Quadratic Irrationals, Quadratic Ideals and Indefinite Quadratic Forms II

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Abstract**—**Let $D \neq 1$ be a positive non-square integer and let $\delta = \sqrt{D}$ or $\frac{1+\sqrt{D}}{2}$ be a real quadratic irrational with trace $t = \delta + \overline{\delta}$ and norm $n = \delta\overline{\delta}$. Let $\gamma = \frac{P+\delta}{Q}$ be a quadratic irrational for positive integers P and Q. Given a quadratic irrational γ, there exist a quadratic ideal $I_\gamma = [Q, \delta + P]$ and an indefinite quadratic form $F_\gamma(x, y) = Q(x - \gamma y)(x - \overline{\gamma} y)$ of discriminant $\Delta = t^2 - 4n$. In the first section, we give some preliminaries form binary quadratic forms, quadratic irrationals and quadratic ideals. In the second section, we obtain some results on γ, I_γ and F_γ for some specific values of Q and P.

Keywords**—**Quadratic irrationals, quadratic ideals, indefinite quadratic forms, extended modular group.

I. PRELIMINARIES.

A real quadratic form (or just a form) F is a polynomial in two variables x, y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2 \tag{1}$$

with real coefficients a, b, c. We denote F briefly by

$$F = (a, b, c).$$

The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by Δ. Moreover F is an integral form if and only if $a, b, c \in \mathbf{Z}$ and F is indefinite if and only if $\Delta > 0$.

Let Γ be the modular group PSL$(2, \mathbf{Z})$, i.e. the set of the transformations

$$z \mapsto \frac{rz + s}{tz + u}, \ r, s, t, u \in \mathbf{Z}, \quad ru - st = 1.$$

Then Γ is generated by the transformations $T(z) = \frac{-1}{z}$ and $V(z) = z + 1$. Let $U = T.V$. Then $U(z) = \frac{-1}{z+1}$. Then Γ has a representation $\Gamma = \left\langle T, U : T^2 = U^3 = I \right\rangle$. So

$$\Gamma = \left\{ g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbf{Z} \ \ , ru - st = 1 \right\}. \tag{2}$$

We denote the symmetry with respect to the imaginary axis with R, that is $R(z) = -\overline{z}$. Then the group $\overline{\Gamma} = \Gamma \cup R\Gamma$ is generated by the transformations R, T, U and has a representation $\overline{\Gamma} = \left\langle R, T, U : R^2 = T^2 = U^3 = I \right\rangle$, and is called the extended modular group. So

$$\overline{\Gamma} = \left\{ g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbf{Z}, \quad ru - st = \pm 1 \right\}. \tag{3}$$

There is a strong connection between the extended modular group and binary quadratic forms (see [5]). Most properties of binary quadratic forms can be given by the aid of the extended modular group. Gauss (1777-1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows: Let $F = (a, b, c)$ be a quadratic form and let $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$. Then the form gF is defined by

$$\begin{aligned} gF(x, y) \ &= \ \left(ar^2 + brs + cs^2\right) x^2 \\ &\quad + (2art + bru + bts + 2csu)\, xy \\ &\quad + \left(at^2 + btu + cu^2\right) y^2, \end{aligned} \tag{4}$$

that is, gF is gotten from F by making the substitution

$$x \to rx + tu, \ y \to sx + uy.$$

Moreover, $\Delta(F) = \Delta(gF)$ for all $g \in \overline{\Gamma}$, that is, the action of $\overline{\Gamma}$ on forms leaves the discriminant invariant. If F is indefinite or integral, then so is gF for all $g \in \overline{\Gamma}$.

