An $H^1$-Galerkin mixed method for the coupled Burgers equation

Xianbiao Jia, Hong Li, Yang Liu, Zhichao Fang

Abstract—In this paper, an $H^1$-Galerkin mixed finite element method is discussed for the coupled Burgers equations. The optimal error estimates of the semi-discrete and fully discrete schemes of the coupled Burgers equation are derived.

Keywords—The coupled Burgers equation; $H^1$-Galerkin mixed finite element method; Backward Euler’s method; Optimal error estimates.

I. INTRODUCTION

WITH the research and development of the mixed finite element methods and $H^1$-Galerkin method, Pani [2] (in 1998) proposed a new mixed finite element method called $H^1$-Galerkin mixed finite element procedure which is applied to a mixed system in $u$ and its flux $q$. The approximating finite element spaces $V_h$ and $W_h$ are allowed to be of differing polynomial degrees. Hence, estimations have been obtained which distinguish the better approximation properties of $V_h$ and $W_h$. Compared to standard mixed methods, the proposed one is not subject to LBB consistency condition. Although we require extra regularity on the solution, a better order of convergence for the flux in $L^2$-norm is obtained. From then on, the method was applied to the evolution integro-differential equation[3][4][5][6], hyperbolic problems[9][10][11][14][16], fourth-order parabolic equation[15], Sobolev equation[7][8], Schrodinger equation[13] and nonlinear evolution equations[17][18][19] and so on. In this paper, we propose $H^1$-Galerkin mixed finite element scheme for the following coupled Burgers equation[1]

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + 2 u_w + (w) = f(x,t), & \quad (x,t) \in \Omega \times J, \\
\frac{\partial v}{\partial t} - \Delta v + 2 v_w + (w) = g(x,t), & \quad (x,t) \in \Omega \times J,
\end{align*}
\]

(1)

where $\Omega = [0,1] \subset \mathbb{R}$ with Lipschitz continuous boundary $\partial \Omega$, $J = (0,T]$ is the time interval with $0 < T < \infty$, $f(x,t), g(x,t)$ are two functions.

II. $H^1$-GALERKIN MIXED FINITE ELEMENT METHOD

Denote the natural inner product on $L^2(I)$ as $\langle \cdot, \cdot \rangle$. Let $H^1_0 = \{ z \in H^1(I) | z(0) = z(1) = 0 \}$. Further, we call the classical Sobolev spaces $W^{m,p}(I)$, $1 \leq p \leq \infty$ as $W^{m,p}$

School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China. Correspondence to: Email: smslh@imu.edu.cn (H. Li); mathliuyang@yahoo.cn (Y. Liu).

Manuscript received April 22, 2012.

with norm $\| \cdot \|_{m,p}$. When $p = 2$, we simply write $W^{m,2}$ as $W^m$ with norm $\| \cdot \|_m$.

With $q = u, \sigma = v$, we reformulate the formulation (1) as the first-order system:

\[
\begin{align*}
u_t - 2u_x + (w) = f(x,t), \\v_t - 2v_x + (w) = g(x,t).
\end{align*}
\]

(2)

To derive the $H^1$-Galerkin mixed finite element method, we consider the following weak formulation of (1): find $u, v, q, \sigma \in [0, T] \rightarrow H^1 \times H^1$ satisfying:

\[
\begin{align*}
\int_0^T \int_{\Omega} u \frac{\partial v}{\partial t} - (u, \Delta v) + 2 u \frac{\partial v}{\partial t} + (w) (v, \Delta v) = \int_0^T \int_{\Omega} f v(t), \\
\int_0^T \int_{\Omega} v \frac{\partial q}{\partial t} - (v, \Delta q) + 2 v \frac{\partial q}{\partial t} + (w) (q, \Delta q) = \int_0^T \int_{\Omega} g q(t).
\end{align*}
\]

(3)

For (3c,d), we have used integration by parts, and the Dirichlet boundary conditions $u(t,0) = u(t,1) = 0, v(t,0) = v(t,1) = 0$.

