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Oscillation theorems for second-order nonlinear neutral dynamic equations with variable delays and damping

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Abstract—In this paper, we study the oscillation of a class of second-order nonlinear neutral damped variable delay dynamic equations on time scales. By using a generalized Riccati transformation technique, we obtain some sufficient conditions for the oscillation of the equations. The results of this paper improve and extend some known results. We also illustrate our main results with some examples.

Keywords—Oscillation theorem, Second-order nonlinear neutral dynamic equation, Variable delay, Damping, Riccati transformation.

I. Introduction

THe aim of this paper is to establish several oscillation theorems for the second-order nonlinear neutral damped variable delay dynamic equation

$$\Big(r(t)\Big(\big[y(t)+p(t)y(\tau(t))\big]^{\Delta}\Big)^{\lambda}\Big)^{\Delta} + \eta(t)\Big(\big[y(t)+p(t)y(\tau(t))\big]^{\Delta\sigma}\Big)^{\lambda} + f(t,y(\delta(t)))=0 \quad (1)$$

on an arbitrary time scale $\mathbb{T}$, where $\lambda > 0$ is a quotient of odd positive integers. Since we are interested in the oscillation of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$. Furthermore, in this paper we shall need the following conditions:

(H_1) $r \in C_{rd}(\mathbb{I}, (0,\infty))$, and $\eta \in C_{rd}(\mathbb{I}, [0,\infty))$, where $\mathbb{I} := [t_0,\infty)$ is a time scale interval in $\mathbb{T}$;

(H_2) $p \in C_{rd}(\mathbb{T}, [0,1])$;

(H_3) $\tau \in C_{rd}(\mathbb{T},\mathbb{T}), \tau(t) \le t$ for $t \in \mathbb{T}$, and $\lim_{t\to\infty} \tau(t) = \infty$;

(H_4) $\delta \in C_{rd}(\mathbb{T},\mathbb{T}), \delta(t) \le t$ for $t \in \mathbb{I}$, and $\lim_{t\to\infty} \delta(t) = \infty$;

(H_5) $f : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(t,u) > 0$ for all $t \in \mathbb{I}$ and for all $u \ne 0$, and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{I}$ such that $|f(t,u)| \ge q(t)|u^{\lambda}|$ for all $t \in \mathbb{I}$ and for all $u \in \mathbb{R}$;

(H_6) $\delta^{\Delta}(t) > 0$ is rd-continuous on $\mathbb{T}$, $\tilde{\mathbb{T}} := \delta(\mathbb{T}) = \{\delta(t) : t \in \mathbb{T}\} \subset \mathbb{T}$ is a time scale, and $(\delta^{\sigma})(t) = (\sigma \circ \delta)(t)$ for all $t \in \mathbb{T}$, where $\sigma(t)$ is the forward jump operator on $\mathbb{T}$ and $(\delta^{\sigma})(t) := (\delta \circ \sigma)(t)$.

Recall that a solution of (1) is a nontrivial real function y such that $y(t) + p(t)y(\tau(t)) \in C^1_{rd}[t_y,\infty)$ and $r(t)([y(t) + p(t)y(\tau(t))]^{\Delta})^{\lambda} \in C^1_{rd}[t_y,\infty)$ for a certain $t_y \ge t_0$ and satisfying (1) for $t \ge t_y$. Our attention is restricted to those solutions of (1) which exist on the half-line $[t_y,\infty)$ and satisfy $\sup\{|y(t)| : t > t_*\} > 0$ for any $t_* \ge t_y$. A solution y of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [2] in 1988 in order to unify continuous and discrete analysis. The cases when a time scale $\mathbb{T}$ is equal to the reals $\mathbb{R}$ or to the integers $\mathbb{Z}$ represent the classical theories of differential and of difference equations. Not only can the theory of so-called "dynamic equations" unify the theories of differential equations and difference equations, but also it is able to extend these classical cases to cases "in between," e.g., to so-called q-difference equations. For the calculus on time scales and the notations used below, we refer to the book by Bohner and Peterson [3]. For advances in dynamic equations on time scales, we refer to [4] and [5].

Dynamic equations on a time scale have many applications in the real world. For instance, it leads itself to insect population models, which are discrete in season (and may follow a difference scheme with variable step-size or are often modeled by continuous dynamic systems), die out in say winter, while their eggs are incubating or dormant, and then in season again, hatching gives rise to a nonoverlapping population (see [3]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [6] discusses several possible applications.

In the past years there has been much research activity concerning the oscillation and asymptotic behavior of solutions of some dynamic equations on time scales, and we refer the reader to the papers [1], [7]–[18] and [20] and the references cited therein. Recently, Agarwal et al. [21] considered a special case of (1), namely, the equation

$$\Big(r(t)\Big(\big[y(t)+p(t)y(t-\tau_0)\big]^{\Delta}\Big)^{\lambda}\Big)^{\Delta} + f(t,y(t-\delta_0)) = 0, \quad (2)$$

where $t \in \mathbb{T}, \tau_0$ and δ_0 are positive constants. For the case when $\lambda > 0$ and $r^{\Delta}(t) \ge 0$, Agarwal et al. [21] obtained several oscillation criteria of (2) by reducing the oscillation properties of (2) to the nonexistence of positive solutions of a delay dynamic inequality. Furthermore, for the case when $\lambda \ge 1$ and the graininess function $\mu(t) := \sigma(t) - t > 0$, Agarwal et al. [21] gave some sufficient conditions for the oscillation

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of all solutions of (2) by developing a Riccati transformation technique.

Subsequently, for the case when $\lambda \geq 1$ is an odd positive integer, Saker [22] did not require the conditions $r^{\Delta}(t) \geq 0$ and $\mu(t) > 0$ and obtained some new oscillation results for (2).

Also in 2006, for the case when $\eta(t) \equiv 0$ and $\lambda \geq 1$ is a quotient of odd positive integers, Wu et al. [23] got several oscillation theorems for (1) and extended and improved some results of Agarwal et al. [21] and Saker [22].

However, the results of Agarwal et al. [21] cannot be applied when the conditions $\mu(t) > 0$ and $r^{\Delta}(t) \geq 0$ do not hold and the results of Saker [22] and Wu et al. [23] are also invalid when $0 < \lambda < 1$; likewise, the results in [21]–[23] cannot be applied when $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{1/\lambda}\Delta t < \infty$. Furthermore, the results in [21]–[23] cannot be applied to (1) when $\eta(t) \not\equiv 0$.

Therefore, it is of great interest to study the oscillation of (1) without necessarily assuming the conditions $\mu(t) > 0$, $r^{\Delta}(t) \geq 0$, $\lambda \geq 1$ and $\eta(t) \equiv 0$.

In this paper, we shall assume $\lambda > 0$ is a quotient of odd positive integers and remove the conditions $\mu(t) > 0$, $r^{\Delta}(t) \geq 0$ and $\eta(t) \equiv 0$. We shall consider two cases: the case when

$$\int_{t_0}^{\infty} \Big(\frac{1}{E(t)r(t)}\Big)^{1/\lambda}\Delta t = \infty \tag{3}$$

holds and the case when (3) does not hold, where

$$E(t) := \mathrm{e}_{\frac{\eta(t)}{r^{\sigma}(t)}}(t, t_0), \tag{4}$$

here $r^{\sigma}(t) := (r \circ \sigma)(t)$. We obtain some oscillation criteria for (1) by applying a generalized Riccati transformation technique. The results in this paper improve and extend the results of Agarwal et al. [21], Saker [22] and Wu et al. [23]. Several examples are provided to illustrate our main results.

In what follows, for convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

We shall need the following lemmas to prove our main results.

Lemma 1. *[Chen [18]] Suppose that (H_6) holds. Let $x : \mathbb{T} \to \mathbb{R}$. If $x^{\Delta}(t)$ exists for all sufficiently large $t \in \mathbb{T}$, then $(x \circ \delta)^{\Delta}(t) = (x^{\Delta} \circ \delta)(t)\delta^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$.*

Lemma 2. *[Chen [18]] Let $g : \mathbb{T} \to \mathbb{R}$ and $\alpha > 0$ be a constant. Furthermore, assume $g^{\Delta}(t) > 0$ and $g(t) > 0$ for all sufficiently large $t \in \mathbb{T}$. Then we have the following:*

(i) If $0 < \alpha < 1$, then $(g^{\alpha})^{\Delta}(t) \geq \alpha(g^{\sigma})^{\alpha-1}(t)g^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$, where $g^{\sigma} := g \circ \sigma$;

(ii) If $\alpha \geq 1$, then $(g^{\alpha})^{\Delta}(t) \geq \alpha g^{\alpha-1}(t)g^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$.