Let F and G be two forms. If there exists a $g \in \overline{\Gamma}$ such that $gF = G$, then F and G are called equivalent. If $detg = 1$, then F and G are called properly equivalent and if $detg = -1$, then F and G are called improperly equivalent. A quadratic form F is said to be ambiguous if it is improperly equivalent to itself. An indefinite quadratic form F of discriminant Δ is said to be reduced if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{5}$$

Mollin (see [1]) considered the arithmetic of ideals in his book. Let $D \neq 1$ be a square free integer and let $\Delta = \frac{4D}{r^2}$, where

$$r = \begin{cases} 2 & D \equiv 1 (mod\ 4) \\ 1 & otherwise. \end{cases} \tag{6}$$

If we set $\mathbf{K} = \mathbf{Q}(\sqrt{D})$, then $\mathbf{K}$ is called a quadratic number field of discriminant $\Delta = \frac{4D}{r^2}$. A complex number is an algebraic integer if it is the root of a monic polynomial with coefficients in $\mathbf{Z}$. The set of all algebraic integers in the complex field $\mathbf{C}$ is a ring which we denote by A. Therefore $A \cap \mathbf{K} = O_\Delta$ is the ring of integers of the quadratic field $\mathbf{K}$ of discriminant Δ. Set

$$w_\Delta = \frac{r - 1 + \sqrt{D}}{r}$$

for r defined in (6). Then w_Δ is called principal surd. We restate the ring of integers of $\mathbf{K}$ as

$$O_\Delta = [1, w_\Delta] = \mathbf{Z}[w_\Delta].$$

In this case $\{1, w_\Delta\}$ is called an integral basis for $\mathbf{K}$. Let $I = [\alpha, \beta]$ denote the $\mathbf{Z}$-module $\alpha\mathbf{Z} \oplus \beta\mathbf{Z}$, i.e., the additive

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abelian group, with basis elements α and β consisting of

$$\{\alpha x + \beta y : x, y \in \mathbf{Z}\}.$$

Note that $O_\Delta = \left[1, \frac{1+\sqrt{D}}{r}\right]$. In this case $w_\Delta = \frac{r-1+\sqrt{D}}{r}$ is called the principal surd. Every principal surd $w_\Delta \in O_\Delta$ can be uniquely expressed as

$$w_\Delta = x\alpha + y\beta,$$

where $x, y \in \mathbf{Z}$ and $\alpha, \beta \in O_\Delta$. We call α, β an integral basis for $\mathbf{K}$, and denote it by $[\alpha, \beta]$. If $\frac{\alpha\overline{\beta}-\beta\overline{\alpha}}{\sqrt{\Delta}} > 0$, then α and β are called ordered basis elements. Recall that two basis of an ideal are ordered if and only if they are equivalent under an element of $\overline{\Gamma}$. If I has ordered basis elements, then we say that I is simply ordered. If I is ordered, then

$$F(x,y) = \frac{N(\alpha x + \beta y)}{N(I)}$$

is a quadratic form of discriminant Δ (Here $N(x)$, denote the norm of x). In this case we say that F belongs to I and write $I \to F$.

Conversely let us assume that

$$G(x,y) = Ax^2 + Bxy + Cy^2 = d(ax^2 + bxy + cy^2)$$

be a quadratic form, where $d = \pm gcd(A,B,C)$ and $b^2-4ac = \Delta$. If $B^2-4AC > 0$, then we get $d > 0$ and if $B^2-4AC < 0$, then we choose d such that $a > 0$. Set

$$I = [\alpha, \beta] = \left[a, \frac{b-\sqrt{\Delta}}{2}\right]$$

for $a > 0$ or

$$I = [\alpha, \beta] = \left[a, \frac{b-\sqrt{\Delta}}{2}\right]\sqrt{\Delta}$$

for $a < 0$ and $\Delta > 0$. Then I is an ordered O_Δ-ideal. Thus to every form G, there corresponds an ideal I to which G belongs and we write $G \to I$. Hence we have a correspondence between ideals and quadratic forms (for further details see [2], [3], [4], [7]).

Theorem 1.1: If $I = [a, b + cw_\Delta]$, then I is a non-zero ideal of O_Δ if and only if $c|b, c|a$ and $ac|N(b + cw_\Delta)$ [1].

