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# Parallel alternating two-stage methods for solving linear system

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*Abstract*—In this paper, we present parallel alternating two-stage methods for solving linear system $Ax = b$, where $A$ is a monotone matrix or an H-matrix. And we give some convergence results of these methods for nonsingular linear system.

*Keywords*—parallel, alternating two-stage, convergence, linear system.

**Mathematics Subject Classification(2000)** 65F10

## I. Introduction

For the solution of the large linear system

$$Ax = b, \quad (1)$$

where $A$ is an $n \times n$ square matrix, and $x$ and $b$ are n-dimensional vectors, the basic iterative method is

$$Mx_{k+1} = Nx_k + b, k = 0, 1 \cdots \quad (2)$$

where $A = M - N$ and $M$ is nonsingular.

Alternating two-stage iterative methods[1] have been studied to approximate the linear system (2) by using a inner iteration. Let $M = P - Q = R - S$ be two splittings of the matrix $M$. In order to approximate (2), for each $k, k = 1, 2 \cdots$, we perform $s(k)$ inner iterations of the general class of iterative methods of the form

$$y_{j-\frac{1}{2}} = P^{-1}Qy_{j-1} + P^{-1}(Nx_{k-1} + b)$$

$$y_j = R^{-1}Sy_{j-\frac{1}{2}} + R^{-1}(Nx_{k-1} + b).\ j = 1, 2, \cdots s(k)$$

Thus, the resulting method is

$$x_k = (R^{-1}SP^{-1}Q)^{s(k)}x_{k-1} + \sum_{j=0}^{s(k)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1} + I)(Nx_{k-1} + b).$$
$$k = 1, 2, \cdots$$

On the other hand, with the development of parallel computation in recent years, the utilization of the parallel algorithms for the solution of large nonsingular linear system has become effective. Now we introduce the parallel alternating two-stage methods.

Given a parallel multisplitting of $A$, s.t.

$$(i) A = M_l - N_l,$$

$$(ii) M_l = P_l - Q_l = R_l - S_l,$$

$$(iii) E_l \geq 0 \, and \sum_{l=1}^{\alpha} E_l = I,$$

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where $l = 1, 2 \cdots \alpha$ and $I$ is the identity matrix.

Suppose that we have a multiprocessor with $\alpha$ processors connected to a host processor, that is, the same number of processors as splittings, and that all processors have the last update vector $x^{k-1}$, then the $lth$ processor only computes those entries of the vector

$$y_{l,j-\frac{1}{2}} = P_l^{-1}Q_l y_{l,j-1} + P_l^{-1}(N_l x_{k-1} + b)$$

$$y_{l,j} = R_l^{-1}S_l y_{l,j-\frac{1}{2}} + R_l^{-1}(N_l x_{k-1} + b).\ j = 1, 2, \cdots s(k)$$

with $y_{l,0} = x_{k-1}$, or equivalently

$$y_{l,j} = R_l^{-1}S_l P_l^{-1}Q_l y_{l,j-1} + R_l^{-1}(S_l P_l^{-1} + I)(N_l x_{k-1} + b).$$
$$j = 1, 2, \cdots s(k),$$

which correspond to the nonzero diagonal entries of $E_l$. The processor then scales these entries so as to be able to deliver the results to the host processor, performing the parallel multisplitting scheme

$$x_k = H(k)x_{k-1} + W(k)b,\ k = 1, 2 \cdots, \quad (3)$$

where

$$H(k) = \sum_{l=1}^{\alpha} E_l[(R_l^{-1}S_l P_l^{-1}Q_l)^{s(l,k)} + \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_l P_l^{-1}Q_l)^j R_l^{-1}(S_l P_l^{-1} + I)N_l]$$

and

$$W(k) = \sum_{l=1}^{\alpha} E_l \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_l P_l^{-1}Q_l)^j (R_l^{-1}S_l P_l^{-1} + R_l^{-1}).$$

Then, we can obtain the next algorithm:

**Algorithm 1**($PATS$):

$$for\, any\, given\, initial\, vector\, x_0$$
$$for\, k = 1, 2 \cdots \, until\, convergent$$
$$for\, l = 1, 2 \cdots \alpha$$
$$y_{l,0} = x_{k-1}$$
$$for\, j = 1, 2 \cdots s(l,k)$$
$$P_l y_{l,j-\frac{1}{2}} = Q_l y_{l,j-1} + (N_l x_{k-1} + b)$$
$$R_l y_{l,j} = S_l y_{l,j-\frac{1}{2}} + (N_l x_{k-1} + b)$$
$$x_k = \sum_{l=1}^{\alpha} E_l y_{l,j}$$

Usually, we say that an parallel alternating two-stage method is stationary when $s(l,k) = s$, for all $l, k$, while an parallel alternating two-stage method is non-stationary if the number of inner iterations changes with the outer iteration $k$. In the following, we call them $SPATS$ method and $NSPATS$ method, respectively.