Let $V_h$ and $W_h$ be finite dimensional subspaces of $H^1_0$ and $H^1$, respectively, with the following approximation properties:

\[
\begin{align*}
in_{V_h} = \inf_{v_h \in V_h} \| v - v_h \|_{L^p} + h \| v - v_h \|_{L^{r+1}} \leq C h^{k+1} \| v \|_{W^{k+1, p}}, v \in H^1_0 \cap W^{k+1, p}, \\
in_{W_h} = \inf_{w_h \in W_h} \| w - w_h \|_{L^p} + h \| w - w_h \|_{L^{r+1}} \leq C h^{k+1} \| w \|_{W^{k+1, p}}.
\end{align*}
\]

(4)

The semidiscrete $H^1$-Galerkin mixed finite element for (3) consists in determining $(u_h, v_h; q_h, \sigma_h): [0, T] \rightarrow V_h \times W_h$ such that:

\[
\begin{align*}
\int_0^T \int_{\Omega} (u_h, \frac{\partial v_h}{\partial t}) + 2 (u_h, \frac{\partial v_h}{\partial t}) = \int_0^T \int_{\Omega} f v_h(x,t), \\
\int_0^T \int_{\Omega} (v_h, \frac{\partial q_h}{\partial t}) + 2 (v_h, \frac{\partial q_h}{\partial t}) = \int_0^T \int_{\Omega} g q_h(x,t),
\end{align*}
\]

(4)

For use in the error analysis, we define the elliptic projection $\tilde{u}, \tilde{v} \in V_h$ by

\[
\begin{align*}
(u_h - \tilde{u}_h, \frac{\partial v_h}{\partial t}) = 0, (v_h - \tilde{v}_h, \frac{\partial u_h}{\partial t}) = 0, (u_h, \omega_h), (\omega_h, v_h) \in V_h.
\end{align*}
\]

(5)
Further, we also define a Ritz projection \( \tilde{q}^h, \tilde{\sigma}^h \in W_h \) with \( \sigma, \varphi \) as the solution of
\[
A(q - \tilde{q}^h, \varphi^h) = 0, A(\sigma - \tilde{\sigma}^h, \psi^h) = 0, \quad \varphi, \psi \in W_h. \tag{6}
\]
where \( A(z, w) = (z_x, w_x) + \lambda(z, w) \). Here \( \lambda \) is chosen appropriately so that \( A \) is \( H^1 \)-coercive, i.e.,
\[
A(w, w) \geq \mu_0 ||w||^2_H, \quad w \in H^1
\]
where \( \mu_0 \) is a positive constant. Moreover, it is not hard to check that \( A(\cdot, \cdot) \) is bounded.

With \( \eta = u - \tilde{u}^h, \tau = v - \tilde{v}^h, \rho = q - \tilde{q}^h, \delta = \sigma - \tilde{\sigma}^h \), the following estimates are well known [19]: for \( j = 0, 1 \)
\[
||\eta||_j + ||\eta_t||_j \leq C H^{k+1-j} ||u||_{k+1} + ||u_t||_{k+1}, \|\tilde{u}^h\|_{\infty} \leq C ||u||_1 \tag{7}
\]
\[
||\tau||_j + ||\tau_t||_j \leq C H^{k+1-j} ||v||_{k+1} + ||v_t||_{k+1}, \|\tilde{v}^h\|_{\infty} \leq C ||v||_1 \tag{8}
\]
\[
||\rho||_j \leq C H^{k+1-j} ||\tilde{q}||_{k+1}, ||\rho_t||_j \leq C H^{k+1-j} ||\tilde{q}_t||_{k+1} \tag{9}
\]
\[
||\delta||_j \leq C H^{k+1-j} ||\tilde{\sigma}||_{k+1}, ||\delta_t||_j \leq C H^{k+1-j} ||\tilde{\sigma}_t||_{k+1} \tag{10}
\]

