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A New Verified Method for Solving Nonlinear Equations

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Abstract—In this paper, verified extension of the Ostrowski method which calculates the enclosure solutions of a given nonlinear equation is introduced. Also, error analysis and convergence will be discussed. Some implemented examples with INTLAB are also included to illustrate the validity and applicability of the scheme.

Keywords—Iinterval analysis, nonlinear equations, Ostrowski method.

I. Introduction

Sometimes it is desired to compute a sharp interval that is guaranteed to enclose a real simple root x^* of $f(x) = 0$, even if rounding errors are taken into account. This can be done using interval analysis tools [1], [4], [6].

An interval Newton method has been developed by Moore [5] for solving nonlinear equations. Here, we develop interval Ostrowski's method and its convergence analysis.

This paper is organized as follows: In the next section, we review some basic concepts and notations of interval analysis. Section 3 introduces interval Ostrowski's method. Section 4 deals with convergence analysis. Finally, in Section 5, some computational results support the interval Ostrowki's method and order of convergence.

II. Basics

We start by repeating some definitions, notations and basic facts; For more information see [1], [5], [6].

Let $[x] = [\underline{x}, \overline{x}]$, $[y] = [\underline{y}, \overline{y}]$ be real compact intervals and $\circ$ one of the basic operations, that is $\circ \in \{+, -, \times, /\}$. Then we define the corresponding operations for intervals $[x]$ and $[y]$ by $[x] \circ [y] = \{x \circ y \,|\, x \in [x], y \in [y]\}$, where we assume $0 \notin [y]$ in case of division.

If $x \in [x]$, we call $[x]$ an *enclosure* of x. If $\underline{x} = \overline{x}$, then the interval $[x]$ is *degenerate* or real number, i.e., $x = [x, x]$.

Standard *interval function* $[\varphi] = [\varphi, \varphi]$ is defined via its range, i.e.,

$$[\varphi] = \varphi([x]) = \{\varphi(x) | x \in [x]\}.$$

Midpoint and *width* of $[x]$ are respectively defined by

$$[x] = \frac{1}{2}(\underline{x} + \overline{x}), \qquad [x] = \overline{x} - \underline{x}.$$

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An interval sequence $\{[x]^k\}$ is *nested* if $[x]^{k+1} \subseteq [x]^k$ for all k.

An interval extension $[f]$ is said to be *Lipschitz* in $[x]^0$ if there is a constant L such that $(f([x])) \leq L\,[x]$ for every $[x] \subseteq [x]^0$.

III. Interval Ostrowski's Method

We now develop *interval Ostrowski's method* (IOM) for finding a simple zero x^* in an interval $[x]^0 = [x^0, x^0]$ for a strictly monotonically increasing or decreasing real function having continuous derivatives of sufficiently high order. This method is always convergent if some conditions are hold. The principle used for construction is due to Ostrowski [7].

Using interval analysis tools as well as classic Ostrowski' method, we consider the following iteration:

$$N[x]^k = [x]^k - \frac{f([x]^k)}{f'([x]^k)}. \tag{1}$$

$$[y]^k = [x]^k \cap N([x]^k), \tag{2}$$

$$S([x]^k, [y]^k) = [y]^k - \frac{f([y]^k)}{f([x]^k) - 2f([y]^k)} \cdot \frac{f([x]^k)}{f'([x]^k)}, \tag{3}$$

$$[x]^{k+1} = [x]^k \cap S([x]^k, [y]^k), \quad k = 0, 1, 2, \ldots, \tag{4}$$

The procedure can be stopped when $[x]^{k+1} = [x]^k$ or $[x]^k \leq \varepsilon$.

IV. Analysis of convergence

In this section, we deal with the convergence analysis of the IOM (4). First, we need [5]

Lemma 4.1: Suppose $\{[x]^k\}$ is such that there is a real number $x \in [x]^k$ for all k. Define $\{[y]^k\}$ by $[y]^1 = [x]^1$ and $[y]^{k+1} = [x]^{k+1} \cap [y]^k$ for all $k = 1, 2, \ldots$. Then $\{[y]^k\}$ is nested, converges, and has the limit

$$x \in [y] = \cap_{k=1}^{\infty} [y]^k.$$

One of the most useful properties of the interval Ostrowski operator S (3) is that we are provided with a means of detecting when a region does not contain a root of f. As this is a common situation, it is important that we can quickly discard a set on the grounds of it containing no roots. Another important contribution from the properties of S is a simple verifiable condition that guarantees the existence of a unique root within an interval. The following theorem addresses these.