Lemma 3. *(Hardy et al. [19]) If X and Y are nonnegative, then*

$$\beta XY^{\beta-1} - X^{\beta} \leq (\beta - 1)Y^{\beta} \quad when\ \beta > 1,$$

where the equality holds if and only if X = Y.

For completeness, we recall the following concepts related to the notion of time scales. More details can be found in [3] and [4].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Some examples of time scales are as follows: the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the positive integers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, $[0,1] \cup [2,3]$, $[0,1] \cup \mathbb{N}$, $h\mathbb{Z} := \{hk : k \in \mathbb{Z}, h > 0\}$ and $\overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}, q > 1\} \cup \{0\}$. But the rational numbers $\mathbb{Q}$, the complex numbers $\mathbb{C}$ and the open interval $(0,1)$ are no time scales. Many other interesting time scales exist, and they give rise to plenty of applications (see [3]).

For $t \in \mathbb{T}$, the forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), here $\emptyset$ denotes the empty set.

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We also need below the set $\mathbb{T}^{\kappa}$: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$, then we define the function $f^{\sigma} : \mathbb{T}^{\kappa} \to \mathbb{R}$ by

$$f^{\sigma}(t) := f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}^{\kappa},$$

i.e., $f^{\sigma} := f \circ \sigma$.

For $a, b \in \mathbb{T}$ with $a < b$, we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals, etc. are defined accordingly.

Fix $t \in \mathbb{T}^{\kappa}$ and let $f : \mathbb{T} \to \mathbb{R}$. Define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, we say that $f^{\Delta}(t)$ is the (delta) derivative of f at t and that f is (delta) differentiable at t.

Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^{\kappa}$. If f is (delta) differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{5}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided it is continuous at each right-dense point in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of all such rd-continuous functions is denoted by

$$C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are (delta) differentiable and whose (delta) derivative is rd-continuous is denoted by

$$C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

We will make use of the following product and quotient rules for the (delta) derivatives of the product fg and the quotient f/g of two (delta) differentiable functions f and g:

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f g^\Delta + f^\Delta g^\sigma \tag{6}$$

and

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{g g^\sigma}, \tag{7}$$

where $g^\sigma = g \circ \sigma$ and $gg^\sigma \neq 0$.

For $a, b \in \mathbb{T}$ and a (delta) differentiable function f, the Cauchy (delta) integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f(t) g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t) g^\sigma(t)\Delta t \tag{8}$$

or

$$\int_a^b f^\sigma(t) g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t) g(t)\Delta t. \tag{9}$$

The infinite integral is defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s.$$

II. Main results

Theorem 1. *Suppose that (H_1)–(H_6) and (3) hold and that $\lambda > 0$ is a quotient of odd positive integers. Moreover, assume that there exists a positive function $\varphi \in C^1_{rd}(\mathbb{I}, \mathbb{R})$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left\{ \varphi(s)Q(s) - \frac{r(\delta(s))(\Phi_+(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\varphi(s)\delta^\Delta(s)]^\lambda} \right\} \Delta s = \infty, \tag{10}$$

where $Q(s) := q(s)[1 - p(\delta(s))]^\lambda$ and $\Phi_+(s) := \max\{0, \varphi^\Delta(s) - \frac{\varphi(s)\eta(s)}{r^\sigma(s)}\}$. Then every solution of (1) is oscillatory.

Proof. Suppose that y is a nonoscillatory solution of (1). Without loss of generality, we may assume that y is an eventually positive solution of (1). Define the function x as follow:

$$x(t) := y(t) + p(t) y(\tau(t)) \quad \text{for } t \in [t_0, \infty). \tag{11}$$

Then from (H_2)–(H_4), we get

$$x(t) \geq y(t) > 0 \quad \text{and} \quad y(\delta(t)) > 0. \tag{12}$$

Therefore, from (1), (H_5), (11) and (12), there exists $t_1 \in [t_0, \infty)$ such that for $t \in [t_1, \infty)$

$$\begin{aligned} \Big(r(t)(x^\Delta(t))^\lambda\Big)^\Delta + \eta(t)\Big(x^{\Delta\sigma}(t)\Big)^\lambda &= -f(t, y(\delta(t))) \\ &\leq -q(t)[y(\delta(t))]^\lambda \\ &< 0. \end{aligned} \tag{13}$$

In view of (4) and the fact that $E^\Delta(t) = E(t)\frac{\eta(t)}{r^\sigma(t)}$, from (13) and (6) we obtain

$$\begin{aligned} &\Big(E(t) r(t)(x^\Delta(t))^\lambda\Big)^\Delta \\ &= \Big(r(t)(x^\Delta(t))^\lambda\Big)^\Delta E(t) + \Big(r(t)(x^\Delta(t))^\lambda\Big)^\sigma E(t)\frac{\eta(t)}{r^\sigma(t)} \\ &= -E(t) f(t, y(\delta(t))) \\ &\leq -E(t) q(t)[y(\delta(t))]^\lambda \\ &< 0 \quad \text{for } t \in [t_1, \infty). \end{aligned} \tag{14}$$

Thus, we get that $E(t)r(t)(x^\Delta(t))^\lambda$ is strictly decreasing on $[t_1, \infty)$ and is eventually of one sign. We claim

$$x^\Delta(t) > 0 \quad \text{for } t \in [t_1, \infty). \tag{15}$$

If not, then there exists $t_2 \in [t_1, \infty)$ such that $x^\Delta(t_2) \leq 0$. Hence, from (H_1) we have $r(t_2)(x^\Delta(t_2))^\lambda \leq 0$. Take $t_3 > t_2$. Since $E(t)r(t)(x^\Delta(t))^\lambda$ is strictly decreasing on $[t_1, \infty)$, it is clear that $E(t_3)r(t_3)(x^\Delta(t_3))^\lambda < E(t_2)r(t_2)(x^\Delta(t_2))^\lambda$. Therefore, for $t \in [t_3, \infty)$ we have $E(t)r(t)(x^\Delta(t))^\lambda \leq E(t_3)r(t_3)(x^\Delta(t_3))^\lambda := c < 0$. Thus, we obtain $x^\Delta(t) \leq c^{\frac{1}{\lambda}}\big(\frac{1}{E(t)r(t)}\big)^{1/\lambda}$ for $t \in [t_3, \infty)$. By integrating both sides of the last inequality from t_3 to t, we get

$$x(t) - x(t_3) \leq c^{\frac{1}{\lambda}} \int_{t_3}^t \Big(\frac{1}{E(s)r(s)}\Big)^{1/\lambda} \Delta s \quad \text{for } t \in [t_3, \infty).$$

Noticing (3) and letting $t \to \infty$, we see $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the fact that $x(t) > 0$. Hence, (15) holds. Thus, from (13) we have for $t \in [t_1, \infty)$

$$\Big(r(t)(x^\Delta(t))^\lambda\Big)^\Delta \leq \Big(r(t)(x^\Delta(t))^\lambda\Big)^\Delta + \eta(t)\Big(x^{\Delta\sigma}(t)\Big)^\lambda < 0. \tag{16}$$

From (11), (12) and (H_2), we conclude

$$\begin{aligned} y(\delta(t)) &= x(\delta(t)) - p(\delta(t)) y(\tau(\delta(t))) \\ &\geq x(\delta(t)) - p(\delta(t)) x(\tau(\delta(t))). \end{aligned} \tag{17}$$

Since $\tau(\delta(t)) \leq \delta(t)$ and $x^\Delta(t) > 0$, we have $x(\tau(\delta(t))) \leq x(\delta(t))$. Therefore, from (17) we obtain

$$\begin{aligned} y(\delta(t)) &\geq x(\delta(t)) - p(\delta(t)) x(\delta(t)) \\ &= x(\delta(t))[1 - p(\delta(t))]. \end{aligned} \tag{18}$$