### III. ERROR ESTIMATES FOR SEMI-DISCRETE SCHEME

For the priori error estimates, we decompose the errors as
\[
u = u - \tilde{u}^h + \tilde{u}^h - u^h = \eta + \xi; \quad v = v^h - \tilde{v}^h + \tilde{v}^h = \tau + \theta; \quad q = q^h - \tilde{q}^h + \tilde{q}^h = \rho + \varphi; \quad \sigma = \sigma^h - \tilde{\sigma}^h + \tilde{\sigma}^h = \delta + \psi
\]

From (3)-(6), we then obtain
\[
\begin{align*}
\langle (\xi, \tilde{\xi}^h) + (\zeta, \tilde{\zeta}^h), \forall \chi_h \in V_h, (a) \\
\langle \theta_x, u^h \rangle = \delta - \xi; \quad \langle \theta_y, w^h \rangle = (\gamma, \tilde{\gamma}^h), \forall \psi_h \in V_h, (b) \\
\langle \zeta, \phi^h \rangle + \langle \zeta, \phi^h \rangle + 2 \langle u - u^h, \phi^h \rangle - \langle \sigma - \sigma^h, \phi^h \rangle = -\langle \rho_t, \phi^h \rangle + \lambda(\rho, \phi^h) \psi_h \in W_h, (c) \\
\langle \gamma^h, \psi^h \rangle + 2 \langle v - v^h, \phi^h \rangle - \langle \sigma - \sigma^h, \phi^h \rangle = -\langle \delta, \psi^h \rangle + \lambda(\delta, \psi^h) \psi_h \in W_h, (d)
\end{align*}
\]

**Theorem 3.1:** Assuming that \( u^h(0) = \tilde{u}^h(0), v^h(0) = \tilde{v}^h(0), q^h(0) = \tilde{q}^h(0), \sigma^h(0) = \tilde{\sigma}^h(0) \), we have
\[
\begin{align*}
\|u - u^h\|^2 + h^2 ||u - u^h||_H^2 &\leq CH^{2\min(k+1, r+1)} \tag{11} \\
\|v - v^h\|^2 + h^2 ||v - v^h||_H^2 &\leq CH^{2\min(k+1, r+1)} \\
\|q - q^h\|^2 + h^2 ||q - q^h||_H^2 &\leq CH^{2\min(k+1, r+1)} \\
\|\sigma - \sigma^h\|^2 + h^2 ||\sigma - \sigma^h||_H^2 &\leq CH^{2\min(k+1, r+1)}
\end{align*}
\]

**Proof:** Since estimates of \( \rho, \delta, \eta \) and \( \tau \) are given, respectively, it is sufficient to estimate \( \xi, \gamma, \zeta, \) and \( \theta \). Choosing \( \chi^h = \zeta \) in (11a) and \( w^h = \theta \) in (11b), using the Cauchy-Schwarz’s inequality and Young’s inequality, we have
\[
\|\xi\| \leq C(||\rho|| + ||\xi||), ||\theta|| \leq C(||\delta|| + ||\gamma||). \tag{12}
\]

Using Poincaré inequality, we have
\[
\|\zeta\| \leq C(||\rho|| + ||\xi||), ||\theta|| \leq C(||\delta|| + ||\gamma||). \tag{13}
\]

Take \( \phi^h = \xi \) in (11c) and use (13) to obtain
\[
(\xi, \xi) + A(\xi, \xi) \leq C(||\rho||^2 + ||\rho_t||^2 + ||\xi||^2) \\
+ C(||\eta||_\infty ||\xi|| + ||\rho|| + ||\xi|| + ||\tau|| + ||\delta|| + ||\gamma||)||\xi||_1 \\
+ C(||u^h||_{\infty} ||\rho|| + ||\xi|| + ||\delta|| + ||\gamma||)||\xi||_1 \\
+ C(||\sigma||_{\infty} ||\xi|| + ||\rho|| + ||\xi||)||\xi||_1 \\
+ C(||u^h||_{\infty} ||\rho|| + ||\xi||)||\xi||_1. \tag{14}
\]

