Generic Filtering of Infinite Sets of Stochastic Signals

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Abstract—A theory for optimal filtering of infinite sets of random signals is presented. There are several new distinctive features of the proposed approach. First, a single optimal filter for processing any signal from a given infinite signal set is provided. Second, the filter is presented in the special form of a sum with \( p \) terms where each term is represented as a combination of three operations. Each operation is a special stage of the filtering aimed at facilitating the associated numerical work. Third, an iterative scheme is implemented into the filter structure to provide an improvement in the filter performance at each step of the scheme. The final step of the scheme concerns signal compression and decompression. This step is based on the solution of a new rank-constrained matrix approximation problem. The solution to the matrix problem is described in this paper. A rigorous error analysis is given for the new filter.

Keywords—Optimal filtering, data compression, stochastic signals.

I. INTRODUCTION

A. Motivation

In this paper, extensions of known approaches to optimal filtering based on the Wiener idea\(^1\) are considered. A theory for a new nonlinear filter which processes infinite sets of random signals is presented. The filter is constructed via an iterative scheme that provides a signal processing improvement with each step. The filter provides simultaneous signal filtering and compression and the subsequent decompression (reconstruction).

There has been significant attention in the literature to filters that process finite sets of random signals but it seems that a filter which is able to process infinite sets of random signals has not been developed. The filter presented in this paper is designed specifically to process infinite sets of random signals. For the case of finite sets of random signals, we show that our filter leads to a lower computational load and better accuracy than the known filters; the improved accuracy is due to the special iteration procedure incorporated into the filter structure (see Section III-E2).

There are three motivations for the proposed method which we now describe.

1) First motivation: infinite sets of signals: Most of the literature on Wiener-like filtering provides an optimal filter for an individual input signal given by a finite random vector\(^2\). This means that if we wish to transform an infinite set of input vector signal into an infinite set \( Y = \{ Y(1), Y(2), \ldots, Y(N) \ldots \} \) of output vector signals using a Wiener-like approach then we have to find a set of corresponding Wiener filters \( \{ F(1), F(2), \ldots, F(N) \ldots \} \) so that each representative \( F_i \) of the filter set relates to a representative \( y(1) \) of the signal-vector set \( Y \). Therefore such a filter cannot be applied if \( X \) and \( Y \) are infinite sets of signals. Moreover, in some situations, a recognizer must be used that will determine to which of the filters \( \{ F(1), F(2), \ldots, F(N) \ldots \} \) each component from \( Y \) should be directed.

Note that even in the case when \( X \) and \( Y \) are finite sets, \( Y = \{ Y(1), Y(2), \ldots, Y(N) \} \) and \( X = \{ x(1), x(2), \ldots, x(N) \} \), and then \( Y \) and \( X \) can be represented as finite vectors, the Wiener approach leads to computation of large covariance matrices. Indeed, if each \( y_i \) has \( n \) components and each \( x_i \) has \( m \) components then the Wiener approach leads to computation of a product of an \( mN \times nN \) matrix and an \( nN \times nN \) matrix and computation of an \( nN \times nN \) pseudo-inverse matrix \([14]\). This requires \( O(2mn^2N^3) \) and \( O(2n^3N^3) \) flops, respectively \([5]\). If \( m, n \) and \( N \) are sufficiently large then the computational work associated with this approach becomes unreasonably hard.

To avoid such drawbacks, we here study an approach that allows us to use only one filter to process any signal from the infinite set \( Y \).

The first question we address in the paper is as follows. Let \( X \) and \( Y \) be infinite sets of signals. How should we construct a single optimal filter \( F : Y \rightarrow X \) which can be applied to each pair of signals \( x, y \in X \times Y \) and which, moreover, transforms each \( y \) to a corresponding \( x \) with associated minimal error?

Surprisingly, perhaps, the answer is based firstly, on a dual representation of signal \( x \) in different spaces and secondly, on the use of the special norm \((3)\) in the statement of the problem. The dual representation means that \( x \) is considered as a single signal in one representation, and on the other hand, as an infinite set of signals in the other, original, representation. A detailed explanation is given in Section VI. Examples of different special cases of the norm \((3)\) used in our statement of the problem are presented in Section VI.

The answer for the first question is provided in Sections II-A, II-C, in Theorems 3 and 4, and in Section III-E. The special norm is given by \((3)\) below.

2) Second motivation: improvement in the filter performance: The performance of filters used for data filtering, compression and subsequent reconstruction, is characterized by the accuracy, the compression ratio and the related computational load. The Karhunen-Loève filter (KLF) \([10], [11]\). 

\(1\) Some references on Wiener-like filtering can be found in \([6], [9], [13], [14], [15]\).

\(2\) We say a random vector \( x \) is finite if each realization \( x = x(\omega) \) has a finite number of scalar components.
It has been shown in [4] that for such problems the solution given in [4] should be used. This solution has been obtained for the case of a minimal norm matrix. In Section III-A below, we extend the solution to a more general case and exploit it in our filter derivation.

B. Contribution

We provide a theory for the new filter which processes infinite sets of random signals. The filter is presented in the form (1) and is based on a new iterative scheme (Section II-C) that ensures the filter accuracy is improved with each step. The final step provides signal compression and decompression (Section III-D). This step is based on the solution of a new rank-constrained matrix approximation problem (Section III-A) following the methodology used in [4]. A rigorous error analysis for the filter is also given (Section III-B).

II. Method description and statement of the problem

A. Filter structure

Let \( (\Omega, \Sigma, \mu) \) be a probability space, where \( \Omega = \{ \omega \} \) is the set of outcomes, \( \Sigma \) a \( \sigma \)-field of measurable subsets in \( \Omega \) and \( \mu : \Sigma \to [0,1] \) an associated probability measure on \( \Sigma \) with \( \mu(\Omega) = 1 \).

Let \( x = \{ x(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}^m) \mid \alpha \in K \} \) where \( K \) is a measurable set [7] in some appropriate \( \sigma \)-field with measure \( \lambda(\alpha) \). Thus \( x \in L^2(\Omega \times K, \mathbb{R}^m) \). We observe an interesting duality in the representation of \( x \) in that \( x \) is a single signal in the space \( L^2(\Omega \times K, \mathbb{R}^m) \) and an infinite set of signals \( \{x(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}^m) \mid \alpha \in K \} \) in the space \( L^2(\Omega, \mathbb{R}^m) \). In similar fashion we write \( y = \{ y(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}^m) \mid \alpha \in K \} \) and \( y \in L^2(\Omega \times K, \mathbb{R}^m) \). We interpret \( x \) as a signal that should be estimated and \( y \) as an observed signal that can be used as an input to the filter \( F_p \) studied below. In an intuitive way \( y \) can be regarded as a noise-corrupted version of \( x \). For example, \( y \) can be interpreted as \( y = x + n \) where \( n \) is white noise. In this paper, we do not restrict ourselves to this simplest version of \( y \) and assume that the dependence of \( y \) on \( x \) and \( n \) is arbitrary.

Let \( x_1 = y \) and let \( x_1, \ldots, x_p \) be a sequence of estimates of \( x \) obtained by the method described in Sections II-C, III-A and III-B below. The filter \( F_p \) is presented in the form

\[
F_p(y) = \sum_{k=1}^{p} G_k H_k P_k(x_1, \ldots, x_k),
\]

where \( P_k : L^2((\Omega \times K)^k, \mathbb{R}^m) \to L^2(\Omega \times K, \mathbb{R}^m) \), \( H_k : L^2(\Omega \times K, \mathbb{R}^m) \to L^2(\Omega \times K, \mathbb{R}^m) \) and \( G_k : L^2(\Omega \times K, \mathbb{R}^m) \to L^2(\Omega \times K, \mathbb{R}^m) \).

The purpose of the components \( G_k \), \( H_k \) and \( P_k \) has been described above in Section I-A2. The components \( P_k \) are defined in Sections II-C and III-C, and components the \( G_k \) and \( H_k \) are determined in Section III-B.

B. Basic ideas

The input to the filter \( F_p \) is a stochastic signal – a random vector \( y \in L^2(\Omega \times K, \mathbb{R}^m) \). We wish to filter and compress \( y \), and to reconstruct a compressed vector that is close to \( x \).
The first is to represent y as a generalized sequence of signals
\[ y = \{ y(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}^m) \mid \alpha \in K \} \]
where (typically) \( K \subset \mathbb{R} \) and to use the special norm (3) in
the statement of the problems in Sections II-C and II-D. The filter
processes each \( y(\cdot, \alpha) \) separately but the filter components are
determined from the entire sets \( x \) and \( y \).