From (13) and (18), there exists $t_4 \in [t_3, \infty)$ such that

$$\begin{aligned} &[r(t)(x^\Delta(t))^\lambda]^\Delta \\ &\leq -\eta(t)(x^{\Delta\sigma}(t))^\lambda - [x(\delta(t))]^\lambda q(t)[1 - p(\delta(t))]^\lambda \end{aligned} \tag{19}$$

for $t \in [t_4, \infty)$. Define the function w by the generalized Riccati substitution

$$w(t) = \varphi(t)\frac{r(t)(x^\Delta(t))^\lambda}{[x(\delta(t))]^\lambda} \quad \text{for } t \in [t_4, \infty). \tag{20}$$

It is easy to see that there exists $t_5 \in [t_4, \infty)$ such that $w(t) > 0$ for $t \in [t_5, \infty)$. By the product rule (6) and the quotient rule

(7), from (20) we get

$$w^{\Delta} = [r(x^{\Delta})^{\lambda}]^{\Delta}\frac{\varphi}{(x\circ\delta)^{\lambda}} + [r(x^{\Delta})^{\lambda}]^{\sigma}\Big[\frac{\varphi}{(x\circ\delta)^{\lambda}}\Big]^{\Delta} = [r(x^{\Delta})^{\lambda}]^{\Delta}\frac{\varphi}{(x\circ\delta)^{\lambda}} + [r(x^{\Delta})^{\lambda}]^{\sigma}\Big[\frac{\varphi^{\Delta}}{(x\circ\delta^{\sigma})^{\lambda}} - \frac{\varphi[(x\circ\delta)^{\lambda}]^{\Delta}}{(x\circ\delta)^{\lambda}(x\circ\delta^{\sigma})^{\lambda}}\Big]. \quad (21)$$

Hence, from (19)–(21) we have

$$w^{\Delta} \le -\varphi\eta\frac{(x^{\Delta\sigma})^{\lambda}}{(x\circ\delta)^{\lambda}} - \varphi Q + \frac{\varphi^{\Delta}}{\varphi^{\sigma}}w^{\sigma} - \varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}[(x\circ\delta)^{\lambda}]^{\Delta}}{(x\circ\delta)^{\lambda}(x\circ\delta^{\sigma})^{\lambda}}, \quad (22)$$

where the function Q is defined as in Theorem 1. From (H_6) we see that $\delta(t)$ is increasing on $\mathbb{T}$. Since $t \le \sigma(t)$, we obtain $\delta(t) \le \delta^{\sigma}(t)$. In view of (15), we have

$$(x\circ\delta)(t) \le (x\circ\delta^{\sigma})(t). \quad (23)$$

From (22), (23) and (20), we get

$$w^{\Delta} \le -\varphi\eta\frac{(x^{\Delta\sigma})^{\lambda}}{(x\circ\delta^{\sigma})^{\lambda}} - \varphi Q + \frac{\varphi^{\Delta}}{\varphi^{\sigma}}w^{\sigma} - \varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}[(x\circ\delta)^{\lambda}]^{\Delta}}{(x\circ\delta)^{\lambda}(x\circ\delta^{\sigma})^{\lambda}} = -\varphi\eta\frac{w^{\sigma}}{\varphi^{\sigma}r^{\sigma}} - \varphi Q + \frac{\varphi^{\Delta}}{\varphi^{\sigma}}w^{\sigma} - \varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}[(x\circ\delta)^{\lambda}]^{\Delta}}{(x\circ\delta)^{\lambda}(x\circ\delta^{\sigma})^{\lambda}} \le -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}[(x\circ\delta)^{\lambda}]^{\Delta}}{(x\circ\delta)^{\lambda}(x\circ\delta^{\sigma})^{\lambda}}, \quad (24)$$

where the function Φ_{+} is defined as in Theorem 1. From (H_6), Lemma 1 and (15), we find

$$(x\circ\delta)^{\Delta} = (x^{\Delta}\circ\delta)\delta^{\Delta} > 0. \quad (25)$$

If $0 < \lambda < 1$, then by taking $g = x\circ\delta$ and $\alpha = \lambda$ and by Lemma 2 (i) and (25) we get

$$[(x\circ\delta)^{\lambda}]^{\Delta} \ge \lambda(x\circ\delta^{\sigma})^{\lambda-1}(x\circ\delta)^{\Delta} = \lambda(x\circ\delta^{\sigma})^{\lambda-1}(x^{\Delta}\circ\delta)\delta^{\Delta}. \quad (26)$$

It follows from (24) and (26) that for $0 < \lambda < 1$

$$w^{\Delta} \le -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}\cdot\lambda(x\circ\delta^{\sigma})^{\lambda-1}(x^{\Delta}\circ\delta)\delta^{\Delta}}{(x\circ\delta)^{\lambda}(x\circ\delta^{\sigma})^{\lambda}} = -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \lambda\delta^{\Delta}\varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}}{(x\circ\delta^{\sigma})^{\lambda+1}}\cdot\frac{(x\circ\delta^{\sigma})^{\lambda}}{(x\circ\delta)^{\lambda}}(x^{\Delta}\circ\delta). \quad (27)$$

If $\lambda \ge 1$, then by taking $g = x\circ\delta$ and $\alpha = \lambda$ and by Lemma 2 (ii) and (25) we have

$$[(x\circ\delta)^{\lambda}]^{\Delta} \ge \lambda(x\circ\delta)^{\lambda-1}(x\circ\delta)^{\Delta} = \lambda(x\circ\delta)^{\lambda-1}(x^{\Delta}\circ\delta)\delta^{\Delta}. \quad (28)$$

It follows from (24) and (28) that for $\lambda \ge 1$

$$w^{\Delta} \le -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}\cdot\lambda(x\circ\delta)^{\lambda-1}(x^{\Delta}\circ\delta)\delta^{\Delta}}{(x\circ\delta)^{\lambda}(x\circ\delta^{\sigma})^{\lambda}} = -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \lambda\delta^{\Delta}\varphi\frac{[r(x^{\Delta})^{\lambda}]^{\sigma}}{(x\circ\delta^{\sigma})^{\lambda+1}}\cdot\frac{(x\circ\delta^{\sigma})}{(x\circ\delta)}(x^{\Delta}\circ\delta). \quad (29)$$

Therefore, from (23), (27) and (29) we get for all $\lambda > 0$

$$w^{\Delta} \le -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \lambda\delta^{\Delta}\varphi\frac{\big(r(x^{\Delta})^{\lambda}\big)^{\sigma}}{(x\circ\delta^{\sigma})^{\lambda+1}}(x^{\Delta}\circ\delta). \quad (30)$$

Form (16), we see that $r(t)(x^{\Delta}(t))^{\lambda}$ is decreasing on $[t_1,\infty)$. Since $\delta(t) \le t \le \sigma(t)$, we have $((r\circ\delta)(x^{\Delta}\circ\delta)^{\lambda})(t) \ge \big(r(x^{\Delta})^{\lambda}\big)^{\sigma}(t)$. Thus, we conclude

$$(x^{\Delta}\circ\delta)(t) \ge \frac{[\big(r(x^{\Delta})^{\lambda}\big)^{\sigma}]^{1/\lambda}(t)}{[(r\circ\delta)]^{1/\lambda}(t)}.$$

Hence, from (30) we obtain

$$w^{\Delta} \le -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \lambda\delta^{\Delta}\varphi\frac{[\big(r(x^{\Delta})^{\lambda}\big)^{\sigma}]^{1+\frac{1}{\lambda}}}{(x\circ\delta^{\sigma})^{\lambda+1}}(r\circ\delta)^{-\frac{1}{\lambda}}. \quad (31)$$