Take \( \psi = \gamma \) in (11d) and use (13) to have
\[
(\gamma, \gamma) + A(\gamma, \gamma) \leq C(||\eta||^2 + ||\gamma||^2) \\
+ C(||u||_{\infty} ||\eta|| + ||\rho|| + ||\xi|| + ||\eta|| + ||\rho|| + ||\xi||)||\gamma||_1 \\
+ C(||u^h||_{\infty} ||\delta|| + ||\eta|| + ||\rho|| + ||\xi||)||\gamma||_1 \\
+ C(||u^h||_{\infty} ||\tau|| + ||\delta|| + ||\gamma||)||\gamma||_1. \tag{15}
\]

Add (14) and (15) to get
\[
\frac{d}{dt}(||\xi||^2 + ||\gamma||^2) + A(\xi, \xi) + A(\gamma, \gamma) \leq C(||\xi||^2 + ||\gamma||^2) + C(||\rho||^2 + ||\rho_t||^2 + ||\xi||^2) \\
+ ||\eta||^2 + ||\gamma||^2 + ||\tau||^2 + \frac{\mu_0}{2}(||\xi||^2 + ||\gamma||^2). \tag{16}
\]

Integrate (16) with respect to \( t \) to get
\[
||\xi||^2 + ||\gamma||^2 + \int_0^t [A(\xi, \xi) + A(\gamma, \gamma)]ds \leq C(\int_0^t (||\xi||^2 + ||\gamma||^2)ds + C(\int_0^t (||\rho||^2 + ||\rho_t||^2 + ||\xi||^2) \\
+ ||\eta||^2 + ||\gamma||^2 + ||\tau||^2)ds + \frac{\mu_0}{2}(\int_0^t (||\xi||^2 + ||\gamma||^2)ds. \tag{17}
\]

For (17), we use the Gronwall lemma and \( A(w, w) \geq \mu_0 ||w||^2_H \) to get
\[
||\xi||^2 + ||\gamma||^2 + \int_0^t (||\xi||^2 + ||\gamma||^2)ds \leq C(\int_0^t (||\rho||^2 + ||\rho_t||^2 + ||\xi||^2) \\
+ ||\eta||^2 + ||\gamma||^2 + ||\tau||^2)ds + \frac{\mu_0}{2}(\int_0^t (||\xi||^2 + ||\gamma||^2)ds. \tag{18}
\]

Choosing \( \phi^h = \xi_t \) in (11c) and using (13), we have
\[
(\xi_t, \xi_t) + \frac{1}{2} \frac{d}{dt} A(\xi, \xi) \leq C(||\rho||^2 + ||\rho_t||^2 + ||\xi||^2) + C(||u^h||_{\infty} ||\rho|| + ||\xi|| + ||\rho_t||^2 + ||\xi||^2 + ||\eta||^2 + ||\tau||^2 + ||\delta||^2 + ||\gamma||^2 + ||\rho|| + ||\xi|| + ||\eta|| + ||\rho|| + ||\xi||)||\xi||_1 \\
+ C(||u^h||_{\infty} ||\rho|| + ||\xi|| + ||\eta|| + ||\xi||)||\xi||_1 + \mu_1 ||\xi||^2. \tag{19}
\]
Take $\psi^h = \gamma_n$ in (11d) to get

$$
(\gamma_n, \gamma_n) + \frac{1}{2} \frac{d}{dt} A(\gamma, \gamma)
\leq C(\|\delta_t^2\|^2 + \|\delta_t\|^2 + \|\tau\|^2) + C\|\sigma\|_0,\infty(\|\rho\| + \|\xi\|)
+ \|\delta_t\| + \|\delta_t\| + \|\tau\| + C\|v^h\|_0,\infty(\|\rho_l\| + \|\xi_l\|)
+ \|\delta_t\| + \|\delta_t\| + \|\tau\| + C\|v^h\|_0,\infty(\|\rho_l\| + \|\xi_l\|)
+ C\|\gamma\|_0,\infty(\|\gamma\|_1 + \|\gamma\|_1 + \mu_2)\|\gamma_t\|^2.
$$