The filter is best illustrated for a particular case when the sets \( x \) and \( y \) are finite (as in Example 4 of Section VI), and we
have, in (1), \( p = 1 \), \( H_1 \), and \( P_1 \) are each the identity mapping,
and \( G_1 \) is determined from problem (4). In such a case, we set
\( y = \{ y(t_k) \in L^2(\Omega, \mathbb{R}^m) \mid t_k \in \mathbb{R} \land k = 1, \ldots, N \} \)
and \( x = \{ x(t_k) \in L^2(\Omega, \mathbb{R}^m) \mid t_k \in \mathbb{R} \land k = 1, \ldots, N \} \).
The estimate \( \tilde{x} \) of \( x \) by the proposed filter is given by
\[ \tilde{x} = [\tilde{x}(t_1), \ldots, \tilde{x}(t_N)] = \hat{G}_1(y(t_1), \ldots, y(t_N)) \]
where \( \hat{G}_1 \) is provided by Theorem 3 of Section III-B and,
and find \( \hat{G}_1 \) that satisfies
\[ \| x - \hat{G}_1(y_1) \|_K^2 = \min_{\hat{G}_1} \| x - \hat{G}_1(y_1) \|_K^2. \]

Let the model for the filter \( F_0 \) be given by (1). The proposed
device for determining operators \( H_1, G_1 \), and \( P_1 \) in (1) is
as follows. The operators \( H_k \), \( G_k \), and \( P_k \) consists of \( p \) successive steps based on the following idea. If \( x_1 = y \) and
the subsequent estimates of \( x \) denoted by \( x_2, \ldots, x_{k-1} \) have been
determined from successive steps then for the \( k \)th step, we define
\[ x_k = \sum_{j=1}^{k-1} \hat{G}_j H_j(y_1, \ldots, y_j) \]
where \( y_j = P_j(x_1, \ldots, x_j) \) and \( P_j \) is conceptually defined in
(1) and where \( \hat{G}_1, \ldots, \hat{G}_k \) are determined from the proposed
error minimization criteria. We propose the following scheme.

First, we introduce the norm \( \| \cdot \|_K^2 \) defined by
\[ \| x \|_K^2 = \frac{1}{K} \int \| x(\omega, \alpha) \|^2 d(\mu(\omega), \lambda(\alpha)), \]
where \( \lambda(K) = \int K d(\lambda(\alpha)) \) is the measure of \( K \) and
where \( \| x(\omega, \alpha) \|^2 \) is the Euclidean norm of \( x(\omega, \alpha) \in \mathbb{R}^m \). More
explanation is given in Section VI.

The proposed scheme consists of a sequence of steps.

Step 1 The initial step. For \( k = 1 \), let \( x_1 = y \) and let \( P_1 = I \)
and \( H_1 = I \) so that \( y_1 := x_1 \) and find \( \hat{G}_1 \) that satisfies
\[ \| x - \hat{G}_1(y_1) \|_K^2 = \min_{\hat{G}_1} \| x - \hat{G}_1(y_1) \|_K^2. \]

Set \( x_2 = \hat{G}_1(x_1). \)

Step \( \ell - 1 \) for \( \ell = 3, \ldots, p. \) The filtering steps. Let \( x_2, \ldots, x_{\ell - 1} \)
be known from the preceding \( \ell - 2 \) steps. For \( k = \ell - 1 \) in
(2) define \( P_{\ell - 1} \) so that \( y_{\ell - 1} = P_{\ell - 1}(x_1, \ldots, x_{\ell - 1}) \) and find
\( \hat{G}_1, \hat{G}_2, \ldots, \hat{G}_{\ell - 1} \) that satisfy
\[ \| x - \sum_{j=1}^{\ell - 1} \hat{G}_j H_j(y_1, \ldots, y_j) \|_K^2 = \min_{\hat{G}_1, \ldots, \hat{G}_{\ell - 1}} \| x - \sum_{j=1}^{\ell - 1} \hat{G}_j H_j(y_1, \ldots, y_j) \|_K^2. \]

Set \( x_\ell = \sum_{j=1}^p \hat{G}_j H_j(y_1, \ldots, y_j). \)

Step \( p. \) The reconstruction step. At the final step, for \( k = p \) in
(2) define \( P_p \) so that \( y_p = P_p(x_1, \ldots, x_p) \) and find \( \hat{G}_1, \hat{G}_2, \ldots, \hat{G}_p \) that satisfy
\[ \| x - \sum_{j=1}^{p} \hat{G}_j H_j(y_1, \ldots, y_j) \|_K^2 = \min_{\hat{G}_1, \ldots, \hat{G}_p} \| x - \sum_{j=1}^{p} \hat{G}_j H_j(y_1, \ldots, y_j) \|_K^2. \]

subject to the rank constraints
\[ \sum_{j=1}^{p} \text{rank } \hat{G}_j \leq r \leq \min\{m, n\} \]
and rank \( \hat{G}_j \leq r_j \)

with \( \sum_{j=1}^{p} r_j = r. \)

We set \( x_{p+1} = \sum_{j=1}^{p} \hat{G}_j H_j(y_1, \ldots, y_j). \)

Due to condition (7), this step provides data compression–
reconstruction as it is shown in Section III-D below.

D. Statement of the problem

The problem we solve follows from the basic structure
presented above. Let \( x \) and \( y \) be random signals where \( x \) is an unobservable signal and \( y \) is an observable input signal.
Both \( x \) and \( y \) are analytically unknown and the only available
information is given by covariance matrices formed from \( x \) and \( y \). We wish to find a filter of the form (1) with \( \hat{G}_1, \ldots, \hat{G}_p \)
determined from the error minimization problems (4)–(7). The
filter must satisfy the requirements (a), (b) and (c) of the Section II-B.

The main differences of the problem given by (1), (4)–(7) from known problems are as follows. First, we are looking for the optimal filter that minimizes the associated error for any signal from an infinite signal set. This is achieved due to the new norm given by (3). The second major difference is the special structure of the filter (1)–(7). Each step of the suggested procedure represents the solution of a best approximation problem that minimizes an associated error. If the final stage Step p is omitted the scheme performs filtering only. If Step p is included then the scheme provides simultaneous signal filtering and compression and the subsequent reconstruction.

Solutions to problems (4)–(7) are given below under assumptions that certain covariance matrices are known. The assumptions are specified in Theorems 3 and 4 below. It is important to note that for new signals constructed during the phase of iterative improvement the required covariance matrices can be constructed from the original data.

E. Advantages of the filter

The advantages of the proposed filter are discussed in Section III-E.

III. MAIN RESULTS

The solution to the problem (6)–(7) is based on the results presented in Section III-A. The solution itself is given in Theorem 4 below.

A. Generic low-rank matrix approximation problem

Let \( C_{m\times n} \) be the set of \( m \times n \) complex valued matrices, and denote by \( R(m, n, k) \subseteq C_{m\times n} \) the variety of all \( m \times n \) matrices of rank \( k \) at most. Fix \( A = [a_{ij}]_{i,j=1}^{m,n} \in C_{m\times n} \). Then \( A^* \in C_{n\times m} \) is the conjugate transpose of \( A \). Let the SVD of \( A \) be given by

\[
A = U_A \Sigma_A V_A^*, \tag{8}
\]

where \( U_A \in C_{m\times m} \) and \( V_A \in C_{n\times n} \) are unitary matrices and

\[
\Sigma_A := \text{diag}(\sigma_1(A), \ldots, \sigma_{\min(m,n)}(A)) \in C_{m\times n}
\]

is a generalized diagonal matrix with singular values \( \sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq 0 \) on the main diagonal. We write

\[
A^* = V_A \Sigma_A^* U_A^*
\]

for the Moore-Penrose pseudo-inverse [1] of the matrix \( A \). In this definition

\[
\Sigma_A^* := \text{diag}(\sigma_1(A)^{\dagger}, \ldots, \sigma_{\min(m,n)}(A)^{\dagger}) \in C_{n\times m}
\]

where we use the notation

\[
\sigma_j(A)^{\dagger} = \begin{cases} 
\sigma_j(A)^{-1} & \text{if } \sigma_j(A) \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