In this paper, our study concentrates on the parallel alternating two-stage method. With this aim, in the next section, we introduce the notation and preliminaries needed in this paper. In section 3, we present convergence conditions of these methods for nonsingular linear systems, when the matrix $A$ of the linear system is monotone or H-matrix. In section 4, we also give two relaxation parallel alternating two-stage methods, $RPATS\,I$ and $RPATS\,II$, and analyze their convergence conditions.

## II. NOTATION AND PRELIMINARIES

We need the following definitions and results.

We say that a vector $x$ is nonnegative, denoted $x \geq 0$, if all of its entries are nonnegative. A nonsingular matrix $A \in Z^{n\times n}$ is an M-matrix if and only if $A$ is a monotone matrix($A^{-1} \geq 0$). For any matrix $A = (a_{ij}) \in R^{n\times n}$, we define its comparison matrix $< A >= (b_{ij})$ by $b_{ii} = |a_{ii}|$, $b_{ij} = -|a_{ij}|$, $i \neq j$. A nonsingular matrix $A$ is said to be an H-matrix if $< A >$ is an M-matrix.

**Lemma 1**: ([2]). Let $A, B \in R^{n\times n}$,

(1) if $A$ is an H-matrix, then $|A^{-1}| \leq < A >^{-1}$, and

(2) if $|A| \leq B$, then $\rho(A) \leq \rho(B)$.

**Lemma 2**: ([3,4]). Let $A \in R^{n\times n}$. A splitting $A = M - N$ is called

(1)regular if $M^{-1} \geq 0$ and $N \geq 0$,

(2)weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$,

(3)H-splitting if $< M > -|N|$ is a nonsingular M-matrix, and

(4)H-compatible splitting if $< A >=< M > -|N|$;

**Lemma 3**: ([3,4]). Let $A = M - N$ be a splitting,

(1)if the splitting is weak regular, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$,

(2)if the splitting is an H-splitting, then $A$ and $M$ are H-matrices and $\rho(M^{-1}N) < \rho(< M >^{-1} |N|) < 1$, and

(3)if the splitting is an H-compatible splitting and $A$ is an H-matrix, then it is an H-splitting and thus convergent.

**Lemma 4**: ([5]). Let $T_1, T_2 \cdots T_k \cdots$ be a sequence of nonnegative matrices in $R^{n\times n}$. If exist a real number $0 \leq \theta < 1$, and a vector $v > 0$ in $R^{n\times n}$, such that $T_j v < \theta v$, $for\, j = 1, 2\cdots$, then $\rho(H_k) \leq \theta^k < 1$, where $H_k = T_k T_{k-1}\cdots T_2 T_1$, and therefore $\lim_{k\to\infty} H_k = 0$.

## III. CONVERGENCE THEOREMS

Firstly, we deal with the convergence of the $PATS$ method when $A$ is a monotone matrix.

**Theorem 1**: Let $A^{-1} \geq 0$. If the outer splitting $A = M_l - N_l$ is regular splitting and the inner splitting $M_l = P_l - Q_l = R_l - S_l$ are weak regular splitting. Then the $PATS$ method converges to $x_* = A^{-1}b$ for any initial vector $x_0$ and for any sequence of inner iteration numbers $s(l,k) \geq 1$, $k = 1, 2\cdots$.