Let δ denote a real quadratic irrational integer with trace $t = \delta + \overline{\delta}$ and norm $n = \delta\overline{\delta}$. Given a real quadratic irrational $\gamma \in \mathbf{Q}(\delta)$, there are rational integers P and Q such that $\gamma = \frac{P+\delta}{Q}$ with $Q|(\delta+P)(\overline{\delta}+P)$. Hence for each

$$\gamma = \frac{P+\delta}{Q}, \tag{7}$$

there is a corresponding $\mathbf{Z}-$module

$$I_\gamma = [Q,\ P+\delta] \tag{8}$$

in fact, this module is an ideal by Theorem 1.1. The conjugate of I_γ is defined as

$$\overline{I}_\gamma = [Q, P+\overline{\delta}].$$

If $I_\gamma = \overline{I}_\gamma$, then I_γ is called ambiguous. The ideal I_γ in (8) is said to be reduced if and only if

$$P + \delta > Q \quad and \quad -Q < P + \overline{\delta} < 0. \tag{9}$$

So I_γ is ambiguous if and only if it contains both $\frac{P+\delta}{Q}$ and $\frac{P+\overline{\delta}}{Q}$, so if and only if

$$\frac{2P}{Q} \in \mathbf{Z}.$$

For the quadratic irrational γ, there exists an indefinite quadratic form

$$F_\gamma(x,y) = Q(x-\gamma y)(x-\overline{\gamma}y). \tag{10}$$

Applying (10), we obtain

$$\begin{aligned} F_\gamma(x,y) &= Q(x-\gamma y)(x-\overline{\gamma}y) \\ &= Q\left[x^2 - xy(\gamma+\overline{\gamma}) + y^2(\gamma\overline{\gamma})\right] \\ &= Q\left[\begin{array}{c} x^2 - xy\left(\frac{P+\delta}{Q} + \frac{P+\overline{\delta}}{Q}\right) \\ +y^2\left(\frac{P+\delta}{Q}.\frac{P+\overline{\delta}}{Q}\right) \end{array}\right] \\ &= Q\left[\begin{array}{c} x^2 - xy\left(\frac{t+2P}{Q}\right) \\ +y^2\left(\frac{P^2+P(\delta+\overline{\delta})+\delta.\overline{\delta}}{Q}\right) \end{array}\right] \\ &= Q\left[x^2 - xy\left(\frac{t+2P}{Q}\right) + y^2\left(\frac{P^2+Pt+n}{Q}\right)\right] \\ &= Qx^2 - (t+2P)xy + \left(\frac{P^2+Pt+n}{Q}\right)y^2. \end{aligned}$$

The discriminant of F_γ is

$$\begin{aligned} \Delta &= [-(t+2P)]^2 - 4Q\left(\frac{P^2+Pt+n}{Q}\right) \\ &= t^2 + 4tP + 4P^2 - 4P^2 - 4Pt - 4n \\ &= t^2 - 4n. \end{aligned}$$

Hence one associates with γ an indefinite quadratic form F_γ defined as above. The opposite of F_γ is hence

$$\overline{F}_\gamma(x,y) = Qx^2 + (t+2P)xy + \left(\frac{n+Pt+P^2}{Q}\right)y^2. \tag{11}$$

II. Quadratics.

In [6], we derived some results concerning the quadratic irrationals γ, quadratic ideals I_γ and indefinite quadratic forms F_γ defined in (7), (8) and (10), respectively. In the present paper we consider the same problem for other values of Q and P.

Let $\delta = \sqrt{D}$ and $Q = 1$. Then $t = 0$ and $n = -D$. Set $P = \frac{-p}{2}$ for primes p such that $p \equiv 1, 5 (mod\ 6)$. Then

$$\gamma_1 = -\frac{p}{2} + \sqrt{D}$$

is a quadratic irrational and hence

$$I_{\gamma_1} = \left[1, \frac{-p}{2} + \sqrt{D}\right] \tag{12}$$

is a quadratic ideal and

$$F_{\gamma_1}(x,y) = x^2 + pxy + \left(\frac{p^2-4D}{4}\right)y^2 \tag{13}$$

is a quadratic form of discriminant $\Delta = 4D$.