Let \( U_A = [u_1 u_2 \ldots u_r] \) and \( V_A = [v_1 v_2 \ldots u_r] \) be the representations of \( U \) and \( V \) in terms of their columns. Let \( r = \text{rank } A \) and write \( U_A = [U_{A1} U_{A2}] \) where \( U_{A1} \in C_{m\times r} \) and \( U_{A2} \in C_{m\times (m-r)} \) and \( V_A = [V_{A1} V_{A2}] \) where \( V_{A1} \in C_{n\times r} \) and \( V_{A2} \in C_{n\times (n-r)} \). Then

\[
P_{A,L} := \sum_{i=1}^{r} u_i u_i^* = U_{A1} U_{A1}^* \quad \text{and} \quad P_{A,R} := \sum_{i=1}^{r} v_i v_i^* = V_{A1} V_{A1}^*
\]

are the orthogonal projections on the ranges of \( A \) and \( A^* \), respectively. We note that \( \Sigma_A = U_A^* V_A \) and we will use the notation

\[
\Sigma_A = \begin{bmatrix} U_{A1}^* & U_{A2}^* \\ U_{A1} & U_{A2} \end{bmatrix} A \begin{bmatrix} V_{A1} & V_{A2} \\ U_{A1}^* V_{A2} & U_{A2} V_{A2} \end{bmatrix} = \begin{bmatrix} A_1 & \odot \\ \odot & \odot \end{bmatrix}
\]

Define

\[
A_k := (A)_k := \sum_{i=1}^{k} \sigma_i(A) u_i v_i^* = U_{Ak} \Sigma_{Ak} V_{Ak}^* \in C_{m\times n}
\]

for \( k = 1, \ldots, \text{rank } A - 1 \), where

\[
U_{Ak} = [u_1 u_2 \ldots u_k], \quad \Sigma_{Ak} = \text{diag}(\sigma_1(A), \ldots, \sigma_k(A)) \quad \text{and} \quad V_{Ak} = [v_1 v_2 \ldots u_k].
\]

For \( k \geq \text{rank } A \), we note that \( A_k := A = (A)_{\text{rank } A} \). For \( 1 \leq k \leq \text{rank } A \), the matrix \( A_k \) is uniquely defined if and only if \( \sigma_k(A) > \sigma_{k+1}(A) \).

Lemma 1: Let \( A \in C_{m\times n} \). If \( R \in C_{m\times m} \) and \( S \in C_{n\times n} \) are unitary matrices then \( R(A)S^* = (RAS^*)_k \).

Proof. If \( A = U_A \Sigma_A V_A^* \) is the singular value decomposition of \( A \) then \( RAS^* = (RU_A \Sigma_A (SV_A)^*)^* \) is the singular value decomposition of \( RAS^* \). Thus

\[
(RAS^*)_k = \sum_{i=1}^{k} \sigma_i(A) (Ru_i)(Sv_i)^* = R \sum_{i=1}^{k} \sigma_i(A) u_i v_i^* S^* = R(A)S^*.
\]

Henceforth \( \| \cdot \|_F \) denotes the Frobenius norm. Below, we provide generalizations of the classical minimization problem due to Eckart and Young [3]. First, we present the result obtained in [4].

Theorem 1: [4] Let matrices \( A \in C_{m\times n}, B \in C_{m\times p} \) and \( C \in C_{n\times q} \) be given. Then

\[
X = B^t (P_{B,L} A P_{C,R}) C^t
\]

is a solution to the minimization problem

\[
\min_{X \in R(p,r,k)} \| A - BXC \|_F,
\]

having the minimal \( \| X \|_F \). This solution is unique if and only if either

\[
k \geq \text{rank } (P_{B,L} A P_{C,R})
\]

or

\[
1 \leq k < \text{rank } (P_{B,L} A P_{C,R}) \quad \text{and} \quad \sigma_k(P_{B,L} A P_{C,R}) > \sigma_{k+1}(P_{B,L} A P_{C,R}).
\]

We will explain this result as briefly as possible and refer the reader to the original article by Friedland and Torokhti [4] for more details. Let \( s = \text{rank } B \) and \( t = \text{rank } C \). Using the
notation introduced above we write $UB = [UB1 U_{B2}]$ where $UB1 \in \mathbb{C}^{m \times x}$ and $UB2 \in \mathbb{C}^{(m-x) \times m}$ and $VC = [VC1 VC2]$ where $VC1 \in \mathbb{C}^{n \times t}$ and $VC2 \in \mathbb{C}^{n \times (n-t)}$. Thus we have

$$\Sigma_B = \left[ \begin{array}{c} \bar{U}_{B1}^* U_{B2}^* \\ \bar{U}_{B2}^* \bar{U}_{B1} \end{array} \right] \left[ \begin{array}{c} V_{B1} \\ V_{B2} \end{array} \right]$$

and

$$\Sigma_C = \left[ \begin{array}{c} \bar{U}_{C1}^* U_{C2}^* \\ \bar{U}_{C2}^* \bar{U}_{C1} \end{array} \right] \left[ \begin{array}{c} V_{C1} \\ V_{C2} \end{array} \right]$$

We will write

$$\bar{A} = U_B^* AV_C = \left[ \begin{array}{c} A_{11} \\ A_{21} \\ A_{22} \end{array} \right]$$

where $A_{11} = UB1^* AV_C 1 \in \mathbb{C}^{x \times t}$, $A_{12} = UB1^* AV_C 2 \in \mathbb{C}^{(m-x) \times t}$ and $A_{22} = UB2^* AV_C 2 \in \mathbb{C}^{(m-x) \times (n-t)}$. The idea behind our explanation of Theorem 1 is simple. The Frobenius norm is not changed when we multiply on the left or the right by a unitary matrix. Thus we have

$$\|A - BXC\|_F = \|U_B^* (A - UB1^* VB2^* UXV_{C1} V_{C2}^*) V_C\|_F$$

$$= \|\bar{A} - \Sigma_B \bar{A} \Sigma_C\|_F$$

where

$$\bar{X} = V_B^* X V_C = \left[ \begin{array}{c} X11 \\ X_{21} \\ X_{22} \end{array} \right].$$

Since

$$\Sigma_B = \left[ \begin{array}{c} B1 \\ B2 \end{array} \right] \text{ and } \Sigma_C = \left[ \begin{array}{c} C1 \\ C2 \end{array} \right]$$

it follows that

$$\|A - BXC\|_F = \|A_{11} - B_{11} X_{11} C_{11}\|_F$$

$$+ \|A_{12}\|_F + \|A_{21}\|_F + \|A_{22}\|_F$$

from which it follows that the solution $X = X_0$ to problem (13) is given by $X_{11} = B_{11}(A_{11})_1 C_{11}(A_{11})_1$, $X_{21} = 0$, $X_{22} = \infty$ and $X_{22} = 0$. We will use the following argument on a number of occasions. We observe $B_{11} = V_{B1}^* B_{11}^* U_{B1}$ and $C_{11} = V_{C1}^* C_{11} U_{C1}$ and hence deduce

$$X_{11} = B_{11}^* (A_{11})_1 C_{11}^*$$

$$= \left[ \begin{array}{c} B_{11}^* \\ 0 \end{array} \right] \left[ \begin{array}{c} I_{x} \\ 0 \end{array} \right] \left[ \begin{array}{c} A_{11} \\ A_{21} \\ A_{22} \end{array} \right] \left[ \begin{array}{c} I_{t} \\ 0 \end{array} \right] \left[ \begin{array}{c} C_{11}^* \\ 0 \end{array} \right]$$

$$= V_{B1}^* B_{11}^* \left[ \begin{array}{c} U_{B1} \\ U_{B2} \end{array} \right] \left[ \begin{array}{c} U_{B1}^* \\ 0 \end{array} \right] \left[ \begin{array}{c} A_{11} \\ 0 \end{array} \right] \left[ \begin{array}{c} C_{11}^* \\ 0 \end{array} \right]$$

$$= V_{B1}^* B_{11}^* (UB1^* UB1) C_{11}^* U_{C1}$$

$$= V_{B1}^* B_{11}^* (P_{B,L} APC_{R}) C_{11}^* U_{C1}$$

We know that $V_{B1}^* V_{B1} = B_{11}^* B_{11}$ and $U_{C1} U_{C1}^* = C_{11}^* C_{11}$ and so

$$V_{B1}^* X_{11} U_{C1} = P_{B,L} B_{11}^* (P_{B,L} APC_{R}) C_{11}^* U_{C1}$$

$$= B_{11}^* (P_{B,L} APC_{R}) C_{11}^* U_{C1}$$

Therefore

$$X_0 = \left[ \begin{array}{c} V_{B1} \\ V_{B2} \end{array} \right] \left[ \begin{array}{c} B_{11}^* (A_{11})_1 C_{11}^* \\ 0 \end{array} \right] \left[ \begin{array}{c} U_{C1} \\ U_{C2} \end{array} \right]$$

$$= B_{11}^* (P_{B,L} APC_{R}) C_{11}^* U_{C1}$$

To find a general solution to problem (13) without the constraint of the minimal $\|X\|_F$ we must have

$$X = V_B^* \tilde{X} V_C^* \text{ with } \tilde{X} = \left[ \begin{array}{c} X_{11} \\ X_{21} \\ X_{22} \end{array} \right].$$

(14)

where $X_{11} = V_{B1}^* B_{11}^* (P_{B,L} APC_{R}) C_{11}^* U_{C1}$ and $X_{21}$, $X_{22}$ are chosen in such a way that $\tilde{X} \in R(p,q,k)$. In Theorem 2 below, we show how to choose $X_{11}$ and $X_{22}$ (see (21)) to satisfy the condition $\tilde{X} \in R(p,q,k)$.