From (20) and (31) there exists $t_6 \in [t_5,\infty)$ such that

$$w^{\Delta}(t) \le -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \lambda\delta^{\Delta}\varphi\Big(\frac{w^{\sigma}}{\varphi^{\sigma}}\Big)^{1+\frac{1}{\lambda}}(r\circ\delta)^{-\frac{1}{\lambda}} = -\varphi Q + \frac{\Phi_{+}}{\varphi^{\sigma}}w^{\sigma} - \lambda\varphi\delta^{\Delta}(r\circ\delta)^{-\frac{1}{\lambda}}(\varphi^{\sigma})^{-\frac{\lambda+1}{\lambda}}(w^{\sigma})^{\frac{\lambda+1}{\lambda}} \quad (32)$$

for $t \in [t_6,\infty)$. Taking $\beta = \frac{\lambda+1}{\lambda}, X = (\lambda\varphi\delta^{\Delta})^{\frac{1}{\beta}}(r\circ\delta)^{-\frac{1}{\lambda\beta}}(\varphi^{\sigma})^{-1}w^{\sigma}$ and $Y = (r\circ\delta)^{\frac{1}{\beta}}(\Phi_{+})^{\lambda}/[\beta^{\lambda}\lambda^{\frac{\lambda}{\beta}}(\varphi\delta^{\Delta})^{\frac{\lambda}{\beta}}]$, by Lemma 3 and (32) we get for $t \in [t_6,\infty)$

$$w^{\Delta}(t) \le -\varphi(t)Q(t) + \frac{r(\delta(t))(\Phi_{+}(t))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\varphi(t)\delta^{\Delta}(t)]^{\lambda}}.$$

Integrating both sides of the last inequality from t_6 to t ($t \ge t_6$), we obtain

$$w(t) - w(t_6) \le -\int_{t_6}^{t}\Big\{\varphi(s)Q(s) - \frac{r(\delta(s))(\Phi_{+}(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\varphi(s)\delta^{\Delta}(s)]^{\lambda}}\Big\}\Delta s.$$

Since $w(t) > 0$ for $t \in [t_5,\infty)$, we have

$$\int_{t_6}^{t}\Big\{\varphi(s)Q(s) - \frac{r(\delta(s))(\Phi_{+}(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\varphi(s)\delta^{\Delta}(s)]^{\lambda}}\Big\}\Delta s \le w(t_6) - w(t) < w(t_6) \quad \text{for } t \in [t_6,\infty).$$

Thus, we get

$$\limsup_{t\to\infty}\int_{t_6}^{t}\Big\{\varphi(s)Q(s) - \frac{r(\delta(s))(\Phi_{+}(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\varphi(s)\delta^{\Delta}(s)]^{\lambda}}\Big\}\Delta s \le w(t_6) < \infty,$$

which contraticts (10). Hence, the proof is complete. □

The following theorem gives a Philos-type oscillation criterion for (1).

Theorem 2. *Assume that $\lambda > 0$ is a quotient of odd positive integers and that (H_1)–(H_6) and (3) hold. Furthermore, suppose that there exist a positive function $\varphi \in C^1_{rd}(\mathbb{I}, \mathbb{R})$ and a function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0\}$, such that*

$$H(t,t) = 0 \quad for \quad t \geq t_0, \quad H(t,s) > 0 \quad for \quad (t,s) \in \mathbb{D}_0,$$

where $\mathbb{D}_0 := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t > s \geq t_0\}$, and H has a nonpositive rd-continuous delta partial derivative $H^{\Delta_s}(t, s)$ on $\mathbb{D}_0$ with respect to the second variable and satisfies

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t-1} \Big\{ H(t,s)\varphi(s)Q(s) - \frac{r(\delta(s))[\varphi(\sigma(s))h_+(t,s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[H(t,s)\varphi(s)\delta^{\Delta}(s)]^{\lambda}} \Big\} \Delta s = \infty, \tag{33}$$

where $Q(s) := q(s)[1 - p(\delta(s))]^{\lambda}$, $\sigma(t)$ is the forward jump operator on $\mathbb{T}$ and $h_+(t,s) := \max\{0, H^{\Delta_s}(t,s) + H(t,s)\frac{\Phi_+(s)}{\varphi^{\sigma}(s)}\}$, here $\Phi_+(s) := \max\{0, \varphi^{\Delta}(s) - \frac{\varphi(s)\eta(s)}{r^{\sigma}(s)}\}$. Then all solutions of (1) are oscillatory.

Proof. Assume that y is a nonoscillatory solution of (1). Without loss of generality, assume that y is an eventually positive solution of (1). Define again the functions x by (11) and w by (20). Proceeding as in the proof of Theorem 1, we see that (32) holds. Multiplying (32) by $H(t, s)$ and integrating from t_6 to $t - 1$ ($t \geq t_6 + 1$), we find

$$\begin{aligned}\int_{t_6}^{t-1} & H(t,s)\varphi(s)Q(s)\Delta s \\ &\leq -\int_{t_6}^{t-1} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_6}^{t-1} H(t,s)\frac{\Phi_+(s)}{\varphi^{\sigma}(s)}w^{\sigma}(s)\Delta s \\ &\quad - \int_{t_6}^{t-1} H(t,s)V(s)(w^{\sigma}(s))^{\frac{\lambda+1}{\lambda}}\Delta s,\end{aligned} \tag{34}$$

where $V(s) := \lambda\varphi(s)\delta^{\Delta}(s)[r(\delta(s))]^{-\frac{1}{\lambda}}(\varphi^{\sigma}(s))^{-\frac{\lambda+1}{\lambda}}$. Applying the integration by parts formula (8), we get

$$\begin{aligned} -\int_{t_6}^{t-1} & H(t,s)w^{\Delta}(s)\Delta s \\ &= \Big[-H(t,s)w(s)\Big]_{s=t_6}^{s=t-1} + \int_{t_6}^{t-1} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \\ &< H(t,t_6)w(t_6) + \int_{t_6}^{t-1} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s.\end{aligned} \tag{35}$$

Substituting (35) in (34), we obtain

$$\begin{aligned}\int_{t_6}^{t-1} & H(t,s)\varphi(s)Q(s)\Delta s \\ &< H(t,t_6)w(t_6) \\ &\quad + \int_{t_6}^{t-1}\Big\{\Big[H^{\Delta_s}(t,s) + H(t,s)\frac{\Phi_+(s)}{\varphi^{\sigma}(s)}\Big]w^{\sigma}(s) \\ &\quad - H(t,s)V(s)(w^{\sigma}(s))^{\frac{\lambda+1}{\lambda}}\Big\}\Delta s \\ &\leq H(t,t_6)w(t_6) + \int_{t_6}^{t-1}\Big[h_+(t,s)w^{\sigma}(s) \\ &\quad - H(t,s)V(s)(w^{\sigma}(s))^{\frac{\lambda+1}{\lambda}}\Big]\Delta s,\end{aligned} \tag{36}$$

where $h_+(t, s)$ is defined as in Theorem 2. Taking $\beta = \frac{\lambda+1}{\lambda}$, $X = [H(t,s)V(s)]^{\frac{1}{\beta}}w^{\sigma}(s)$ and $Y = [h_+(t,s)]^{\lambda}/\{\beta^{\lambda}[H(t,s)V(s)]^{\frac{\lambda}{\beta}}\}$, by Lemma 3 and (36) we have

$$\begin{aligned}\int_{t_6}^{t-1} & H(t,s)\varphi(s)Q(s)\Delta s \\ &< H(t,t_6)w(t_6) + \int_{t_6}^{t-1}\Theta(t,s)\Delta s,\end{aligned} \tag{37}$$

where

$$\Theta(t,s) := \frac{[h_+(t,s)]^{\lambda\beta}}{\lambda\beta^{\lambda\beta}[H(t,s)V(s)]^{\lambda}} = \frac{r(\delta(s))[\varphi(\sigma(s))h_+(t,s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[H(t,s)\varphi(s)\delta^{\Delta}(s)]^{\lambda}}.$$

Since $H^{\Delta_s}(t,s) \leq 0$ on $\mathbb{D}_0$, we obtain $H(t,t_6) \leq H(t,t_0)$ for $t > t_6 \geq t_0$. Hence, for $t > t_6 + 1 \geq t_0 + 1$, it follows from (37) that

$$\begin{aligned}\int_{t_6}^{t-1} & \Big[H(t,s)\varphi(s)Q(s) - \Theta(t,s)\Big]\Delta s \\ &< H(t,t_6)w(t_6) \leq H(t,t_0)w(t_6).\end{aligned} \tag{38}$$

For $t > s \geq t_0$, we have $0 < H(t,s) \leq H(t,t_0)$ and $0 < \frac{H(t,s)}{H(t,t_0)} \leq 1$. Thus, from (38) we get