Theorem 2.1: γ_1 is equivalent to its conjugate $\overline{\gamma}_1$ for every primes $p \equiv 1, 5(mod\ 6)$.

Proof: Recall that two real numbers α and β are said to be equivalent if there exists a $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$ such that

$$g\alpha = \beta \Leftrightarrow \frac{r\alpha + s}{t\alpha + u} = \beta.$$

The conjugate of γ_1 is $\overline{\gamma}_1 = \frac{-p}{2} - \sqrt{D}$. Now consider the equation

$$g\overline{\gamma}_1 = \gamma_1 \Leftrightarrow \frac{r\left(\frac{-p}{2} - \sqrt{D}\right) + s}{t\left(\frac{-p}{2} - \sqrt{D}\right) + u} = \frac{\frac{-p}{2} + \sqrt{D}}{1} \tag{14}$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$. One solution of (14) is

$$g = \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix} \in \overline{\Gamma}.$$

So γ_1 is equivalent to its conjugate $\overline{\gamma}_1$. ■

Theorem 2.2: I_{γ_1} is ambiguous for every $p \equiv 1, 5(mod\ 6)$.

Proof: We know that an ideal I_γ is ambiguous if it is equal to its conjugate $\overline{I}_\gamma$, or in other words, if and only if $\frac{\delta+P}{Q} + \frac{\overline{\delta}+P}{Q} = \frac{t+2P}{Q} \in \mathbf{Z}$. For $\delta = \sqrt{D}$ we have $t = 0$, and hence

$$\frac{t + 2P}{Q} = \frac{2(-p/2)}{1} = -p \in \mathbf{Z}. \tag{15}$$

Therefore I_{γ_1} is ambiguous. ■

From above two theorems we can give the following corollary.

Corollary 2.3: F_{γ_1} is properly equivalent to its opposite $\overline{F}_{\gamma_1}$ and is ambiguous for every $p \equiv 1, 5(mod\ 6)$.

Proof: It is clear that F_{γ_1} is properly equivalent to its opposite $\overline{F}_{\gamma_1}$ by (15) since $\frac{t+2P}{Q} = -p \in \mathbf{Z}$. We know as above that an indefinite quadratic form F_γ is ambiguous if and only if the quadratic irrational γ is equivalent to its conjugate $\overline{\gamma}$. We proved in Theorem 2.1 that γ_1 is equivalent to its conjugate $\overline{\gamma}_1$. So F_{γ_1} is ambiguous for every $p \equiv 1, 5(mod\ 6)$. ■

Now we can give the following theorem.

Theorem 2.4: Let F_{γ_1} be the quadratic form in (13). Then

1) If $p \equiv 1(mod\, 6)$, say $p = 1 + 6k$ for a positive integer $k \geq 1$, then F_{γ_1} is reduced if and only if $D \in [9k^2 + 3k + 1,\, 9k^2 + 9k + 2] - \{9k^2 + 6k + 1\}$.
2) If $p \equiv 5(mod\, 6)$, say $p = 5 + 6k$ for a positive integer $k \geq 1$, then F_{γ_1} is reduced if and only if $D \in [9k^2 + 15k + 7,\, 9k^2 + 21k + 12] - \{9k^2 + 12k + 9\}$.

In both cases the number of these reduced forms is p.