**Theorem 2:** If the requirement that $\|X\|_F$ should be minimized is omitted then the solution to problem (13) is not unique. In this case the general solution to problem (13) can be written in the form

$$X = B_{11}^* (P_{B,L} APC_{R}) C_{11}^* + K,$$

(15)

where

$$K = \left[ \begin{array}{c} P_{B,R} \\ I - P_{B,R} \end{array} \right] \left[ \begin{array}{c} 0 \\ K_{21} \\ K_{22} \end{array} \right] \left[ \begin{array}{c} P_{C,L} \\ I - P_{C,L} \end{array} \right].$$

(16)

$$K_{12} = V_{B1}^* X_{11} Q U_{C2}^*,$$

$$K_{21} = V_{B2}^* P X_{11} U_{C1}^*,$$

$$K_{22} = V_{B2}^* P X_{11} U_{C2}^*,$$

and where $P \in \mathbb{C}^{x \times x}$ and $Q \in \mathbb{C}^{x \times (x-q)}$ are arbitrary.

**Proof.** To preserve rank $\tilde{X} = X_{11}$ we should choose $X_{12}$, $X_{21}$ and $X_{22}$ from (14) in a compatible form. To be specific there must exist matrices $P \in \mathbb{C}^{x \times x}$ and $Q \in \mathbb{C}^{x \times (x-q)}$ such that

$$\left[ \begin{array}{c} I \\ -P \\ I \end{array} \right] \left[ \begin{array}{c} X_{11} \\ X_{21} \\ X_{22} \end{array} \right] \left[ \begin{array}{c} I \\ -Q \\ I \end{array} \right] = \left[ \begin{array}{c} X_{11} \\ \infty \end{array} \right],$$

from which it follows that

$$X_{12} = X_{11} Q,$$

$$X_{21} = P X_{11}$$

and $X_{22} = P X_{11} Q$ (18)

where $X_{11} = V_{B1}^* B_{11}^* (P_{B,L} APC_{R}) C_{11}^*$ $U_{C1}$. Since

$$X = V_B^* \tilde{X} V_C^* = V_B^* \left[ \begin{array}{c} X_{11} \\ X_{21} \\ X_{22} \end{array} \right] U_{C1}^* + V_B^* \left[ \begin{array}{c} 0 \\ X_{21} \\ X_{22} \end{array} \right] U_{C2}^*$$

we have

$$X = B_{11}^* (P_{B,L} APC_{R}) C_{11}^* + V_{B1} X_{12} U_{C1}^* + V_{B2} X_{22} U_{C2}^*$$

We note that $V_{B1} = V_{B1}^* V_{B1} = P_{B,R} V_{B1}$, $U_{C1} = U_{C1}^* U_{C1} U_{C1} = U_{C1}^* P_{C,L}$, $V_{B2} = V_{B2} X_{22} U_{C2}^* = \left( I_{p} - \right.$
\( P_{B,R}V_{B2} \) and \( U_{C2}^* = U_{C2}U_{C2}^* = U_{C2}(I_q - P_{C,L}) \) and hence
\[
X = B^1(P_{B,L}AP_{C,R})kC^† + (P_{B,L}R)B1X11QU_{C2}^*(I_q - P_{C,L}) + (I_p - P_{B,R})V_{B2}PX_{11}U_{C1}P_{C,L}
\]
\[
= B^1(P_{B,L}AP_{C,R})kC^† + (P_{B,L}R)B1K12(I_q - P_{C,L}) + (I_p - P_{B,R})K22(I_q - P_{C,L})
\]
where \( K12 = V_{B2}X11QU_{C2}^*, K21 = V_{B2}PX_{11}U_{C1}, K22 = V_{B2}PX_{11}QU_{C2}^* \) and \( X_{11} = V_{B2}B^1(P_{B,L}AP_{C,R})kC^†U_{C1}^† \).

The following Corollary will be used in the next Section.

Corollary 1: Let \( p = m, q = n \) and \( B = I \) and define
\( \tilde{A} = AV_C = [A_1 A_2] \) where \( A_1 = AV_{C1} \in \mathbb{C}^{m \times t} \) and \( A_2 = AV_{C2} \in \mathbb{C}^{m \times (n-t)} \). The general solution to the problem
\[
\min_{X \in \mathbb{R}^{m \times n \times k}} \| A - XC \|_F
\]
is given by
\[
X = (AP_{C,R})kC^† + (AP_{C,R})kC^†U_{C1}QU_{C2}^*(I_q - P_{C,L}).
\]
(20)
where \( Q \in \mathbb{C}^{n \times (n-t)} \) is arbitrary.

Proof Let \( \tilde{X} = XU_C = X[U_{C1} U_{C2}] = [X_1 X_2] \). Since
\[
\| A - XC \|_F = \| A - \tilde{X}SC \|_F = \| A_1 - X_1C_1 \|_F + \| A_2 \|_F
\]
it follows that the solution to problem (19) with minimum value for \( \| X \|_F \) is given by \( X = (A_1PC,R)kC^† \). Now consider the problem without the constraint requiring a minimum value for \( \| X \|_F \). To preserve rank \( \tilde{X} = X \) we should choose \( X_2 \) in a compatible form. To be specific there must exist a matrix \( Q \in \mathbb{C}^{n \times (q-t)} \) such that
\[
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
\begin{bmatrix}
I & -Q \\
\text{O} & I
\end{bmatrix}
= \begin{bmatrix}
X_1 \\
\text{O}
\end{bmatrix}
\]
from which it follows that
\[
X_2 = X_1Q = \text{O}.
\]
(21)
By adapting the general argument used in the previous theorem we can see that \( X_1 = (AP_{C,R})kC^†U_{C1} \) and \( X_2 = (AP_{C,R})kC^†U_{C1}Q \). Therefore
\[
X = \tilde{X}U_C = X_1U_{C1} + X_2U_{C2}
\]
and so \( X = (AP_{C,R})kC^† + (AP_{C,R})kC^†U_{C1}QU_{C2}^* \). We note that \( U_{C2}^* = U_{C2}U_{C2}^* = U_{C2}(I_q - P_{C,L}) \) and hence
\[
X = (AP_{C,R})kC^† + (AP_{C,R})kC^†U_{C1}QU_{C2}^*(I_q - P_{C,L}).
\]

Remark 1: The Eckart-Young theorem [3] follows from Theorem 2 as a particular case when \( p = m, q = n, B = I_m \) and \( C = I_n \).

B. Solution of the problems (4)–(5) and (6)–(7)

Let \( x = \{ x(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}) \mid \alpha \in K \} \) \( \in L^2(\Omega \times K, \mathbb{R}) \) and \( y = \{ y(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}) \mid \alpha \in K \} \in L^2(\Omega \times K, \mathbb{R}) \) where \( K \) is a measurable set with finite measure \( \lambda(K) \) in some sigma field of measurable sets [7].

