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Fuzzy T-Neighborhood Groups Acting on Sets

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***Abstract*—**In this paper, The T-G-action topology on a set acted on by a fuzzy T-neighborhood (T-neighborhood, for short) group is defined as a final T-neighborhood topology with respect to a set of maps. We mainly prove that this topology is a T-regular T-neighborhood topology.

***Keywords*—**Fuzzy set, Fuzzy topology, Triangular norm, Separation axioms.

I. Introduction

A T-neighborhood topology on a set can be defined by several method e.g., via closures, interiors, filters, etc. Sometimes a T-neighborhood topology constructed out of given T-neighborhood topologies may be useful. In the classical theory of topological groups, when a topological group *G* acts on a set *X*, it confers a topology on *X*, called the G-action topology on *X*. In this paper we develop a fuzzy extension of that notion, in the case *G* is a T-neighborhood group. Varity of useful characterizations of this T-neighborhood topology are considered. We show that the T-G-action topology τ_X^{T-G} coincides with the final T-neighborhood topology τ_f introduced on *X* by a set of functions $\left\{\hat{g}\right\}$;

$$\hat{g} : G \to X .$$

II. Definition and Preliminaries

Definition 2.1. [8] A topological group *G* acts on a non-empty set *X*, if to each $g \in G$ and each $x \in X$ there corresponds a unique element *gx* such that

$g_2(g_1x) = (g_2g_1)x \quad \forall\ x \in X$ and $g_1, g_2 \in G$

$ex = x$.

When *G* acts on a set *X*, two families of functions can be defined as follows:

To each $g \in G$, we define $\hat{g} : X \to X$,

$\hat{g}(x) = gx$.

To each $x \in X$, we define $\hat{x} : G \to X$,

$\hat{x}(g) = gx$.

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We will use two important theorems which introduced in [7]. The first gives necessary and sufficient conditions for a group structure and T-neighborhood system to be compatible, and the second gives necessary and sufficient conditions for a filter to be the T-neighborhood filter of *e* in a T-neighbourhood group.

Theorem 2.1 [7] Let *(G, .)* be a group and β a T-neighborhood base on *G*. Then *(G, .,t(β))* is a T-neighborhood group if and only if the following are fulfilled:

(a) For every $a \in G$ we have

$$\beta\ (a) = \{\zeta_a(\mu) \ /\ \mu \in \beta\ (e)\}$$

$$(\text{res . } \beta\ (a) = \{R_a(\mu) \ /\ \mu \in \beta\ (e)\} \text{ and}$$

$\beta\ (a) = \{\zeta_a(\mu) \ /\ \mu \in \beta\ (e)\}$ is a T-neighborhood base at *a*.

(b) For all $\mu \in \beta\ (e)$ and for all $\varepsilon \in I_0$ there exists $\nu \in \beta\ (e)$ such that $\nu - \varepsilon \le \mu^{-1}$, i.e., *r* is continuous at *e*.

(c) For all $\mu \in \beta\ (e)$ and for all $\varepsilon \in I_0$ there exists $\nu \in \beta\ (e)$ such that $\nu .\ \nu - \varepsilon \le \mu$, i.e., *m* is continuous at *(e, e)*.

(d) For all $\mu \in \beta\ (e)$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exist $\nu \in \beta\ (e)$ such that $1_x.\ \nu.1_x^{-1} - \varepsilon \le \mu$, i.e., int_x is continuous at *e*.

Where $\zeta_x : G \to G : z \mapsto x z$ (resp. $R_x : G \to G : z \mapsto z x$) is the left (resp. right) translation.

Theorem 2.2 [7] Let *(G, .)* be a group and $\mathfrak{I}$ a family of fuzzy subset of *G* such that the following hold:

(a) $\mathfrak{I}$ is a filterbasis, such that $\mu(e) = 1$ for all $\mu \in \mathfrak{I}$.

(b) For all $\mu \in \mathfrak{I}$ and for all $\varepsilon \in I_0$ there exists $\nu \in \mathfrak{I}$ such that $\nu - \varepsilon \le \mu^{-1}$.

(c) For all $\mu \in \mathfrak{I}$ and for all $\varepsilon \in I_0$ there exists $\nu \in \mathfrak{I}$ such that $\nu .\ \nu - \varepsilon \le \mu$.