Theorem 4.2: Suppose f is a continuous differentiable on an initial interval $[x]^0$, and $0 \notin' ([x]^k)$ for $k = 0, 1, 2, \ldots$.

(1) If $x^* \in [x]^0$ and $S([x]^k, [y]^k) \subseteq [x]^k$, then $[x]^k$ contains exactly one zero of f. Also
$$x^* \in [x]^* = \lim_{k\to\infty} [x]^k.$$

(2) If $[x]^k \cap S([x]^k, [y]^k) = \emptyset$, then $[x]^k$ does not contain any zero of f.

`Proof.`

Part (1) Since $0 \notin f'([x]^k)$, then $f'(x) \neq 0$ for all $x \in [x]^k$ and therefore f is monotonic on $[x]^k$. In other words, it has at most one zero in $[x]$. Hence, it is sufficient to find a zero $x^* \in [x]^k$. Since $S([x]^k, [y]^k) \subseteq [x]^k$, using the Lemma (4.1), so f has exactly one root in $[x]^k$ and $x^* \in [x]^* = \lim_{k\to\infty} [x]^k$.

Part (2) Now, suppose x^* is a zero of f and $x^* \in [x]^0$, then previous part results $x^* \in S([x]^k, [y]^k)$. Consequently $x^* \in [x]^k \cap S([x]^k, [y]^k)$ which is contradiction. So the proof is completed.

We need [5]:

Lemma 4.3: If $[f]$ is defined for $[x] \subseteq [x]^0$, then $[f]$ is Lipschitz in $[x]^0$; In other words:
$$(f([x])) \leq L\,[x]. \tag{5}$$

We proved that the sequence (4) converges to x^* if the assumptions of the Theorem (4.2) are hold. Now we want to show that if IOM converges, then the order of its convergence is asymptotically 4.

Theorem 4.4: Assume that f is continuous differentiable function on initial interval $[x]^0$ with $0 \notin f'([x]^0)$, and f has a unique simple root $x^* \in [x]^0$. Then, if $S([x]^k, [y]^k) \subseteq [x]^k$, the sequence (4) has convergent rate four, i.e., there exists a constant γ such that
$$[x]^{k+1} \leq \gamma \left([x]^k\right)^4. \tag{6}$$

`Proof.` By Mean Value Theorem, since $f(x^*) = 0$, we have
$$f([x]^k) = f'(\xi)\left([x]^k - x^*\right), \tag{7}$$
where ξ is between $[x]^k$ and x^*. Since $S([x]^k, [y]^k) \subseteq [x]^k$, thus from (3), (4) and (7), we have
$$[x]^{k+1} = [y]^k - \frac{\left([y]^k - x^*\right) f'(\xi_1) \left([x]^k - x^*\right) f'(\xi_2)}{\left(f([x]^k) - 2f([y]^k)\right) f'([x]^k)}, \tag{8}$$
where ξ_1 is between $[y]^k$and x^* and ξ_2 is between $[x]^k$ and x^*. Also,
$$[x]^{k+1} = \frac{|[y]^k - x^*||f'(\xi_1)||[x]^k - x^*||f'(\xi_2)|}{|f([x]^k) - 2f([y]^k)|}\left(\frac{1}{f'([x]^k)}\right). \tag{9}$$

We have $m[x]^k, x^* \in [x]^k$, therefore
$$|[x]^k - x^*| \leq [x]^k. \tag{10}$$

Furthermore, since $[y]^k$ is generated from interval Newton iteration (2),
$$|[y]^k - x^*| \leq [y]^k \leq \left([x]^k\right)^2. \tag{11}$$

Also, from Lemma (4.3) we have
$$\left(\frac{1}{f'([x]^k)}\right) \leq [x]^k. \tag{12}$$

Let $|f'(\xi_i)| \leq \gamma_i,\ i = 1, 2$ and $|f([x]^k) - 2f([y]^k)| \leq \gamma_3$. Considering (9–12), we have
$$[x]^{k+1} \leq \frac{\gamma_1\gamma_2}{\gamma_3}\left([x]^k\right)^4 = \gamma\left([x]^k\right)^4, \tag{13}$$
where $\gamma = \gamma_1\gamma_2/\gamma_3$, and the proof is completed.