We write \( x = (x(1), \ldots, x(n))^T \) and \( y = (y(1), \ldots, y(n))^T \) where \( x(i) \in L^2(\Omega \times K, \mathbb{R}) \) for \( i = 1, \ldots, m \) and \( y(j) \in L^2(\Omega \times K, \mathbb{R}) \) for \( j = 1, \ldots, n \) are real valued random variables. Let
\[
(x(i, j)) = \frac{1}{\lambda(K)} \int_{\Omega \times K} x(i(\omega, \alpha))y(j(\omega, \alpha)) d(\mu(\omega), \lambda(\alpha))
\]
and
\[
(y(i, j)) = \frac{1}{\lambda(K)} \int_{\Omega \times K} y(i(\omega, \alpha))y(j(\omega, \alpha)) d(\mu(\omega), \lambda(\alpha))
\]
and define covariance matrices
\[
R[xy]^T = [(x(i, j))] \in \mathbb{R}^{m \times n}
\]
(24)
and
\[
R[yy]^T = [(y(i, j))] \in \mathbb{R}^{n \times n}.
\]
(25)

If \( M \in \mathbb{R}^{m \times n} \) then we define a corresponding operator \( M : L^2(\Omega \times K, \mathbb{R}) \rightarrow L^2(\Omega \times K, \mathbb{R}) \) by the formula
\[
[M[y]](\omega, \alpha) = M[y(\omega, \alpha)].
\]
(26)
Henceforth all such operators will be denoted by a calligraphic letter.

We wish
(i) to give explicit solutions to the problems (4)–(5) and (6)–(7), and
(ii) to show that the error associated with the proposed method decreases as the number \( k \) of steps (4)–(7) increases.

First, we give a method for the determination of the operators \( \mathcal{H}_1, \ldots, \mathcal{H}_k \) in the filter model (1).

Definition 1: The random vectors \( \{v_i\}_{i=1}^{k} \subset L^2(\Omega \times K, \mathbb{R}) \) are called pairwise orthogonal if
\[
R[v_i v_j^T] = \text{O} \in \mathbb{R}^{n \times n}
\]
(27)
for \( i \neq j \) with \( i, j = 1, \ldots, k \).

Lemma 2: Let \( y_i = \mathcal{P}_i(x_1, \ldots, x_i) \) and \( y_i = \mathcal{H}_i(y_1, \ldots, y_i) \) for each \( i = 1, 2, \ldots, k \) where the linear operators \( \mathcal{H}_i : L^2(\Omega \times K, \mathbb{R}) \rightarrow L^2(\Omega \times K, \mathbb{R}) \) are defined inductively by the generalized Gram-Schmidt algorithm
\[
v_1 = y_1 \quad \text{and} \quad v_i = y_i - \sum_{i=1}^{i-1} \mathcal{L}_i(v_i)
\]
(28)
with \( \mathcal{L}_i : L^2(\Omega \times K, \mathbb{R}) \rightarrow L^2(\Omega \times K, \mathbb{R}) \) defined by \( \mathcal{L}_i(v_i) = L_i v_i \) where
\[
L_i = \mathcal{R}[y_i v_i^T]R[v_i v_i^T] + M_i(I - \mathcal{R}[v_i v_i^T]R[v_i v_i^T])
\]
(29)
and \( M_i \in \mathbb{R}^{n \times n} \) is arbitrary for each \( i = 2, \ldots, k \). Then the vectors \( v_1, v_2, \ldots, v_k \) are pairwise orthogonal in \( L^2(\Omega \times K, \mathbb{R}) \).
Proof. The proof follows directly from (28), (29) and Definition 1 on the basis of the relation $R[y, y]^T = R[y, v]^T R[v, v]^T$. We refer the reader to Torokhti and Howlett [14], p. 168 for additional information.

Remark 2: In practice it is usual to take $M_{id} = \emptyset$.

Remark 3: The Gram-Schmidt formula can be rearranged to give

$$y_1 = v_1 \quad \text{and} \quad y_i = v_i + \sum_{t=1}^{i-1} L_{it} v_t$$

for each $i = 2, \ldots, k$. Thus we can write

$$y^{(i)} = (I_m + L_i) y^{(i)}$$

from which it follows easily that

$$v^{(i)} = (I_m + L_i)^{-1} y^{(i)}$$

is well defined. This formula shows clearly that $v_i = \mathcal{H}_i(y_1, \ldots, y_i) = \mathcal{H}_1 y_1 + \cdots + \mathcal{H}_{i-1} y_{i-1} + y_i$.

The matrices $R[xv]^T$, $R[y, v]^T$ and $R[v, v]^T$ are assumed known. Estimation of these matrices is discussed in Section 5.3 of the book by Torokhti and Howlett [14].

Theorem 3: For $k = 1, \ldots, p - 1$ the solution $\tilde{G}_j$ to problems (4)–(5) is


where $M_j$ is arbitrary for each $j = 1, \ldots, k - 1$ and $v_j = \mathcal{H}_j(y_1, \ldots, y_j)$ has been determined for each $j = 1, 2, \ldots, k$ by Lemma 2. The associated error is

$$\|x - x_h\|^2_{K, \Omega} = \text{tr}\left( R[xx]^T - \sum_{j=1}^{k-1} R[xv] R[y, v]^T R[y, x] \right).$$

Proof. We write $\delta_k = \|x - x_h\|^2_{K, \Omega}$. We have $\|x\|^2_{K, \Omega} = \text{tr}(R[xx]^T)$. Therefore, for $k = 1, \ldots, p$,

$$\delta_k = \text{tr}\left( R \left[ x - \sum_{j=1}^{k-1} \mathcal{G}_j(v_j) \right] \left[ x - \sum_{j=1}^{k-1} \mathcal{G}_j(v_j) \right]^T \right)$$

$$= \text{tr}\left( R[xx]^T - \sum_{j=1}^{k-1} (R[xv]^T G_j^T + G_j R[y, x]) \right)$$

$$+ \text{tr}\left( \sum_{j=1}^{k-1} \mathcal{G}_j(v_j) \left[ \sum_{j=1}^{k-1} G_j(v_j) \right]^T \right) .$$

In (31),

$$R \left[ \sum_{j=1}^{k-1} \mathcal{G}_j(v_j) \left[ \sum_{j=1}^{k-1} G_j(v_j) \right]^T \right] = \sum_{j=1}^{k-1} G_j R[v, v]^T G_j^T$$

owing to the orthogonality of vectors $v_1, v_2, \ldots, v_k$. Thus,

$$\delta_k = \| R[y, v]^T \|^2_{F}$$

$$\sum_{j=1}^{k-1} \| R[xv]^T R[v, v]^T \|^{1/2} + \sum_{j=1}^{k-1} J_j(G_j)$$

where

$$J_j(G_j) = \| A_j - G_j C_j \|^2_{F} .$$

Because the terms $J_j(G_1), \ldots, J_k(G_k)$ in the sum $\sum_{j=1}^{k} J_j(G_j)$ are determined independently we have

$$\min_{G_1, \ldots, G_k} \sum_{j=1}^{k} J_j(G_j) = \sum_{j=1}^{k} \min_{G_j} J_j(G_j) .$$

Therefore, for $k = 1, \ldots, p - 1$, the minimum in (4)–(5) is achieved if


and if we multiply on the right by $(R[v, v]^T 1/2)^T R[v, v]^T$ we can see that


This is clearly true. Therefore the optimal value $\tilde{G}_j$ is given by


where $M_j$ is arbitrary for all $j = 1, 2, \ldots, k - 1$.

Remark 4: We remind the reader that there is no rank restriction on the matrix operators during the filtration procedure. Thus Theorem 3 solves a matrix optimization problem without rank restriction.

Define

$$A_j = R[xv]^T R[v, v]^T 1/2$$

and let

$$A_j = U_{A_j} \Sigma_{A_j} V_{A_j}^T$$

and $C_j = U_{C_j} \Sigma_{C_j} V_{C_j}^T$ be the singular value decompositions for $A_j$ and $C_j$ respectively. We use a similar notation to that used in (8). To derive a solution to the problem (6)–(7) define

$$U_{C_j} = [U_{C_j}, U_{C_j} 2]$$

and $V_{C_j} = [V_{C_j}, V_{C_j} 2]$ and let

$$A_j = A_j V_j C_j$$

and $D_j = \text{diag}(\sigma_1(C_j), \ldots, \sigma_t(C_j)) \in \mathbb{C}^{t_j \times t_j}$ with $t_j = \text{rank} C_j$ and

$$K_j = (\tilde{A}_j)_{t_j} D_j^{-1} Q_j U_{A_j}^T (I - C_j C_j^T)$$

where $Q_j$ is arbitrary. The singular values of matrix $A_j$ are denoted by $\sigma_k(A_j)$. The matrix

$$(A_j)_{r_j} = U_{A_j} r_j \Sigma_{A_j} r_j V_{A_j}^T$$

(36)
is determined similarly to \((A)_k\) given by (10) with the replacement of \(A\) and \(k\) by \(A_1\) and \(r_1\), respectively.