Proof: 1) Let $p \equiv 1(mod\, 6)$, say $p = 1 + 6k$ and let F_{γ_1} be reduced. Then by (5), we get

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta} \;\Leftrightarrow\; \left|\sqrt{4D} - 2|1|\right| < p < \sqrt{4D}$$
$$\Leftrightarrow\; 2\sqrt{D} - 2 < p < 2\sqrt{D}. \tag{16}$$

Applying (16), we find that

$$D > \frac{p^2}{4} = \frac{1}{4} + 3k + 9k^2 \Leftrightarrow D \geq 9k^2 + 3k + 1$$

and

$$D < \frac{(p+2)^2}{4} = \frac{9}{4} + 9k + 9k^2 \Leftrightarrow D \leq 9k^2 + 9k + 2.$$

So we have

$$9k^2 + 3k + 1 \leq D \leq 9k^2 + 9k + 2.$$

But $D = 9k^2 + 6k + 1 = (3k+1)^2$ is a square. So we have to omit it (since D must be a square-free positive integer). Therefore we have

$$D \in [9k^2 + 3k + 1,\, 9k^2 + 9k + 2] - \{9k^2 + 6k + 1\}.$$

The converse is also true, that is, if $D \in [9k^2+3k+1,\, 9k^2+9k+2] - \{9k^2+6k+1\}$, then F_{γ_1} is reduced. Further the number of these reduced forms is

$$9k^2 + 9k + 2 - (9k^2 + 3k + 1) = 6k + 1 = p.$$

2) Let $p \equiv 5(mod\, 6)$, say $p = 5+6k$ and let F_{γ_1} be reduced. Then by (16), we get

$$D > \frac{p^2}{4} = \frac{25}{4} + 15k + 9k^2 \Leftrightarrow D \geq 9k^2 + 15k + 7$$

and

$$D < \frac{(p+2)^2}{4} = \frac{49}{4} + 21k + 9k^2 \Leftrightarrow D \leq 9k^2 + 21k + 12.$$

So we have

$$9k^2 + 15k + 7 \leq D \leq 9k^2 + 21k + 12.$$

But $D = 9k^2 + 18k + 9 = (3k+3)^2$ is a square. So we have to omit it. Therefore we have

$$D \in [9k^2 + 15k + 7,\, 9k^2 + 21k + 12] - \{9k^2 + 18k + 9\}.$$

Conversely if $D \in [9k^2+15k+7,\, 9k^2+21k+12] - \{9k^2+18k+9\}$, then clearly F_{γ_1} is reduced. The number of these reduced forms is

$$9k^2 + 21k + 12 - (9k^2 + 15k + 7) = 6k + 5 = p.$$

■

Now let $\delta = \frac{1+\sqrt{D}}{2}$ and $Q = 1$. Then $t = 1$ and $n = \frac{1-D}{4}$. Set $P = \frac{-(p+1)}{2}$ for primes p such that $p \equiv 1, 5(mod\, 6)$. Then

$$\gamma_2 = \frac{-p + \sqrt{D}}{2}$$

is a quadratic irrational and hence

$$I_{\gamma_2} = \left[1, \frac{-p+\sqrt{D}}{2}\right] \tag{17}$$

is a quadratic ideal and

$$F_{\gamma_2}(x,y) = x^2 + pxy + \left(\frac{p^2-D}{4}\right)y^2 \tag{18}$$

is a quadratic form of discriminant $\Delta = D$.

Theorem 2.5: γ_2 is equivalent to its conjugate $\overline{\gamma}_2$ for every $p \equiv 1,5(mod\,6)$.

Proof: The conjugate of γ_2 is $\overline{\gamma}_2 = \frac{-p-\sqrt{D}}{2}$. Now consider the equation

$$g\overline{\gamma}_2 = \gamma_2 \Leftrightarrow \frac{r\left(-p-\sqrt{D}\right)+s}{t\left(-p-\sqrt{D}\right)+u} = \frac{-p+\sqrt{D}}{1} \tag{19}$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$. One solution of (19) is

$$g = \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix} \in \overline{\Gamma}.$$

So γ_2 is equivalent to its conjugate $\overline{\gamma}_2$. ■

Theorem 2.6: I_{γ_2} is ambiguous for every $p \equiv 1,5(mod\,6)$.