(20)

Add (19) and (20), integrate with respect to time $t$ and use the Gronwall lemma to get

$$
\mu_3 \int_0^t \left( \|\xi\|^2 + \|\gamma\|^2 \right) ds + \|\xi\|^2 + \|\gamma\|^2
\leq C \int_0^t \left( \|\rho\|^2 + \|\rho_l\|^2 + \|\delta_t\|^2 + \|\tau\|^2 \right) ds.
$$

(21)

Combining (12), (13), (18) and (21), we have

$$
\|\xi\|^2 \leq \|\gamma\|^2 \leq C(\|\rho\|^2 + \|\xi\|^2)
\leq C(\|\rho\|^2 + \|\rho_l\|^2 + \|\delta_t\|^2 + \|\tau\|^2) ds.
$$

(22)

$$
\|\theta\|^2 \leq \|\gamma\|^2 \leq C(\|\delta_t\|^2 + \|\gamma\|^2)
\leq C(\|\delta_t\|^2 + \|\rho_l\|^2 + \|\delta_t\|^2 + \|\tau\|^2) ds.
$$

(23)

Using (13), (21)-(23), (7)-(10) with the triangle inequality, we obtain the conclusion.

IV. FULLY-DISCRETIZE ERROR ESTIMATES

For the backward Euler procedure, let $0 = t_0 < t_1 < \cdots < t_M = T$ be a given partition of the time interval $[0, T]$ with step length $\Delta t = T/M$, for some positive integer $M$. For a smooth function $\phi$ on $[0, T]$, define $\phi^h(\tau) = \phi(t_\tau)$ and $\partial_t \phi^h = (\phi^h - \phi^{h-1})/\Delta t$.

Let $U^n, V^n, Q^n$ and $Z^n$, respectively, be the approximations of $u, v, q$ and $\sigma$ at $t = t_n$ which we shall define through the following scheme. Given $\{U^{n-1}, V^{n-1}, Q^{n-1}, Z^{n-1}\}$ in $V_h \times W_h$, we now determine $\{U^n, V^n, Q^n, Z^n\}$ in $V_h \times W_h$ satisfying

$$
\begin{align*}
(U^n, \chi^n) &= (Q^n, \chi^n), \forall \chi^n \in V_h, (a) \\
(V^n, \sigma^n) &= (Z^n, \sigma^n), \forall \sigma^n \in W_h, (b) \\
(\mathcal{Q}Q^n, \phi^n) + (Q^n, \phi^n) &= 2(U^n Q^n, \phi^n) - (Q^n V^n, \phi^n) \\
- (U^n Z^n, \phi^n) &= - (f^n, \phi^n), \forall \phi^n \in W_h, (c) \\
(\mathcal{Q}Z^n, \psi^n) + (Z^n, \psi^n) &= 2(V^n Z^n, \psi^n) - (Z^n U^n, \psi^n) \\
- (Q^n V^n, \psi^n) &= - (g^n, \psi^n), \forall \psi^n \in W_h, (d) \\
\end{align*}
$$

(24)

we now split the errors

$$
\begin{align*}
u(t_n) - U^n &= \hat{u}(t_n) + \tilde{u}(t_n) - U^n = \eta^n + \zeta^n \\
v(t_n) - V^n &= \hat{v}(t_n) + \tilde{v}(t_n) - V^n = \tau^n + \theta^n \\
q(t_n) - Q^n &= \hat{q}(t_n) + \tilde{q}(t_n) + Q^n = \rho^n + \xi^n \\
\sigma(t_n) - Z^n &= \hat{\sigma}(t_n) + \tilde{\sigma}(t_n) - Z^n = \sigma^n + \gamma^n
\end{align*}
$$