**Theorem 4:** The solution to problem (6)–(7) is given by

\[
\hat{G}_j = (A_j)_{r_j} C_j^T + K_j
\]

(37)

and the associated error is

\[
\|x - x_{p+1}\|_{K, \Omega}^2 = \|R^{1/2}[xx^T]\|^2 - \sum_{j=1}^{p} r_j \sum_{k=1}^{m} |\sigma_k(A_j)|^2. \tag{38}
\]

**Proof.** We write \(\epsilon = \|x - x_{p+1}\|_{K, \Omega}^2\). By an argument analogous to that which established (32) and (34) we have

\[
\epsilon = \|R^{1/2}[xx^T]\|^2 - \sum_{j=1}^{p} \|R[xx^T]\|^2 = \sum_{j=1}^{p} J_j(G_j) \tag{39}
\]

and

\[
\min_{\text{rank } G_j \leq r_j} \sum_{j=1}^{p} \|A_j - G_j C_j\|^2_2 = \sum_{j=1}^{p} \min_{\text{rank } G_j \leq r_j} \|A_j - G_j C_j\|^2_2. \tag{40}
\]

By Corollary 1 the solution to

\[
\min_{\text{rank } G_j \leq r_j} \|A_j - G_j C_j\|^2_2
\]

is attained when

\[
G_j = (A_j P_{C_j, r_j})_{r_j} V_{C_j}^T C_j + K_j = (A_j)_{r_j} C_j + K_j = \hat{G}_j.
\]

We refer to [4] for more details. If \(\hat{G}_j\) is substituted instead of \(G_j\) in (39), then we have

\[
\epsilon = \|R^{1/2}[xx^T]\|^2 - \sum_{j=1}^{p} \|R[xx^T]\|^2 - \sum_{j=1}^{p} \|A_j\|^2 - \sum_{j=1}^{p} \sum_{k=1}^{m} |\sigma_k(A_j)|^2
\]

(41)

\[
\|R^{1/2}[xx^T]\|^2 = \sum_{j=1}^{p} \|A_j\|^2 + \sum_{j=1}^{p} \sum_{k=1}^{m} |\sigma_k(A_j)|^2
\]

(42)

since \((A_j)_{r_j} C_j\) is defined in [14].

**Remark 5:** In (38) the number \(p\) is the number of steps in the procedure described by (4)–(7). It follows from Theorem 4 that the error given by (38) decreases as \(p\) increases.

**Remark 6:** It follows from Theorems 3 and 4 that the components \(\hat{G}_j\) and \(\hat{G}_j\)'s of the proposed filter are computed separately and require computation of \(m \times n\) and \(n \times n\) matrices for \(\hat{G}_j\) and \(m \times r_j\) and \(r_j \times n\) matrices for \(\hat{G}_j\). To the best of our knowledge, a filter that is able to process infinite signal sets is not previously known. Therefore we compare the computational loads needed for our proposed filters with those for known filters in the case of finite sets of signals only. In this case most known nonlinear filters [14] require computation of much larger matrices than those mentioned above. The procedures presented in Theorems 3 and 4 are advantageous from a numerical point of view because they require much less computational work.

**C. Choice of operators \(\mathcal{P}_k\) for (1)–(7)**

The purpose of using the operators \(\mathcal{P}_k\) in our filter has been discussed in Section I-A2. Here, we illustrate the choice of \(\mathcal{P}_k\) with some possible particular suggestions.

For instance, \(\mathcal{P}_k\) could be chosen using the Hadamard product as in [12], [13] or in the form

\[
[\mathcal{P}_k(x_1, \ldots, x_N)](\omega, \alpha) = \frac{1}{\gamma} \sum_{j=1}^{k} \gamma_j x_j(\omega, \alpha)
\]

where \(\gamma = \sum_{j=1}^{k} \gamma_j\) and \(\gamma_j \in \mathbb{R}\) is a constant.

Another possible choice for \(\mathcal{P}_k\) is a time shifting operator similar to that used in [9]. For the case considered in Example 4 of Section VI, we define

\[
\mathcal{P}_k(x_1, \ldots, x_N) = \mathcal{P}_k(x_k(\cdot, t_1), \ldots, x_k(\cdot, t_N)) = [x_k(\cdot, t_1 - \Delta_i), \ldots, x_k(\cdot, t_N - \Delta_N)]
\]

with \(\Delta_k \in \mathbb{R}\) for all \(k = 1, \ldots, p\).

Alternatively \(\mathcal{P}_k\) could be chosen as a \(k\)-linear operator. In particular, \(\mathcal{P}_k\) could be a \(k\)-linear integral operator. We cite [2] in this regard.

**D. Device for data compression and reconstruction**

Let us denote \(D_j(1) = U_{A_j} \Sigma A_j r_j\) and \(D_j(2) = V_{A_j} C_j^T\).

See (35) and (36) in this regard. In (37) the matrix \(K_j\) is arbitrary. Let us assume that \(K_j = 0\). Then the filter (1) based on solutions (30) and (37) to problems (4)–(5) and (6)–(7), respectively, is given by

\[
F(y) = \sum_{j=1}^{p} D_j(1) D_j(2) v_j
\]

where \(v_j = H_j x_j\). Note that \(D_j(1) \in \mathbb{R}^{m \times r_j}\) and \(D_j(2) \in \mathbb{R}^{r_j \times n}\). Thus \(D_j(2)\) provides compression of a vector \(v_j\) with \(n\) components to a vector \(D_j(2) v_j\) with \(r_j\) components while \(D_j(1)\) provides decompression (reconstruction) from a vector with \(r_j\) components to a vector with \(m\) components. The compression ratio is given by

\[
c = \frac{r}{\min(m, n)}, \quad \text{where } r = r_1 + \ldots + r_p. \tag{43}
\]

We reiterate that \(m\) and \(n\) are the numbers of components in signals \(x\) and \(y\), respectively, and \(r\) is the number of components in the compressed data. By the condition (7), \(r \leq \min(m, n)\), i.e. the compressed data should be ‘shorter’ than \(x\) and \(y\). See also Section 1 in this regard. In Section 4 below, we consider examples of our filters with compression ratio \(c = 25/116\) and \(c = 9/116\).
E. Advantages associated with filtering based on scheme (4)–(7)

1) General advantages: The proposed filter consists of the two parts. The first one is the filtering of a random signal \( y \) based on the concatenation of solutions to \( p - 1 \) unconstrained error minimization problems given by (4)–(5). This filter is important in its own right since it can be considered as a stand alone filter aimed at optimal filtering of stochastic signals.

The second part is based on the solution of the constrained error minimization problem given by (6)–(7). As a result, while this part is still filtering the input, it also compresses and decompresses the input. The latter procedure is realized via the solution of the problem (6)–(7) and is described in Section III-B.

The overall advantages of our proposed approach are the ability to determine a single optimal filter to process an infinite set of signals and the progressive decrease in the filtering error with an increase in the number of steps (4)–(7).

The mathematical advantages include simplification of the numerical procedure for determining the filter components \( G_1, \ldots, G_6 \) in (4)–(6) which we address in Remark 6 above, the rigorous theoretical justification of the results given in Sections III-A and III-B and the effective procedure for signal compression and decompression presented in Section III-D.

In the following Section, we consider other specific advantages by comparison with known filters.

2) Specific advantage: comparison with the known filters: As we have mentioned in Section I-A1, to the best of our knowledge, the previously known filters can process only finite sets of random signals. Our filter processes infinite sets of random signals. Therefore, a comparison between the known filters and the proposed one can only be done for the particular case when each set \( x \) and \( y \) (see Section II-A) consists of one signal only. In such a case, the results obtained above are presented in terms of the norm (46), \( \| \cdot \|_G \) (see Section VI below). The norm (46) is a particular case of the norm (3), \( \| \cdot \|_G \).

The errors associated with the Karhunen-Loève filter (KLF) [14] and the Wiener filter [14] follow from Theorems above if \( p = 1, k = 1 \) and the norm \( \| \cdot \|_G \) is replaced with the norm \( \| \cdot \|_G \). The errors presented in Theorems above are less than those for the KLF and Wiener filter if \( p = 2, 3, \ldots \) and \( k = 2, 3, \ldots \). Thus our filter provides improved accuracy when compared to the KLF and Wiener filter.

The compression ratio of the KLF is \( \eta / \min \{m, n\} \) where \( \eta \) is the number of components in the compressed signal. Thus, the compression ratio (43) of our filter is better than that of the KLF if \( r_1 + \ldots + r_p < \eta \).