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{t_0}^{t-1}\Big[H(t,s)\varphi(s)Q(s) - \Theta(t,s)\Big]\Delta s \\ &= \frac{1}{H(t,t_0)}\Big(\int_{t_0}^{t_6} + \int_{t_6}^{t-1}\Big)\Big[H(t,s)\varphi(s)Q(s) - \Theta(t,s)\Big]\Delta s \\ &< \int_{t_0}^{t_6}\frac{H(t,s)}{H(t,t_0)}\varphi(s)Q(s)\Delta s + w(t_6) \\ &\leq \int_{t_0}^{t_6}\varphi(s)Q(s)\Delta s + w(t_6) \quad \text{for } t > t_6 + 1 \geq t_0 + 1.\end{aligned}$$

Therefore, we find

$$\begin{aligned}\limsup_{t\to\infty} & \frac{1}{H(t,t_0)}\int_{t_0}^{t-1}\Big[H(t,s)\varphi(s)Q(s) - \Theta(t,s)\Big]\Delta s \\ &\leq \int_{t_0}^{t_6}\varphi(s)Q(s)\Delta s + w(t_6) < \infty,\end{aligned}$$

which implies a contradiction to (33). Thus, this completes the proof. □

Let $H(t, s) = (t - s)^m, (t, s) \in \mathbb{D}$, where $m \geq 1$ is a constant, then $H^{\Delta_s}(t, s) \leq -m(t - \sigma(s))^{m-1}$ for $(t, s) \in \mathbb{D}_0$ (see Saker [22]). Therefore, from (35) we obtain

$$\begin{aligned} -\int_{t_6}^{t-1} & H(t,s)w^{\Delta}(s)\Delta s \\ &< H(t,t_6)w(t_6) + \int_{t_6}^{t-1}[-m(t-\sigma(s))^{m-1}]w^{\sigma}(s)\Delta s.\end{aligned} \tag{39}$$

By replacing (35) with (39) and using methods similar to those of the proof of Theorem 2, we obtain the following Kamenev-type oscillation criterion for (1).

Theorem 3. *Assume that $\lambda > 0$ is a quotient of odd positive integers and that (H_1)–(H_6) and (3) hold. Furthermore, suppose that there exist a constant $m \geq 1$ and a positive function $\varphi \in C^1_{rd}(\mathbb{I}, \mathbb{R})$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t-1} \Big\{ (t-s)^m \varphi(s) Q(s) - \frac{r(\delta(s))\big[\varphi(\sigma(s))K_+(t,s)\big]^{\lambda+1}}{(\lambda+1)^{\lambda+1}\big[(t-s)^m \varphi(s)\delta^\Delta(s)\big]^\lambda} \Big\} \Delta s = \infty, \quad (40)$$

where $Q(s) := q(s)[1-p(\delta(s))]^\lambda$, $\sigma(t)$ is the forward jump operator on $\mathbb{T}$ and $K_+(t,s) := \max\{0, (t-s)^m \frac{\Phi_+(s)}{\varphi(\sigma(s))} - m(t-\sigma(s))^{m-1}\}$, here $\Phi_+(s) := \max\{0, \varphi^\Delta(s) - \frac{\varphi(s)\eta(s)}{r^\sigma(s)}\}$. Then all solutions of (1) are oscillatory.

Remark 1. From Theorems 1–3, we can obtain many different sufficient conditions for the oscillation of (1) with different choices of $\varphi(t)$, $H(t,s)$ and m.

For example, let $\varphi(s) = s$, then Theorem 1 yields the following results.

Corollary 1. *Suppose that (H_1)–(H_6) and (3) hold and that $\lambda > 0$ is a quotient of odd positive integers. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[sQ(s) - \frac{r(\delta(s))(\xi_+(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[s\delta^\Delta(s)]^\lambda} \Big] \Delta s = \infty, \quad (41)$$

where $Q(s) := q(s)[1-p(\delta(s))]^\lambda$ and $\xi_+(s) := \max\{0, 1 - \frac{s\eta(s)}{r^\sigma(s)}\}$, then every solution of (1) is oscillatory.

Let $\varphi(s) = 1$, then Theorem 1 yields the following well-known Leighton-Winter theorem.

Corollary 2. *Assume that (H_1)–(H_6) and (3) hold and that $\lambda > 0$ is a quotient of odd positive integers. If*

$$\int_{t_0}^{\infty} q(s)[1-p(\delta(s))]^\lambda \Delta s = \infty, \quad (42)$$

then every solution of (1) is oscillatory.

Let $\varphi(s) = 1$, then from Theorem 3 we obtain the following results.

Corollary 3. *Suppose that $\lambda > 0$ is a quotient of odd positive integers and that (H_1)–(H_6) and (3) hold. Furthermore, suppose that there exists a constant $m \geq 1$ such that*

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t-1} (t-s)^m q(s)[1-p(\delta(s))]^\lambda \Delta s = \infty.$$

Then all solutions of (1) are oscillatory.

Next, we consider the case when

$$\int_{t_0}^{\infty} \Big(\frac{1}{E(t)r(t)} \Big)^{1/\lambda} \Delta t < \infty \quad (43)$$

holds, where $E(t)$ is defined as in (4). Obviously, (43) implies that (3) does not hold.

Theorem 4. *Suppose that (H_1)–(H_6) and (43) hold and that $\lambda > 0$ is a quotient of odd positive integers. Let $\varphi(t)$ be defined as in Theorem 1 such that (10) holds. Moreover, assume that there exists a constant $\varepsilon > 0$ such that for every constant $C \geq t_0$,*

$$\limsup_{t\to\infty} \int_C^t \Big\{ \frac{1}{E(u)r(u)} \int_C^u E(s)q(s) \cdot [1-(1+\varepsilon)p(\delta(s))]^\lambda \Delta s \Big\}^{1/\lambda} \Delta u = \infty. \quad (44)$$

Then every solution of (1) is oscillatory or converges to zero as $t \to \infty$.

Proof. Assume that y is a nonoscillatory solution of (1). Without loss of generality, assume that y is an eventually positive solution of (1). Define again the functions x by (11) and w by (20). There are two cases for the sign of $x^\Delta(t)$. The proof when $x^\Delta(t)$ is eventually positive is similar to that of Theorem 1 and hence is omitted.

Next, assume $x^\Delta(t) \leq 0$. Proceeding as in the proof of Theorem 1, we obtain that (12) and (14) hold. Thus we get $\lim_{t\to\infty} x(t) := L \geq 0$ and $x(t) \geq L$. We claim $L = 0$. Assume not, i.e., $L > 0$, then we now show that this leads to a contradiction to (44). By the properties of limit, for $\varepsilon > 0$ we have $L \leq x(t) < (1+\varepsilon)L$. Hence, from ($H_3$) and ($H_4$) we get $L \leq x(\delta(t)) < (1+\varepsilon)L$ and $L \leq x(\tau(\delta(t))) < (1+\varepsilon)L$. Then from ($H_2$)–($H_4$), (11) and (12) we obtain

$$\begin{aligned} y(\delta(t)) &= x(\delta(t)) - p(\delta(t))y(\tau(\delta(t))) \\ &\geq x(\delta(t)) - p(\delta(t))x(\tau(\delta(t))) \\ &\geq L[1-(1+\varepsilon)p(\delta(t))]. \end{aligned} \quad (45)$$

Take $t_7 \in [t_0, \infty)$ such that $x^\Delta(t) \leq 0$ for $t \in [t_7, \infty)$. From (14) and (45), there exists $t_8 \in [t_7, \infty)$ such that

$$[E(t)r(t)(x^\Delta(t))^\lambda]^\Delta \leq -L^\lambda E(t)q(t)[1-(1+\varepsilon)p(\delta(t))]^\lambda$$

for $t \in [t_8, \infty)$. Integrating both sides of the last inequality from t_8 to t ($t \geq t_8$), we have

$$\begin{aligned} & E(t)r(t)(x^\Delta(t))^\lambda \\ & \leq E(t_8)r(t_8)(x^\Delta(t_8))^\lambda \\ & \quad - L^\lambda \int_{t_8}^t E(s)q(s)[1-(1+\varepsilon)p(\delta(s))]^\lambda \Delta s \\ & \leq -L^\lambda \int_{t_8}^t E(s)q(s)[1-(1+\varepsilon)p(\delta(s))]^\lambda \Delta s. \end{aligned}$$