(d) For all $\mu \in \mathfrak{I}$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exists $\nu \in \mathfrak{I}$ such that $1_x.\ \nu.1_{x^{-1}} - \varepsilon \le \mu$.

Then there exists a unique T-neighborhood system β such that $\mathfrak{I}$ is T-neighborhood basis for the T-neighbourhood system at *e*, *β(e)* and β is compatible with the group structure. This T-neighbourhood system is given by

$$\beta(x) = \{1_x.\mu \ /\ \mu \in \mathfrak{I}\}^{-1} = \{\mu\ .1_x / \mu \in \mathfrak{I}\}^{-1},\ x \in G.$$

III. T-Neighborhood Topologies Induced by T-Neighborhood Group Actions on Set

Definition 3.1. Let *(G, .)* be a group acting on a set *X*, then for all $\Gamma \in I^G$, $\mu \in I^X$, $g \in G$ and $x \in X$ we define for all $y \in X$

$$\Gamma\mu(y) = \sup\ \{\Gamma\ (g) T\ \mu(x) : (g, x) \in G \times X \text{ and } gx = y\} \qquad (1)$$

Proposition 3.1. Let $(G, .)$ be a group acting on a set X and $\Psi, \Gamma \in I^G$, $\mu \in I^X$. Then

(a) $\Psi(\Gamma\mu) \leq (\Psi.\Gamma)\mu$ In particular
$\Psi(\Gamma\mu)(y) \leq (\Psi.\Gamma)\mu(y)$

(b) $\Gamma 1_M = \bigvee_{x \in M} \Gamma 1_x$

(c) $\Gamma 1_M(y) = \sup\{\Gamma(g): g \in G \text{ and } g^{-1}y \in M\}$

(d) $\Gamma 1_x(y) = \sup\{\Gamma(g): g \in G \text{ and } gx = y\}$

(e) $1_g\mu(y) = \sup\{\mu(x): x \in X \text{ and } gx = y\}$
$= \mu(g^{-1}y)$

Proof: (b)-(e) follow immediately from Definition 3.1.

(a) For any $y \in X$:

$\Psi(\Gamma\mu)(y) = \sup\{\Psi(g)\, T\, \Gamma\mu(x): (g, x) \in G\times X, gx = y\}$
$= \sup\{\Psi(g)\, T \sup\{\Gamma(h)\, T\, \mu(z): hz = x\}: gx = y\}$
$= \sup\{\Psi(g)\, T\, \Gamma(h)\, T\, \mu(z): ghz = y\}$

$(\Psi \odot_T \Gamma)\mu(y) = \sup\{(\Psi \odot_T \Gamma(k)\, T\, \mu(z): kz = y\}$
$= \sup\{\sup\{\Psi(g)\, T\, \Gamma(h): gh = k\}\, T\, \mu(z): kz = y\}$
$= \sup\{\Psi(g)\, T\, \Gamma(h)\, T\, \mu(z): (g, h, z) \in G\times G\times X \text{ and } ghz = y\}$

Hence $\Psi(\Gamma\mu)(y) = (\Psi \odot_T \Gamma)\mu(y) \leq (\Psi.\Gamma)\mu(y)$.

If both Γ, μ are crisp, then $\Gamma\mu$ is also crisp and is given by
$\Gamma\mu = \{gx: g \in \Gamma \text{ and } x \in \mu\}$.

Note that $\Gamma\mu$, $\Gamma 1_x$, $1_g\mu \in I^X$ and $\Gamma 1_x(y) = 0$ if $y \notin$ orbit of x.

Theorem 3.1. Let G be a T-neighborhood group acting on a set X, and let $\Re$ be a fundamental system of G at e. For each $x \in X$, let $\beta_x = \{\Gamma 1_x: \Gamma \in \Re\} \in I^X$. Then $\{\beta_x\}_{x\in X}$ is a T-neighborhood basis on X. The resulting T-neighborhood space is denoted by τ_X^{T-G}. Its fuzzy closure operator $^- : I^X \to I^X$ is given by: For all $\eta \in I^X$, $x \in X$:

$$\bar{\eta}(x) = \inf_{\Gamma\in\Re} \sup_{g\in G} \Gamma(g) T \eta(gx) \tag{2}$$

Proof. First, we verify that $\{\beta_x\}_{x\in X}$ is a T-neighborhood basis in X. Let $x \in X$, Γ, $\Psi \in \Re$, $\mu = \Gamma 1_x \in \beta_x$, $\lambda = \Psi 1_x \in \beta_x$

(i) $\mu(x) = \Gamma 1_x(x) = \sup\{\Gamma(g): g \in G \text{ and } gx = x\}$
$\geq \Gamma(e) = 1$ (Because $ex = x$).