Proof: Recall that $t = 1$ for $\delta = \frac{1+\sqrt{D}}{2}$. So

$$\frac{t+2P}{Q} = \frac{1+2(\frac{-(p+1)}{2})}{1} = -p \in \mathbf{Z}.$$

Therefore I_{γ_2} is ambiguous. ■

From above two theorems we can give the following corollary.

Corollary 2.7: F_{γ_2} is properly equivalent to its opposite $\overline{F}_{\gamma_2}$ and is ambiguous for every $p \equiv 1,5(mod\,6)$.

Proof: We know that an indefinite quadratic form F_γ is ambiguous if and only if the quadratic irrational γ is equivalent to its conjugate $\overline{\gamma}$. We proved in Theorem 2.5 that γ_2 is equivalent to its conjugate $\overline{\gamma}_2$. So F_{γ_2} is ambiguous for every $p \equiv 1,5(mod\ 6)$. ■

Now we can give the following theorem.

Theorem 2.8: Let F_{γ_2} be the quadratic form in (18). Then

1) If $p \equiv 1(mod\,6)$, say $p = 1+6k$ for a positive integer $k \geq 1$, then F_{γ_2} is reduced if and only if $D \in [36k^2 + 12k+2,\ 36k^2+36k+8] - \{36k^2+24k+4\}$.
2) If $p \equiv 5(mod\,6)$, say $p = 5+6k$ for a positive integer $k \geq 1$, then F_{γ_2} is reduced if and only if $D \in [36k^2 + 60k+26,\ 36k^2+84k+48] - \{36k^2+72k+36\}$.

In both cases the number of these reduced forms is $4p+2$.

Proof: 1) Let $p \equiv 1(mod\,6)$, say $p = 1+6k$ and let F_{γ_2} be reduced. Then by (5), we get

$$\begin{aligned} \left|\sqrt{\Delta}-2|a|\right| < b < \sqrt{\Delta} \quad &\Leftrightarrow \quad \left|\sqrt{D}-2|1|\right| < p < \sqrt{D} \\ &\Leftrightarrow \quad \sqrt{D}-2 < p < \sqrt{D}. \end{aligned} \tag{20}$$

Applying (20) we get

$$D > p^2 = 1+12k+36k^2 \Leftrightarrow D \geq 36k^2+12k+2$$

and

$$D < (p+2)^2 = 9+36k+36k^2 \Leftrightarrow D \leq 36k^2+32k+8.$$

So

$$36k^2+12k+2 \leq D \leq 36k^2+36k+8.$$

But $D = 36k^2+24k+4 = (6k+2)^2$ is a square. So we have to omit it. Therefore we have

$$D \in [36k^2+12k+2,\ 36k^2+36k+8] - \{36k^2+24k+4\}.$$

Conversely if $D \in [36k^2+12k+2,\ 36k^2+36k+8] - \{36k^2+24k+4\}$, then F_{γ_2} is reduced. Further the number of these reduced forms is

$$36k^2+36k+8-(36k^2+12k+2) = 24k+6 = 4p+2.$$

2) Let $p \equiv 5(mod\,6)$, say $p = 5+6k$ and let F_{γ_2} be reduced. Then by (20), we get

$$D > p^2 = 25+60k+36k^2 \Leftrightarrow D \geq 36k^2+60k+26$$

and

$$D < (p+2)^2 = 49+84k+36k^2 \Leftrightarrow D \leq 36k^2+84k+48.$$

So we have

$$36k^2+60k+26 \leq D \leq 36k^2+84k+48.$$

But $D = 36k^2+72k+36 = (6k+6)^2$ is a square. So we have to omit it. Therefore we have

$$D \in [36k^2+60k+26,\ 36k^2+84k+48] - \{36k^2+72k+36\}.$$

The converse is also true, that is, if $D \in [36k^2+60k+26,\ 36k^2+84k+48] - \{36k^2+72k+36\}$, then F_{γ_2} is reduced. The number of these reduced forms is

$$36k^2+84k+48-(36k^2+60k+26) = 24k+22 = 4p+2.$$

■

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