Therefore, we get

$$x^\Delta(t) \leq -L\Big\{ \frac{1}{E(t)r(t)} \int_{t_8}^t E(s)q(s) \cdot [1-(1+\varepsilon)p(\delta(s))]^\lambda \Delta s \Big\}^{1/\lambda} \quad \text{for } t \in [t_8, \infty).$$

Integrating both sides of the last inequality from t_8 to t, we obtain for $t \in [t_8, \infty)$

$$x(t) \le x(t_8) - L \int_{t_8}^{t} \Big\{ \frac{1}{E(u)r(u)} \int_{t_8}^{u} E(s)q(s) \cdot [1-(1+\varepsilon)p(\delta(s))]^\lambda \Delta s \Big\}^{1/\lambda} \Delta u \quad \text{for } t \in [t_8, \infty).$$

Hence, we have

$$\int_{t_8}^{t} \Big\{ \frac{1}{E(u)r(u)} \int_{t_8}^{u} E(s)q(s)[1-(1+\varepsilon)p(\delta(s))]^\lambda \Delta s \Big\}^{\frac{1}{\lambda}} \Delta u \le \frac{x(t_8)-x(t)}{L} < \frac{x(t_8)}{L} \quad \text{for } t \in [t_8, \infty)$$

and

$$\limsup_{t\to\infty} \int_{t_8}^{t} \Big\{ \frac{1}{E(u)r(u)} \int_{t_8}^{u} E(s)q(s) \cdot [1-(1+\varepsilon)p(\delta(s))]^\lambda \Delta s \Big\}^{1/\lambda} \Delta u \le \frac{x(t_8)}{L} < \infty,$$

which yields a contradiction to (44). Therefore we have $L = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$. In view of (12), we obtain $\lim_{t\to\infty} y(t) = 0$. The proof is complete. □

Theorem 5. *Suppose that* (H_1)–(H_6) *and (43) hold and that* $\lambda > 0$ *is a quotient of odd positive integers. Let* $\varphi(t)$ *and* $H(t,s)$ *be defined as in Theorem 2 such that (33) holds. Moreover, assume that there exists a constant* $\varepsilon > 0$ *such that for every constant* $C \ge t_0$*, (44) holds. Then every solution of (1) is oscillatory or tends to zero as* $t \to \infty$*.*

Proof. Assume that y is a nonoscillatory solution of (1). Without loss of generality, assume that y is an eventually positive solution of (1). Define again the functions x by (11) and w by (20). There are two cases for the sign of $x^\Delta(t)$. The proof when $x^\Delta(t)$ is eventually positive is similar to that of Theorem 2 and hence is omitted.

Next, assume $x^\Delta(t) \le 0$. In this case, the proof is similar to that of the proof of Theorem 4 and therefore is omitted. The proof is complete. □

Theorem 6. *Suppose that* (H_1)–(H_6) *and (43) hold and that* $\lambda > 0$ *is a quotient of odd positive integers. Let* m *and* $\varphi(t)$ *be defined as in Theorem 3 such that (40) holds. Moreover, assume that there exists a constant* $\varepsilon > 0$ *such that for every constant* $C \ge t_0$*, (44) holds. Then every solution of (1) is oscillatory or tends to zero as* $t \to \infty$*.*

Proof. The proof is similar to that of Theorem 5 and thus is omitted. The proof is complete. □

III. Examples

Example 1. Consider the following second-order neutral damped variable delay dynamic equation

$$\Big(\frac{t^{\lambda-1}}{\phi(t)}\Big(\Big[y(t) + \frac{1}{1+e^t} y(\tau(t))\Big]^\Delta\Big)^\lambda\Big)^\Delta + \frac{(\sigma(t))^{\lambda-1}}{t^2 \phi^\sigma(t)}\Big(\Big[y(t) + \frac{1}{1+e^t} y(\tau(t))\Big]^{\Delta\sigma}\Big)^\lambda + \alpha \frac{[1+e^{-\delta(t)}]^\lambda}{t^2}\Big(y(\delta(t))\Big)^\lambda = 0 \quad \text{for } t \in \mathbb{T}. \tag{46}$$

In (46), $\lambda > 0$ is a quotient of odd positive integers, $\phi(t) = e_{\frac{1}{t^2}}(t, t_0)$, $\phi^\sigma(t) := (\phi \circ \sigma)(t)$, $r(t) = \frac{t^{\lambda-1}}{\phi(t)}$, $\eta(t) = \frac{(\sigma(t))^{\lambda-1}}{t^2 \phi^\sigma(t)}$, $p(t) = \frac{1}{1+e^t}$, $\alpha > 0$ is a constant and $f(t,y) = \alpha \frac{[1+e^{-\delta(t)}]^\lambda}{t^2} y^\lambda$.

If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$, $\delta(t) = t - hk_0$ and $\tau(t) = t - hk_1$, where $h > 0$ is a constant and k_0, k_1 are arbitrary positive integers, then by taking $t_0 = h(k_0+1)$ and $q(t) = \alpha \frac{[1+e^{-\delta(t)}]^\lambda}{t^2}$ we see that (H_1)–(H_6) and (3) hold. To apply Corollary 1, it remains to satisfy the condition (41). We have

$$Q(s) := q(s)[1-p(\delta(s))]^\lambda = \alpha \frac{[1+e^{-\delta(s)}]^\lambda}{s^2[1+e^{-\delta(s)}]^\lambda} = \frac{\alpha}{s^2}.$$

Since $E(t) := e_{\frac{\eta(t)}{r^\sigma(t)}}(t, t_0) = e_{\frac{1}{t^2}}(t, t_0)$, we get $E^\Delta(t) = \frac{1}{t^2} E(t) > 0$ for $t \in [t_0, \infty)$. Thus, we obtain that $E(t)$ is increasing on $[t_0, \infty)$. Hence, for $t \in [t_0, \infty)$ we have $E(t) \ge E(t_0) = 1$ and $0 < \frac{1}{E(t)} \le 1$. Since $\delta(t) \ge t_0$ for $t \ge (2k_0+1)h$, we get $0 < r(\delta(t)) = \frac{(\delta(t))^{\lambda-1}}{E(\delta(t))} \le (\delta(t))^{\lambda-1}$ for $t \ge (2k_0+1)h$. Furthermore, we have

$$0 \le \xi_+(s) := \max\{0, 1 - \frac{s\eta(s)}{r^\sigma(s)}\} \le 1 \quad \text{for } s \in [t_0, \infty).$$

Therefore, we obtain

$$\limsup_{t\to\infty} \int_{(2k_0+1)h}^{t} \Big[sQ(s) - \frac{r(\delta(s))(\xi_+(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[s\delta^\Delta(s)]^\lambda}\Big]\Delta s \ge \limsup_{t\to\infty} \int_{(2k_0+1)h}^{t} \Big[sQ(s) - \frac{(\delta(s))^{\lambda-1}}{(\lambda+1)^{\lambda+1}[s\delta^\Delta(s)]^\lambda}\Big]\Delta s = \limsup_{t\to\infty} \int_{(2k_0+1)h}^{t} \Big(\frac{\alpha}{s} - \frac{(s-hk_0)^{\lambda-1}}{(\lambda+1)^{\lambda+1}s^\lambda}\Big)\Delta s. \tag{47}$$

Let $\alpha - \frac{1}{(\lambda+1)^{\lambda+1}} > 0$. Since $\lim_{s\to\infty}\Big(\frac{\alpha}{s} - \frac{(s-hk_0)^{\lambda-1}}{(\lambda+1)^{\lambda+1}s^\lambda}\Big)/(\frac{1}{s}) = \alpha - \frac{1}{(\lambda+1)^{\lambda+1}}$ and $\int_{(2k_0+1)h}^{\infty} \frac{1}{s}\Delta s = \infty$, we have $\int_{(2k_0+1)h}^{\infty}\Big(\frac{\alpha}{s} - \frac{(s-hk_0)^{\lambda-1}}{(\lambda+1)^{\lambda+1}s^\lambda}\Big)\Delta s = \infty$. Therefore, from (47) we obtain

$$\limsup_{t\to\infty} \int_{(2k_0+1)h}^{t} \Big[sQ(s) - \frac{r(\delta(s))(\xi_+(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[s\delta^\Delta(s)]^\lambda}\Big]\Delta s = \infty,$$

which implies that (41) holds. Hence, by Corollary 1 every solution of (46) is oscillatory when $\alpha - \frac{1}{(\lambda+1)^{\lambda+1}} > 0$.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}}$, $\delta(t) = {q_0}^{-k_0} t$ and $\tau(t) = {q_0}^{-k_1} t$, where $\overline{q_0^{\mathbb{Z}}} := q_0^{\mathbb{Z}} \cup \{0\}$, $q_0^{\mathbb{Z}} := \{q_0^k : k \in \mathbb{Z}\}$, $q_0 > 1$ is a constant and k_0, k_1 are arbitrary positive integers, then by taking $t_0 = q_0$ and $q(t) = \alpha \frac{[1+e^{-\delta(t)}]^\lambda}{t^2}$ we see that (H_1)–(H_6) and (3) hold. To apply Corollary 1, it remains to satisfy the condition (41).