**Proof**: Let $x_* = A^{-1}b$ and $e_k = x_k - x_*$ be the error vector at the $kth$ outer iteration of the $PATS$ method. It is easy to prove that $x_*$ is a fixed point of (3). Therefore,

$$e_k = H(k)H(k-1)\cdots H(1)e_0,\; k = 1, 2\cdots.$$

We suppose that $H(k) = \sum_{l=1}^{\alpha} E_l T_l(k)$, where

$$T_l(k) = (R_l^{-1}S_lP_l^{-1}Q_l)^{s(l,k)} + \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R_l^{-1}(S_lP_l^{-1}+I)N_l,$$

we can easily have

$$T_l(k) \geq 0 \;\; and \;\; H(k) \geq 0.$$

Now we consider the vector $e = (1, 1\cdots 1)^\top$, and suppose $x = A^{-1}e$, since $A^{-1} \geq 0$, and $A^{-1}$ have no rows that all entries are zero, then $x > 0$. Similarly, $R_l^{-1}(S_lP_l^{-1} + I) > 0$. From

$$(I - M_l^{-1}N_l)x = M_l^{-1}Ax = M_l^{-1}e,$$

it follows that

$$\begin{aligned}
T_l(k)x =& [(R_l^{-1}S_lP_l^{-1}Q_l)^{s(l,k)} \\
&+ \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R_l^{-1}(S_lP_l^{-1}+I)N_l]x \\
=& [I - (I - (R_l^{-1}S_lP_l^{-1}Q_l)^{s(l,k)})(I - M_l^{-1}N_l)]x \\
=& x - \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R_l^{-1}(S_lP_l^{-1}+I)(P_l-Q_l)M_l^{-1}e \\
=& x - \sum_{j=0}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R_l^{-1}(S_lP_l^{-1}+I)M_lM_l^{-1}e \\
=& x - R_l^{-1}(S_lP_l^{-1}+I)e \\
&- \sum_{j=1}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R_l^{-1}(S_lP_l^{-1}+I)e
\end{aligned}$$

therefore,

$$\begin{aligned}
H(k)x =& x - \sum_{l=1}^{\alpha} E_lR_l^{-1}(S_lP_l^{-1}+I)e \\
&- \sum_{l=1}^{\alpha} E_l \sum_{j=1}^{s(l,k)-1}(R_l^{-1}S_lP_l^{-1}Q_l)^j R_l^{-1}(S_lP_l^{-1}+I)e \\
<& x - \sum_{l=1}^{\alpha} E_lR_l^{-1}(S_lP_l^{-1}+I)e.
\end{aligned}$$

Since $T_l(k)x \geq 0$, we have $x - \sum_{l=1}^{\alpha} E_lR_l^{-1}(S_lP_l^{-1}+I)e < x$. Therefore, there exists $0 \leq \theta < 1$, such that

$$x - \sum_{l=1}^{\alpha} E_lR_l^{-1}(S_lP_l^{-1}+I)e \leq \theta x,$$

then

$$H(k)x < \theta x,\; k = 1, 2\cdots,$$

and from Lemma 4 it follows that

$$\lim_{k\to\infty} \tilde{H}(k) = 0,$$

where $\tilde{H}(k) = H(k)H(k-1)\cdots H(1)$.

So we have

$$\lim_{k\to\infty} e(k) = 0,$$

and the proof is complete.

Now, we study the convergence of the $NSPATS$ method when $A$ is an H-matrix, and therefore not necessarily a monotone matrix.

**Theorem 2**: Let $A$ is an H-matrix. If the outer splitting $A = M_l - N_l$ is H-splitting and the inner splitting $M_l = P_l - Q_l = R_l - S_l$ are H-compatible splitting. Then the $PATS$ method converges to $x_* = A^{-1}b$ for any initial vector $x_0$ and for any sequence of inner iteration numbers $s(l,k) \geq 1$, $k = 1, 2\cdots$.

**Proof**: By Lemma 3(2), the matrices $P_l$ and $R_l$ are H-matrices.

and using Lemma 1(1) we can obtain,

$$\begin{aligned}|H(k)| =& |\sum_{l=1}^{\alpha} E_l T_l(k)| \leq \sum_{l=1}^{\alpha} E_l |T_l(k)| \\ &\leq \sum_{l=1}^{\alpha} E_l[(|R_l^{-1}||S_l||P_l^{-1}||Q_l|)^{s(l,k)} \\ &+ \sum_{j=0}^{s(l,k)-1} (|R_l^{-1}||S_l||P_l^{-1}||Q_l|)^j |R_l^{-1}|(|S_l||P_l^{-1}| + I)|N_l|] \\ &\leq \sum_{l=1}^{\alpha} E_l[(< R_l^{-1} > |S_l| < P_l^{-1} > |Q_l|)^{s(l,k)} \\ &+ \sum_{j=0}^{s(l,k)-1} (< R_l^{-1} > |S_l| < P_l^{-1} > |Q_l|)^j \\ &\times < R_l^{-1} > (|S_l| < P_l^{-1} > + I)|N_l|] \\ &= \tilde{H}(k).\end{aligned}$$