V. Numerical implementation

In this section, we apply IOM to solve some examples [2], [3]. Also we compare the computed results with interval Newton's method. Computational results support the IOM theory discussed in this paper. We used INTLAB to carry out numerical results [8].

The results of this example also show that the interval Ostrowski method is faster than the interval Newton method.

Subsection text here.

TABLE I
TEST FUNCTIONS AND THEIR ROOTS

Test functions	Roots
$f_1(x) = \exp(x) - 4x^2$	4.30658472822069882
$f_2(x) = x^2 - \exp(x) - 3x + 2$	0.25753028543986072
$f_3(x) = \exp(-x) + \cos(x)$	1.746139530408012285
$f_4(x) = x^2 - 3$	1.7320508075688772
$f_5(x) = \sin^2(x) - x^2 + 1$	1.4044916482153411
$f_6(x) = (x+2)\exp(x) - 1$	−0.44285440100238854
$f_7(x) = x^5 + x^4 + 4x^2 - 15$	1.3474280989683043
$f_8(x) = \cos(x) - x$	0.73908513321516067
$f_9(x) = x^5 - 10$	1.5848931924611134

TABLE II
INTERVAL NEWTON AND OSTROWSKI SOLUTIONS

$f_i(x)$	$[x]^0$	Iterations NM	SM	Enclosure
f_1	$[4, 5]$	6	3	$4.306584728220_{6934}^{7050}$
	$[4, 4.5]$	5	3	
	$[4.2, 4.3]$	4	2	
f_2	$[0, 1]$	5	3	0.2575302854398608_7^8
	$[0, .5]$	4	2	
	$[2.4, 2.6]$	3	2	
f_3	$[1, 2]$	4	3	1.746139530408012_2^5
	$[1.5, 2]$	3	2	
	$[1.6, 1.8]$	3	2	
f_4	$[1, 2]$	5	3	1.732050807568877_2^4
	$[1.5, 2]$	4	2	
	$[1.6, 1.8]$	4	2	
f_5	$[1, 2]$	5	3	1.404491648215341_1^6
	$[1, 1.5]$	5	3	
	$[1.4, 1.5]$	4	3	
f_6	$[-1, 0]$	5	3	-0.442854401002388_5^4
	$[-.5, 0]$	4	3	
	$[-.5, -.4]$	3	2	
f_7	$[1, 2]$	5	3	1.34742809896830_4^5
	$[1, 1.5]$	5	3	
	$1.3, 1.4]$	3	2	
f_8	$[0, 1]$	5	3	0.7390851332151606_7^8
	$[.5, 1]$	4	2	
	$[.6, .7]$	4	2	
f_9	$[1, 2]$	5	4	1.584893192461113_4^6
	$[1.5, 2]$	4	2	
	$[1.5, 1.6]$	4	3	

VI. Conclusion

In this paper, a new enclosure method, interval Ostrowski method, was introduced to find the interval solution of a given nonlinear equation. A fundamental distinction between the interval Osrowski method and the ordinary Osrowski method is that the former uses computation with sets instead of computation with points. Again, this permits us to find all zeros of a function in a given staring interval. Whereas the ordinary Ostrowski method is prone to erratic behavior, the interval version practically always converge. The difference in performance of the two methods can be dramatic. This method has the local order of convergence equal to 4 like classic Osrowski method. Moreover, necessary and sufficient conditions about the convergency were discussed in details. Also, error bound and convergence rate were studied. To verify the theory, this algorithm was then tested using some examples via INTLAB. Furthermore, the suggested method was compared with the interval Newton method. As expected, according to the discussed theory, this method was better than the interval Newton method.

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