Since $\delta(t) \geq t_0$ for $t \geq {q_0}^{1+k_0}$, we obtain $0 < r(\delta(t)) = \frac{(\delta(t))^{\lambda-1}}{E(\delta(t))} \leq (\delta(t))^{\lambda-1}$ for $t \geq {q_0}^{1+k_0}$. Thus, if $\alpha - \frac{{q_0}^{k_0}}{(\lambda+1)^{\lambda+1}} > 0$, then we obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{{q_0}^{1+k_0}}^{t}\Big[sQ(s)-\frac{r(\delta(s))(\xi_+(s))^{\lambda+1}}{(\lambda+1)^{\lambda+1}[s\delta^{\Delta}(s)]^{\lambda}}\Big]\Delta s\\
&\geq \limsup_{t\to\infty}\int_{{q_0}^{1+k_0}}^{t}\Big[sQ(s)-\frac{(\delta(s))^{\lambda-1}}{(\lambda+1)^{\lambda+1}[s\delta^{\Delta}(s)]^{\lambda}}\Big]\Delta s\\
&= \limsup_{t\to\infty}\int_{{q_0}^{1+k_0}}^{t}\Big(s\frac{\alpha}{s^2}-\frac{({q_0}^{-k_0}s)^{\lambda-1}}{(\lambda+1)^{\lambda+1}(s{q_0}^{-k_0})^{\lambda}}\Big)\Delta s\\
&= \limsup_{t\to\infty}\int_{{q_0}^{1+k_0}}^{t}\Big(\alpha-\frac{{q_0}^{k_0}}{(\lambda+1)^{\lambda+1}}\Big)\frac{1}{s}\Delta s=\infty,
\end{aligned}$$

which yields that (41) holds. Therefore, by Corollary 1 every solution of (46) is oscillatory when $\alpha - \frac{{q_0}^{k_0}}{(\lambda+1)^{\lambda+1}} > 0$.

Example 2. Consider the following second-order neutral damped variable delay dynamic equation

$$\begin{aligned}
&\Big(\frac{t^{\lambda-1}}{\phi(t)}\Big(\Big[y(t)+\frac{t^2}{1+t^2}y(\tau(t))\Big]^{\Delta}\Big)^{\lambda}\Big)^{\Delta}\\
&\quad+\frac{t^3(\sigma(t))^{\lambda-1}}{\phi^{\sigma}(t)}\Big(\Big[y(t)+\frac{t^2}{1+t^2}y(\tau(t))\Big]^{\Delta\sigma}\Big)^{\lambda}\\
&\quad+\alpha\frac{[1+\delta^2(t)]^{\lambda}}{t}\Big(y(\delta(t))\Big)^{\lambda}=0 \quad \text{for } t\in\mathbb{T}. \qquad (48)
\end{aligned}$$

In (48), $\lambda > 0$ is a quotient of odd positive integers, $\phi(t) = e_{t^3}(t,t_0)$, $\phi^{\sigma}(t) := (\phi\circ\sigma)(t)$, $r(t) = \frac{t^{\lambda-1}}{\phi(t)}$, $\eta(t) = \frac{t^3(\sigma(t))^{\lambda-1}}{\phi^{\sigma}(t)}$, $p(t) = \frac{t^2}{1+t^2}$, $\alpha > 0$ is a constant and $f(t,y) = \alpha\frac{[1+\delta^2(t)]^{\lambda}}{t}y^{\lambda}$.

If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, $\delta(t) = t - hk_0$ and $\tau(t) = t - hk_1$, where $h\mathbb{Z}, k_0$ and k_1 are defined as in Example 1, then by taking $t_0 = h(k_0+1)$ and $q(t) = \alpha\frac{[1+\delta^2(t)]^{\lambda}}{t}$ we see that (H_1)–(H_6) and (3) hold. We will apply Corollary 2. It remains to satisfy the condition (42). We have

$$\begin{aligned}
\int_{t_0}^{\infty}q(s)[1-p(\delta(s))]^{\lambda}\Delta s &= \int_{t_0}^{\infty}\alpha\frac{[1+\delta^2(s)]^{\lambda}}{s[1+\delta^2(s)]^{\lambda}}\Delta s\\
&= \int_{t_0}^{\infty}\frac{\alpha}{s}\Delta s=\infty, \qquad (49)
\end{aligned}$$

which implies that (42) holds. Hence, by Corollary 2 every solution of (48) is oscillatory.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}}$, $\delta(t) = {q_0}^{-k_0}t$ and $\tau(t) = {q_0}^{-k_1}t$, where $\overline{q_0^{\mathbb{Z}}}, k_0$ and k_1 are defined as in Example 1, then by taking $t_0 = q_0$ and $q(t) = \alpha\frac{[1+\delta^2(t)]^{\lambda}}{t}$ we see that (H_1)–(H_6) and (3) hold. We will apply Corollary 2. In this case, we still have (49), which implies that (42) holds. Thus, by Corollary 2 every solution of (48) is oscillatory.

Example 3. Consider the following second-order neutral damped variable delay dynamic equation

$$\begin{aligned}
&\Big(\frac{t^{\lambda+1}}{\phi(t)}\Big(\Big[y(t)+\frac{3}{4}(1-\frac{1}{\sqrt{1+t^2}})y(\tau(t))\Big]^{\Delta}\Big)^{\lambda}\Big)^{\Delta}\\
&\quad+\frac{t^2(\sigma(t))^{\lambda+1}}{\phi^{\sigma}(t)}\Big(\Big[y(t)+\frac{3}{4}(1-\frac{1}{\sqrt{1+t^2}})y(\tau(t))\Big]^{\Delta\sigma}\Big)^{\lambda}\\
&\quad+(t+\alpha)^{\lambda}\Big(y(\delta(t))\Big)^{\lambda}=0 \quad \text{for } t\in\mathbb{T}. \qquad (50)
\end{aligned}$$

In (50), $\lambda > 0$ is a quotient of odd positive integers, $\phi(t) = e_{t^2}(t,t_0)$, $\phi^{\sigma}(t) := (\phi\circ\sigma)(t)$, $r(t) = \frac{t^{\lambda+1}}{\phi(t)}$, $\eta(t) = \frac{t^2(\sigma(t))^{\lambda+1}}{\phi^{\sigma}(t)}$, $p(t) = \frac{3}{4}(1-\frac{1}{\sqrt{1+t^2}})$, $\alpha > 0$ is a constant and $f(t,y) = (t+\alpha)^{\lambda}y^{\lambda}$.

