tau_n}{\partial z})^2 - 3(\frac{\partial \tau_n}{\partial x})^2 \\ = \quad & -3\lambda_1[(d_{-1}, d_0, d_1, d_2, \bullet)\,(\bullet) - (d_{-1}, d_0, \bullet)\,(d_1, d_2, \bullet) \\ & + (d_0, d_2, \bullet)\,(d_{-1}, d_1, \bullet) - (d_0, d_1, \bullet)\,(d_{-1}, d_2, \bullet)] \\ = \quad & 0, \end{aligned} \tag{89}$$

where we have made use of the equation (16) with $m = 2$. This shows that the Pfaffian $\tau_n = (1, 2, ..., 2n)$ with the conditions (24) solves the generalized bilinear B-type Kadomtsev–Petviashvili equation (20), which ends the proof. ■

ACKNOWLEDGMENT

First, I wish to thank A. Ariston for all his guidance. Additionally, I would like to acknowledge all of those people who helped make this conference possible. This work was supported in part by the Alexandria University, Alexandria, Egypt.

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