Using (5)-(6) and (24), we obtain the following error equation

$$
\begin{align*}
&\begin{cases}
\xi^n + \chi^n = (\rho^n + \xi^n, \chi^n), \forall \chi^n \in V_h, (a) \\
\sigma^n + \omega^n = (\delta^n + \gamma^n, \omega^n), \forall \omega^n \in W_h, (b) \\
(\partial_t \xi^n, \phi^n) + (\xi^n, \phi^n) + 2(u^n q^n - Q^n V^n, \phi^n) \\
- (q^n V^n - Q^n V^n, \phi^n) - (u^n \sigma^n - U^n Z^n, \phi^n) \\
= - (\tau^n + \partial_t \rho^n, \phi^n) + \lambda(\rho^n, \phi^n), \forall \phi^n \in W_h, (c) \\
(\partial_t \gamma^n, \psi^n) + (\gamma^n, \psi^n) + 2(u^n \sigma^n - U^n Z^n, \psi^n) \\
- (\sigma^n u^n - Z^n U^n, \psi^n) - (\gamma^n v^n - Q^n V^n, \psi^n) \\
= - (\varepsilon^n + \partial_t \delta^n, \psi^n) + \lambda(\delta^n, \psi^n), \forall \psi^n \in W_h, (d)
\end{cases}
\end{align*}
$$

(25)

where $\varepsilon^n = \eta(t_n) - \hat{\eta}(t_n), c^n = \tau(t_n) - \hat{\gamma}(t_n)$. Theorem 4.1: With $Q^n(0) = \hat{q}(0), Z^n(0) = \hat{\eta}(0)$ and $1 \leq J \leq M$, we have

$$
\begin{align*}
\|\xi^n - Q^n(\sigma^n - U^n Z^n, \phi^n) \\
+ h\|u^n q^n - Q^n V^n, \phi^n\| + h\|\tau^n + \partial_t \rho^n, \phi^n\| \leq C h^{\min(r+1,k+1)}
\end{align*}
$$

(26)

Proof: Take $\chi^n = \psi^n + \theta^n$ in (25a,b,d) and use the Poincare inequality, we have

$$
\begin{align*}
\|\xi^n\|^2 &\leq C(\|\rho^n\|^2 + \|\xi^n\|^2). \quad \|\theta^n\|^2 \leq C(\|\delta^n\|^2 + \|\gamma^n\|^2).
\end{align*}
$$

(27)

Using the Poincare inequality, we have

$$
\begin{align*}
\|\xi^n\|^2 &\leq C(\|\rho^n\|^2 + \|\xi^n\|^2). \quad \|\theta^n\|^2 \leq C(\|\delta^n\|^2 + \|\gamma^n\|^2).
\end{align*}
$$

(28)

Note that $\mathcal{Q}Q^n, \phi^n \geq \mathcal{Q}Q^n, \phi^n \geq \mathcal{Q}Q^n, \phi^n$.

and using (27), we have

$$
\begin{align*}
\frac{1}{2} \mathcal{Q}Q^n, \phi^n + \mathcal{Q}Q^n, \phi^n + \|\xi^n\|^2 + \|\xi^n\|^2
\leq C(\|\xi^n\|^2 + \|\eta^n\|^2 + \|\delta^n\|^2 + \|\varepsilon^n\|^2) + C(\|\xi^n\|^2 + \|\xi^n\|^2 + \|\xi^n\|^2 + \|\xi^n\|^2)
\end{align*}
$$

(29)

Note that $\mathcal{Q}Q^n, \phi^n \geq \mathcal{Q}Q^n, \phi^n \geq \mathcal{Q}Q^n, \phi^n$.