A comparison with other known filters [14] can be done in a similar way and leads to similar conclusions.

IV. NUMERICAL EXAMPLES AND SIMULATIONS

The following elementary Example 1 illustrates the proposed filter performance in the case of infinite sets of signals.

**Example 1.** Let
\[
x = \{x(\omega, t) = [x^2, \omega^2]\}^T | \omega \in [0, 1], t \in [0, 1]\}
\]

and
\[
y = \{y(\omega, t) = [0.8\omega^2, 1.2\omega^2]^T | \omega \in [0, 1], t \in [0, 1]\}.
\]

That is \( x \) and \( y \) are represented by infinite sets of signals. For such signals, the norm (3) is reduced to the particular case given by (48). Accordingly, matrices \( R[xy] \) and \( R[yy] \) given by (22)–(24) are determined in terms of the norm (48). For instance, the entries of \( R[yy] \) are
\[
(x^{(i)}, y^{(j)}) = \int_0^1 \int_0^1 x^{(i)}(\omega, t)y^{(j)}(\omega, t) d\omega dt.
\]

Therefore
\[
R[xy] = \begin{bmatrix}
0.053 & 0.075 \\
0.050 & 0.080
\end{bmatrix},
R[yy] = \begin{bmatrix}
0.011 & 0.060 \\
0.060 & 0.096
\end{bmatrix}
\]

and
\[
R[yy] = \begin{bmatrix}
-37.738 & 23.5849 \\
23.5849 & 4.3239
\end{bmatrix}.
\]

Let us first consider the simplest case of our filter presented by (1) with \( p = 1, H_1 \) and \( P_1 \) given by the identity, and
\[ G_1 = G_1 \] where \( G_1 \) is determined by (30) with \( M_1 = \emptyset \). Then for any \( \omega \in [0, 1] \) and \( t \in [0, 1] \), the estimate of signals \( x(\omega, t) \) by the considered particular case of our filter is given by
\[
\hat{x}(\omega, t) = R[xy]^T[R[yy]^T]^{-1}[0.8\omega^2, 1.2\omega^2]^T. \quad (44)
\]

Note that (44) does not coincide with an estimate by the generalized Wiener filter which can process a fixed random signal-vector [14], but is not able to process infinite sets of signals. Differences between (44) and an estimate by the generalized Wiener filter [14] are matrices \( R[xy] \) and \( R[yy] \) determined in terms of the norm (47) used in (44), and the term \( [0.8\omega^2, 1.2\omega^2]^T \).

Table 1 contains magnitudes of the error \( \Delta_1 = ||x(\omega, t) - \hat{x}(\omega, t)||_2^2 \) with respect to some particular values of \( \omega \) and \( t \).

Next, let us now consider other simplest case of the proposed filter when (1), as before, \( p = 1, H_1 \) and \( P_1 \) are the identity, but \( G_1 = G_1 \) where \( G_1 \) is determined by (37) with \( r_1 = 1 \) and \( K_1 = \emptyset \). Then for any \( \omega \in [0, 1] \) and \( t \in [0, 1] \), the estimate of signals \( x(\omega, t) \) by this particular case of our filter is given by
\[
\hat{x}(\omega, t) = (A_1)_{r_1} C_1^T[0.8\omega^2, 1.2\omega^2]^T \quad (45)
\]

where \( A_1 = R[xy]^T[R[yy]^T]^{1/2} \), \( C_1 = R[yy]^T^{1/2} \) and \( (A_1)_{r_1} \) is determined by (36). Compression and decompression is realized by \( (A_1)_{r_1} C_1^T \) as those described in Section III-D above. For the same values of \( \omega \) and \( t \) as those in Table 1, the error \( \Delta_2 = ||x(\omega, t) - \hat{x}(\omega, t)||_2^2 \) is worse than the error \( \Delta_1 \) by approximately 20%. This is because \( \hat{x}(\omega, t) \) is determined with the truncated SVD given by (36).

The estimate \( \hat{x}(\omega, t) \) does not coincide with an estimate by the KLF [14] for reasons which are similar to those described above for \( \hat{x}(\omega, t) \) when making comparison with the generalized Wiener filter.

Of course, in many instances, matrices \( R[xy] \) and \( R[yy] \) are unknown and should be estimated. Some estimation methods can be found in [14].

**Example 2.** Here, we consider a case where \( x \) and \( y \) are represented by finite signal sets and illustrate advantages of
the proposed approach over the known methods. This case has been discussed briefly in Section I-A1.

Let $x = \{x_{1(1)}, x_{2(1)}, \ldots, x_N(1)\}$ and $y = \{y_{1(1)}, y_{2(2)}, \ldots, y_N(N)\}$, where $N = 8$ and $x_{j(1)}, y_{j(1)} \in L^2(\Omega, \mathbb{R}^n)$ with $n = 116$ for each $j = 1, \ldots, 8$. Random vectors $y_{j(1)}$ and $x_{j(1)}$ are interpreted as an input of the filter and its 'idealistic' output, respectively. That is $x_{j(1)}$ is the signal that should be estimated by the filter. In the example, $x_{j(1)}$ and $y_{j(1)}$ are simulated as digital images presented by $116 \times 256$ matrices $X_{j(1)}$ and $Y_{j(1)}$, respectively. The columns of matrices $X_{j(1)}$ and $Y_{j(1)}$ represent a realization of signals $x_{j(1)}$ and $y_{j(1)}$, respectively.

Each picture $X_{j(1)}$ in the sequence $X_{1(1)}, \ldots, X_{8(1)}$ has been taken at a certain time $t_j$ with $j = 1, \ldots, 8$. Images $Y_{1(1)}, \ldots, Y_{8(8)}$ have been simulated from $X_{1(1)}$ to $X_{8(1)}$ in the form $Y_{j(1)} = X_{j(1)} \ast x\zeta t_{j}$ for each $j = 1, \ldots, 8$. Here, $\ast$ means the Hadamard product and $x\zeta t_{j}$ is a $116 \times 256$ matrix whose elements are uniformly distributed in the interval $(0, 1)$. Images $X_{1(1)}, \ldots, X_{8(1)}$ are presented in Fig. 1. Image $Y_{3(1)}$ is given in Fig. 2 (a) as an example of images $Y_{1(1)}, \ldots, Y_{8(1)}$. Other images $Y_{j(1)}$ with $j = 1, \ldots, 8$, $j \neq 3$ are similar.

For finite signal sets $x$ and $y$ considered in this example, the norm (47) is used. As a consequence, for $j = 1, 2$, matrix $R[R[y_{j}]^{2}]$ is defined by (30) and (37) is presented by

$$R[R[y_{j}]^{2}] = \{r_{jk}\}_{j, k = 1}^{116, 256}$$

with

$$r_{jk} = \frac{1}{8} \sum_{k=1}^{8} \int_{\Omega} x^{(1)}(\omega, t_k) y^{(j)}(\omega, t_k) d\mu(\omega).$$

The matrix $R[R[y_{j}]^{2}]$ is determined similarly.

First, the simplest case of our filter defined from (4) by Theorem 3 with $p = 1$ has been applied to the considered signal sets. Then $R[R[y_{1}]^{2}] = R[R[y_{2}]^{2}]$ and $R[R[y_{1}]^{2}] = R[R[y_{2}]^{2}]$.

In (1), $G_{1}$ is determined as $\hat{G}_{1}$ by (30). An estimate of $X_{1(3)}$ by this filter is denoted by $\hat{X}_{1(3)}$ and the filter itself is denoted by $\hat{F}_{1}$.

To illustrate the performance of the proposed filters associated with different steps of the scheme (5)-(7), their related versions have also been applied to the given signal sets. Estimates of $X_{1(3)}$ are denoted as follows:

- $\hat{X}_{2(3)}$ is the estimate by $\hat{F}_{2}$ defined by (5) and Theorem 3 with $p = 3$;
- $\hat{X}_{2(3)}$ is the estimate by $\hat{F}_{2}$ defined by (6)-(7) and Theorem 4 with $p = 2$, $r_{1} = 10$ and $r_{2} = 15$, with the compression ratio $c = 25/116$; and
- $\hat{X}_{2(3)}$ is the estimate by $\hat{F}_{2}$ defined by (6)-(7) and Theorem 4 with $p = 2$, $r_{1} = 4$ and $r_{2} = 3$, with the compression ratio $c = 9/116$.