If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, $\delta(t) = t - hk_0$ and $\tau(t) = t - hk_1$, where $h\mathbb{Z}, k_0$ and k_1 are defined as in Example 1, then by taking $t_0 = h(k_0+1)$ and $q(t) = (t+\alpha)^{\lambda}$ we see that (H_1)–(H_6) and (43) hold. We will apply Theorem 4. It remains to satisfy the conditions (10) and (44). Let $\alpha \geq \sqrt{(hk_0)^2+1}$. Taking $\varphi(s) = 1$, then the left side of (10) becomes

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_0}^{t}Q(s)\Delta s\\
&= \limsup_{t\to\infty}\int_{t_0}^{t}q(s)[1-p(\delta(s))]^{\lambda}\Delta s\\
&= \limsup_{t\to\infty}\int_{t_0}^{t}(s+\alpha)^{\lambda}\Big[\frac{1}{4}+\frac{3}{4}\frac{1}{\sqrt{1+(s-hk_0)^2}}\Big]^{\lambda}\Delta s\\
&\geq \big(\frac{3}{4}\big)^{\lambda}\limsup_{t\to\infty}\int_{t_0}^{t}(s+\alpha)^{\lambda}\Big[\frac{1}{\sqrt{1+(s-hk_0)^2}}\Big]^{\lambda}\Delta s\\
&\geq \big(\frac{3}{4}\big)^{\lambda}\limsup_{t\to\infty}\int_{t_0}^{t}\Delta s=\infty, \qquad (51)
\end{aligned}$$

which implies that (10) holds. Take $\varepsilon = \frac{1}{3}$. For any constant $C \geq t_0$, Take $M \in \mathbb{T}$ such that $M \geq 2C$. Since $E(t) := e_{\frac{\eta(t)}{r^{\sigma}(t)}}(t,t_0) = e_{t^2}(t,t_0) \geq 1$ for $t \in [t_0,\infty)$, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{C}^{t}\Big\{\frac{1}{E(u)r(u)}\int_{C}^{u}E(s)q(s)\cdot\\
&\qquad\qquad [1-(1+\varepsilon)p(\delta(s))]^{\lambda}\Delta s\Big\}^{1/\lambda}\Delta u\\
&\geq \limsup_{t\to\infty}\int_{C}^{t}\Big\{\frac{1}{u^{\lambda+1}}\int_{C}^{u}(s+\alpha)^{\lambda}\Big[1-(1+\frac{1}{3})\frac{3}{4}\cdot\\
&\qquad\qquad (1-\frac{1}{\sqrt{1+(s-hk_0)^2}})\Big]^{\lambda}\Delta s\Big\}^{1/\lambda}\Delta u\\
&= \limsup_{t\to\infty}\int_{C}^{t}\Big\{\frac{1}{u^{\lambda+1}}\int_{C}^{u}(s+\alpha)^{\lambda}\cdot\\
&\qquad\qquad \Big[\frac{1}{\sqrt{1+(s-hk_0)^2}}\Big]^{\lambda}\Delta s\Big\}^{1/\lambda}\Delta u\\
&\geq \limsup_{t\to\infty}\int_{C}^{t}\Big(\frac{1}{u^{\lambda+1}}\int_{C}^{u}\Delta s\Big)^{1/\lambda}\Delta u\\
&= \limsup_{t\to\infty}\int_{C}^{t}\frac{(u-C)^{1/\lambda}}{u^{1+\frac{1}{\lambda}}}\Delta u\\
&= \int_{C}^{M}\frac{(u-C)^{1/\lambda}}{u^{1+\frac{1}{\lambda}}}\Delta u+\limsup_{t\to\infty}\int_{M}^{t}\frac{(u-C)^{1/\lambda}}{u^{1+\frac{1}{\lambda}}}\Delta u\\
&\geq \int_{C}^{M}\frac{(u-C)^{1/\lambda}}{u^{1+\frac{1}{\lambda}}}\Delta u+\limsup_{t\to\infty}\int_{M}^{t}\big(\frac{1}{2}\big)^{1/\lambda}\frac{1}{u}\Delta u\\
&= \infty, \qquad (52)
\end{aligned}$$

which implies that (44) holds. Thus, if $\alpha \geq \sqrt{(hk_0)^2+1}$ then by Theorem 4 every solution of (50) is oscillatory or converges to zero as $t \to \infty$.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}}$, $\delta(t) = {q_0}^{-k_0}t$ and $\tau(t) = {q_0}^{-k_1}t$, where $\overline{q_0^{\mathbb{Z}}}, k_0$ and k_1 are defined as in Example 1, then by taking $t_0 = q_0$ and $q(t) = (t+\alpha)^{\lambda}$ we see that (H_1)–(H_6) and (43) hold.

We will apply Theorem 4. It remains to satisfy the conditions (10) and (44). Let $\alpha \geq q_0^{k_0}$. Take $\varphi(s) = 1$ and $\varepsilon = \frac{1}{3}$. By using methods similar to those of the proof of (51) and (52), we obtain

$$\limsup_{t\to\infty} \int_{t_0}^{t} Q(s)\Delta s = \limsup_{t\to\infty} \int_{t_0}^{t} q(s)[1-p(\delta(s))]^{\lambda}\Delta s = \infty$$

and

$$\limsup_{t\to\infty} \int_{C}^{t} \Big\{ \frac{1}{E(u)r(u)} \int_{C}^{u} E(s)q(s) \cdot [1-(1+\varepsilon)p(\delta(s))]^{\lambda}\Delta s \Big\}^{1/\lambda} \Delta u = \infty, \qquad (53)$$

which imply that (10) and (44) hold. Hence, if $\alpha \geq q_0^{k_0}$ then by Theorem 4 every solution of (50) is oscillatory or tends to zero as $t \to \infty$.

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References

[1] A. Zafer, On oscillation and nonoscillation of second-order dynamic equations, Appl. Math. Lett. 22 (2009) 136–141.
[2] S. Hilger, Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten (PhD thesis), Universität Würzburg, Würzburg, 1988.
[3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.
[4] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[5] R. P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, Dynamic equations on time scales: a survey, J. Comput. Appl. Math. 141 (2002) 1–26.
[6] V. Spedding, Taming Nature's Numbers, New Scientist, 179 (2003) 28–31.
[7] L. Erbe, T. S. Hassan and A. Peterson, Oscillation criteria for nonlinear damped dynamic equations on time scales, Appl. Math. Comput. 203 (2008) 343–357.
[8] T. S. Hassan, Oscillation criteria for half-linear dynamic equations on time scales, J. Math. Anal. Appl. 345 (2008) 176–185.
[9] L. Ou, Atkinson's super-linear oscillation theorem for matrix dynamic equations on a time scale, J. Math. Anal. Appl. 299 (2004) 615–629.
[10] A. D. Medico and Q. Kong, Kamenev-type and interval oscillation criteria for second-order linear differential equations on a measure chain, J. Math. Anal. Appl. 294 (2004) 621–643.
[11] M. Bohner and S. H. Saker, Oscillation criteria for perturbed nonlinear dynamic equations, Math. Comput. Modelling 40 (2004) 249–260.
[12] D. -X. Chen, Oscillation and asymptotic behavior for nth-order nonlinear neutral delay dynamic equations on time scales, Acta Appl. Math. 109 (2010) 703–719.
[13] D. -X. Chen and J. -C. Liu, Asymptotic behavior and oscillation of solutions of third-order nonlinear neutral delay dynamic equations on time scales, Can. Appl. Math. Q. 16 (2008) 19–43.
[14] O. Došlý and S. Hilger, A necessary and sufficient condition for oscillation of the Sturm-Liouville dynamic equation on time scales, J. Comput. Appl. Math. 141 (2002) 147–158.
[15] B. G. Zhang and Z. Shanliang, Oscillation of second-order nonlinear delay dynamic equations on time scales, Comput. Math. Appl. 49 (2005) 599–609.
[16] Y. Şahiner, Oscillation of second-order delay differential equations on time scales, Nonlinear Anal. 63 (2005) 1073–1080.
[17] L. Erbe, A. Peterson and S. H. Saker, Hille and Nehari type criteria for third-order dynamic equations, J. Math. Anal. Appl. 329 (2007) 112–131.
[18] D. -X. Chen, Oscillation of second-order Emden-Fowler neutral delay dynamic equations on time scales, Math. Comput. Modelling 51 (2010) 1221–1229.
[19] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities (second ed.), Cambridge University Press, Cambridge, 1988.
[20] L. Erbe, A. Peterson and S. H. Saker, Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales, J. Comput. Appl. Math. 181 (2005) 92–102.
[21] R. P. Agarwal, D. O'Regan and S. H. Saker, Oscillation criteria for second-order nonlinear neutral delay dynamic equations, J. Math. Anal. Appl. 300 (2004) 203–217.
[22] S. H. Saker, Oscillation of second-order nonlinear neutral delay dynamic equations on time scales, J. Comput. Appl. Math. 187 (2006) 123–141.
[23] H. -W. Wu, R. -K. Zhuang and R. M. Mathsen, Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations, Appl. Math. Comput. 178 (2006) 321–331.