Then we have

$$|H(k)||H(k-1)| \cdots |H(1)| \leq |\tilde{H}(k)||\tilde{H}(k-1)| \cdots |\tilde{H}(1)|.$$

Moreover, $\tilde{H}(k)$ is the iteration matrix of $NASTSM$ method for the matrix $(< M_l > -|N_l|)$ with the regular splittings $< M_l > -|N_l|$ and $< M_l >=< P_l > -|Q_l| =< R_l > -|S_l|$. From the definitions of H-splitting, H-compatible splitting and Theorem 1, we can obtain

$$\begin{aligned}&\lim_{k\to\infty} |H(k)||H(k-1)| \cdots |H(1)| \\ &\leq \lim_{k\to\infty} |\tilde{H}(k)||\tilde{H}(k-1)| \cdots |\tilde{H}(1)| \\ &= 0.\end{aligned}$$

So we have

$$\lim_{k\to\infty} e(k) = 0,$$

and the proof is complete.

## IV. Relaxation iteration methods

Now we introduce the relaxation factor to the parallel alternating two-stage methods. Then, we can obtain the following two algorithms:

**Algorithm 2**($RPATS\,I$):

$$\begin{aligned}&for\,any\,given\,initial\,vector\,x_0\,and\,\omega \in (0,1) \\ &\quad for\,k = 1,2\cdots\,until\,convergent \\ &\quad\quad for\,l = 1,2\cdots\alpha \\ &\quad\quad y_{l,0} = x_{k-1} \\ &\quad\quad\quad for\,j = 1,2\cdots s(l,k) \\ &\quad\quad P_l y_{l,j-\frac{1}{2}} = Q_l y_{l,j-1} + (N_l x_{k-1} + b) \\ &\quad\quad R_l y_{l,j} = S_l y_{l,j-\frac{1}{2}} + (N_l x_{k-1} + b) \\ &\quad\quad x_k = \omega \sum_{l=1}^{\alpha} E_l y_{l,j} + (1-\omega) x_{k-1}\end{aligned}$$

**Algorithm 3**($RPATS\,II$):

$$\begin{aligned}&for\,any\,given\,initial\,vector\,x_0\,and\,\omega \in (0,1) \\ &\quad for\,k = 1,2\cdots\,until\,convergent \\ &\quad\quad for\,l = 1,2\cdots\alpha \\ &\quad\quad y_{l,0} = x_{k-1} \\ &\quad\quad\quad for\,j = 1,2\cdots s(l,k) \\ &P_l y_{l,j-\frac{1}{2}} = \omega(Q_l y_{l,j-1} + (N_l x_{k-1} + b)) + (1-\omega) P_l y_{l,j-1} \\ &R_l y_{l,j} = \omega(S_l y_{l,j-\frac{1}{2}} + (N_l x_{k-1} + b)) + (1-\omega) R_l y_{l,j-\frac{1}{2}} \\ &\quad\quad x_k = \sum_{l=1}^{\alpha} E_l y_{l,j}\end{aligned}$$

For relaxation parallel alternating two-stage methods($RPATS\,I$ and $RPATS\,II$), we can similarly get the following convergence theorems.

**Theorem 3**: Let $A^{-1} \geq 0$. If the outer splitting $A = M_l - N_l$ is regular splitting and the inner splitting $M_l = P_l - Q_l = R_l - S_l$ are weak regular splitting, and $0 < \omega < 1$. Then the $RPATS\,I$ and $RPATS\,II$ methods converge to $x_* = A^{-1}b$ for any initial vector $x_0$ and for any sequence of inner iteration numbers $s(l,k) \geq 1$, $k = 1,2\cdots$.

**Theorem 4**: Let $A$ is an H-matrix. If the outer splitting $A = M_l - N_l$ is H-splitting and the inner splitting $M_l = P_l - Q_l = R_l - S_l$ are H-compatible splitting, and $0 < \omega < 1$. Then the $RPATS\,I$ and $RPATS\,II$ methods converge to $x_* = A^{-1}b$ for any initial vector $x_0$ and for any sequence of inner iteration numbers $s(l,k) \geq 1$, $k = 1,2\cdots$.

The proofs of the Theorem 3 and Theorem 4 are similar to the proofs of Theorem 1 and Theorem 2, so we omit them.

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