We point out again that by the proposed method, the same filter is applied to each pair of signals $Y_{j(1)}$ and $X_{j(1)}$ for $j = 1, \ldots, 8$. That is the form presented by (1), (4)-(7) and Theorems 3, 4 is invariant with respect to different pairs $(Y_{j(1)}, X_{j(1)})$. In particular the matrices determined by (30) and (37) remain the same regardless of which pair $(Y_{j(1)}, X_{j(1)})$ is processed.

The known filters based on the Wiener filtering approach [14] are specifically constructed for each pair $(Y_{j(1)}, X_{j(1)})$. As a result, such filters incur much greater computational load. Therefore, by comparison with known filters, this important difference should be borne in mind. We denote by $\hat{X}_{2(3)}$ and $X_{KLF(3)}$ the estimate of $X_{3(3)}$ by the Wiener filter and by the KLF with rank=25, respectively. The compression ratio of the KLF with rank=25 is $c = 25/116$.

In Fig. 2, estimates of signal $X_{3(3)}$ by different filters are presented. Numerical results related to estimation of a signal $X_{3(3)}$ are given in Tables 2 and 3. Results associated with estimates of other signals $X_{j(3)}$ with $j = 1, \ldots, 8$, $j \neq 3$ are similar.

In particular, while the error associated with the filter $\hat{F}_{1}$ is $\|X_{3(3)} - \hat{X}_{1(3)}\|^{2} = 0.9893c + 07$, the error $\|X_{3(3)} - \hat{X}_{2(3)}\|^{2}$ associated with the filter $\hat{F}_{2}$ is significantly lesser. This numerical illustration relates to Remark 5.

Next, we observe that for the same compression ratio, $c = 25/116$, the error $\|X_{3(3)} - \hat{X}_{2(3)}\|^{2}$ associated with the filter $\hat{F}_{2}$ is four times less than the error $\|X_{3(3)} - X_{KLF(3)}\|^{2}$ associated with the KLF.

It is interesting to compare compression ratios of those filters in the case when associated error are similar. In particular, it follows from the first and third columns of Table 3, that while the errors associated with the filter $\hat{F}_{2}$ and the KLF are almost the same, the compression ratio of the filter $\hat{F}_{2}$ (i.e. $c = 9/116$) is more than two times better that the KLF compression ratio (which is $c = 25/116$).

V. CONCLUSION

We have provided the theory for a new approach to filtering, compression and decompression for infinite sets of stochastic signals. Distinctive features of the proposed approach are as follows. While we consider processing infinite signal sets, the proposed filter is nevertheless fixed for all signal pairs. The filter is nonlinear and consists of $p$ steps. Each step contains three components represented by three consecutive operations. The various operations are determined at each step by an iterative scheme designed to improve filter performance as the number of steps increases. Signal compression and decompression is provided by the final step of the scheme. The final step is based on a new method [4] for the best rank-constrained matrix approximation. The error associated with our filter decreases when the number of steps in the iterative scheme increases.

VI. APPENDIX A: PARTICULAR CASES OF THE NORM (3)

Here, we consider some particular norms that lead to the norm (3) used in our statement of the problem in Sections II-C–II-D.

Example 3. Let $x \in L^{2}(\Omega, \mathbb{R}^{m})$. Then we set

$$\|x\|_{3}^{2} = \left(\int_{\Omega} \|x(\omega)\|_{3}^{2} d\mu(\omega)\right)^{2}. \tag{46}$$

Note, that most of results related to Wiener-like optimal filtering have been obtained using the norm (46). Some relevant references can be found in [14].

Example 4. Let $x = \{x_{k}, t_{k}\} \in L^{2}(\Omega, \mathbb{R}^{m}) | t_{k} \in \mathbb{R}, \forall k = 1, \ldots, N\}$. Thus, $x \in L^{2}(\Omega \times [t_{1}, \ldots, t_{N}], \mathbb{R}^{m})$. In
The norms given by (46)–(49) are generalized in the following way. Let \( x = \{ x(\cdot) \in L^2(\Omega, \mathbb{R}^m) \mid \tau \in [a, b] \subset \mathbb{R} \} \) where \( \mathbb{C}^q \) is a q-dimensional cube in \( \mathbb{R}^q \). We put

\[
\| x \|_{C^q, \Omega}^2 = \frac{1}{V} \int_{C^q} \int_{\Omega} \| x(\omega, \tau) \|_2^2 \, d\mu(\omega) \, d\tau = \frac{1}{V} \int_{C^q} \int_{\Omega} \| x(\cdot, \tau) \|_2^2 \, d\mu(\omega) \, d\tau,
\]

where \( V = \int_{C^q} \, d\tau \). Then \( \| x \|_{C^q, \Omega}^2 \) follows from \( \| x \|_{C^q, \Omega}^2 \) if the set \( x \) consists of a single signal, \( x(\cdot, \tau) \), with \( \tau \) fixed.

The norms given by (46)–(49) are generalized in the following way. Let \( x = \{ x(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}^m) \mid \alpha \in K \} \) where \( K \) is a measurable set [7] with a measure \( \lambda(\alpha) \). Thus, \( x(\cdot, \alpha) \) is a single signal in the space \( L^2(\Omega \times K, \mathbb{R}^m) \) and is the infinite set of signals \( \{ x(\cdot, \alpha) \in L^2(\Omega, \mathbb{R}^m) \mid \alpha \in K \} \) in the space \( L^2(\Omega, \mathbb{R}^m) \).

Let us set the norm by

\[
\| x \|_{K, \Omega}^2 = \frac{1}{\lambda(K)} \int_K \int_{\Omega} \| x(\omega, \alpha) \|_2^2 \, d\mu(\omega) \, d\lambda(\alpha),
\]

where \( \lambda(K) = \int_K \, d\lambda(\alpha) \). Then the norms given by (46)–(49) follow from \( \| x \|_{K, \Omega}^2 \) for a suitable choice of \( K \) and \( \lambda(\alpha) \). In particular, \( \| x \|_{K, \Omega}^2 \) follows from \( \| x \|_{K, \Omega}^2 \) if \( x \) consists of a single signal only, \( x(\cdot, \alpha) \), with \( \alpha \) fixed.

**REFERENCES**


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Fig. 1. Signals $X_1, \ldots, X_8$ to be estimated from observed data.

Table I

| $|\Delta_1|$ | $0.86 \times 10^{-4}$ | $1.16 \times 10^{-2}$ | $1.13 \times 10^{-2}$ | $1.35 \times 10^{-2}$ | $2.90 \times 10^{-3}$ |
|---|---|---|---|---|---|
| $\omega$ | 0.5 | 0.4 | 0.4 | 0.2 | 0.9 |
| $t$ | 0.5 | 0.7 | 0.3 | 0.8 | 0.9 |
Fig. 2. Illustration to signal $X^{(3)}$ estimation by different filters. Here, $X_{W}^{(3)}$ is the estimate by the Wiener filter, $\tilde{X}^{(3)}_2$ is the estimate that follows from (5) with $p = 3$, $X_{KLF}^{(3)}$ is the estimate by KLF with $c = 25/116$, $\hat{X}^{(3)}_2$ is the estimate that follows from (6)-(7) with $c = 25/116$, and $\bar{X}^{(3)}_2$ is the estimate that follows from (6)-(7) with $c = 9/116$.

Table II
Error estimations of signal $X^{(3)}$.
Here, $X_{W}^{(3)}$ is the estimate by the Wiener filter, $\tilde{X}^{(3)}_2$ is the estimate (5) with $p = 3$.

<table>
<thead>
<tr>
<th></th>
<th>$|X^{(3)} - X_{W}^{(3)}|^2$</th>
<th>$|X^{(3)} - \tilde{X}^{(3)}_2|^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0.7555e + 07$</td>
<td>$1.5089e - 07$</td>
</tr>
</tbody>
</table>

Table III
Error estimations of signal $X^{(3)}$ by the KLT and (5) with $p = 3$, for some different compression ratios $c$.

<table>
<thead>
<tr>
<th></th>
<th>$|X^{(3)} - X_{KLF}^{(3)}|^2$</th>
<th>$|X^{(3)} - \hat{X}^{(3)}_2|^2$</th>
<th>$|X^{(3)} - \bar{X}^{(3)}_2|^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c = 25/116$</td>
<td>$8.8739e + 06$</td>
<td>$2.1730e + 06$</td>
<td>$8.2173e + 06$</td>
</tr>
<tr>
<td>$c = 9/116$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>