and combining (29), we have

$$
\begin{align*}
\|\xi^n\|^2 + \|\xi^n\|^2 + \|\gamma^n\|^2
+ 2\Delta t(\|\xi^n\|^2 + \|\xi^n\|^2) \leq C(\|\rho^n\|^2 + \|\eta^n\|^2 + \|\xi^n\|^2 + \|\eta^n\|^2 + \|\xi^n\|^2)
+ C\Delta t(\|\rho^n\|^2 + \|\xi^n\|^2 + \|\xi^n\|^2).
\end{align*}
$$

(30)
Sum (30) from \(n = 1\) to \(J\) \((1 \leq J \leq M)\) and use Gronwall lemma to get

\[
(1 - C\Delta t)(\|\xi^J\|^2 + \|\gamma^J\|^2) + 2\Delta t \sum_{n=1}^{J} (\|\xi^n_t\|^2 + \|\gamma^n_t\|^2) \\
\leq C(\|\xi^0\|^2 + \|\gamma^0\|^2) + C\Delta t \sum_{n=1}^{J} (\|\pi^n\|^2 + \|\epsilon^n\|^2) \\
+ \|\delta^n\|^2 + \|\tau^n\|^2) + C\Delta t \sum_{n=1}^{J} (\|\pi^n\|^2 + \|\epsilon^n\|^2) \\
+ C\Delta t \sum_{n=1}^{J} (\|\sigma^n\|^2 + \|\theta^n\|^2).
\]

Use

\[
\|\sigma^n\|^2 \leq \frac{h^2(r+1)}{\Delta t} \int_{t_{n-1}}^{t_n} \|q_{tt}\|^2 + ds,
\]

\[
\|\delta^n\|^2 \leq \frac{h^2(r+1)}{\Delta t} \int_{t_{n-1}}^{t_n} \|q_{tt}\|^2 + ds,
\]

\[
\|\pi^n\|^2 \leq C\Delta t \int_{t_{n-1}}^{t_n} \|q_{tt}\|^2 + ds,
\]

\[
\|\epsilon^n\|^2 \leq C\Delta t \int_{t_{n-1}}^{t_n} \|q_{tt}\|^2 + ds.
\]

and (7)-(10) to obtain

\[
(\|\xi^J\|^2 + \|\gamma^J\|^2) + 2\Delta t \sum_{n=1}^{J} (\|\xi^n_t\|^2 + \|\gamma^n_t\|^2) \\
\leq Ch^{2\min(r+1,k+1)}(\|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^\infty(H^{k+1})}) \\
+ \|v\|_{L^\infty(H^{k+1})} + \|v_t\|_{L^\infty(H^{k+1})} + \|q\|_{L^2(H^{k+1})} \\
+ \|\sigma\|_{L^2(H^{k+1})} + \|\theta\|_{L^2(L^2)} + \|\sigma_t\|_{L^2(L^2)} + \|\theta_t\|_{L^2(L^2)}.
\]

Using (26)-(27), we have

\[
\|\xi^J\|_1 + \|\gamma^J\|_1 \\
\leq Ch^{r+1}(\|q\|_{L^\infty(H^{r+1})} + \|\sigma\|_{L^\infty(H^{r+1})}) \\
+ Ch^{\min(r+1,k+1)}(\|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^\infty(H^{k+1})}) \\
+ \|v\|_{L^\infty(H^{k+1})} + \|v_t\|_{L^\infty(H^{k+1})} + \|q\|_{L^2(H^{k+1})} \\
+ \|\sigma\|_{L^2(H^{k+1})} + \|\theta\|_{L^2(H^{k+1})} + \|\sigma_t\|_{L^2(L^2)} + \|\theta_t\|_{L^2(L^2)}.
\]

We apply the triangle inequality to get the conclusion.

ACKNOWLEDGMENT

This work is supported by National Natural Science Fund (No. 11061021), the Scientific Research Projection of Higher Schools of Inner Mongolia (No. NJ10006) and YSF of Inner Mongolia University (No. ND0702)

REFERENCES