(ii) There exists $\Lambda \in \Re: \Gamma \wedge \Psi \geq \Lambda$. Hence
$\mu \wedge \lambda = \Gamma 1_x \wedge \Psi 1_x \geq \Lambda 1_x$,
which is in β_x.

(iii) T-kernel condition:

Recall that $\{\Re 1_g\}_{g\in G}$ is a T-neighborhood basis of the T-neighborhood group G Theorem 2.2 . Let, as before, $\mu = \Gamma 1_x \in \beta_x$. By the T-kernel condition for $\Gamma \in \Re$, for all $\varepsilon \in I_0$ there exists a family $\{\Gamma_g 1_g \in \Re_g\}_{g\in G}$ such that for all $g, k \in G$

$$\Gamma_e(k)\ T\ (\Gamma_k 1_k)(g) \leq \Gamma(g) + \varepsilon\,. \tag{3}$$

We take $v_x = \Gamma_e 1_x$. For each $y \in X$, if $y \notin$ orbit of x, take for v_y any element of $\beta_y = \Re_y$.

If $y \in$ orbit of x, choose some $h \in G$ such that $y = hx$, and

$$\delta + \Gamma_e(h) \geq \sup\{\Gamma_e(k): kx = y\} \tag{4}$$

where $\delta \in I_0$ is a real number that satisfies
$(b + \delta)\ T\ (c + \delta) \leq (b\ T\ c) + \varepsilon$
for all b, $c \in I$. Such δ exists by the uniform continuity of T. Take $v_y = \Gamma_h 1_y \in \beta_y$. Then, if $y \notin$ orbit of x, we find for all $z \in X$ that

$$2\varepsilon + \mu(z) \geq \upsilon_x(y)\ T\ \upsilon_y(z)$$

because then $v_x(y) = (\Gamma_e 1_x)(y) = 0$. And when $y \in$ orbit of x, we find for all $z \in X$:

$2\varepsilon + \mu(z) = 2\varepsilon + (\Gamma 1_x)(z)$
$= \varepsilon + \sup\{\varepsilon + \Gamma(g): gx = z\}$
$\geq \varepsilon + \sup\{\Gamma_e(h)\ T\ (\Gamma_h 1_h)(g): gx = z\}$ by (3)
$\geq (\Gamma_e(h) + \delta)\ T \sup\{(\Gamma_h 1_h)(g): gx = z\}$
$\geq \sup\{\Gamma_e(k): kx = y\}\ T \sup\{(\Gamma_h)(gh^{-1}): (gh^{-1})(hx) = z\}$

by (4) Since $hx = y$, then

$2\varepsilon + \mu(z) \geq (\Gamma_e 1_x)(y)\ T \sup\{(\Gamma_h)(t): ty = z\}$
$= (\Gamma_e 1_x)(y)\ T\ (\Gamma_h 1_y)(z)$
$= v_x(y)\ T\ v_y(z)$.

Thus, the kernel condition holds for $\mu \in \beta_x$ in both cases of y. Finally, for all $\eta \in I^X$

$$\bar{\eta}(x) = \inf_{\mu\in\beta} \sup_{y\in X} \mu(y)\ T\ \eta(y)$$
$$= \inf_{\Gamma\in\Re} \sup_{y\in X} \eta(y)\ T\ (\Gamma 1x)(y)$$
$$= \inf_{\Gamma\in\Re} \sup_{y\in orbitex} \eta(y)\ T \sup\{\Gamma(g): g \in G \text{ and } gx = y\}.$$

Because if $y \notin$ orbit x, then $(\Gamma 1_x)(y) = 0$. Thus,

$$\bar{\eta}(x) = \inf_{\Gamma\in\Re} \sup_{g\in G} \eta(gx)\ T\ \Gamma(g),$$

Rendering (2).

Proposition 3.2. Let $\Gamma \in I^G$, $\wp \subset I^G$, $g \in G$, $x \in X$ then

$(\Gamma.1_g)1_x = (\Gamma 1_{gx}) \in I^X$, and hence
$(\wp.1_g\})1_x = \wp 1_{gx} \subset I^X$.

Proof:

$((\Gamma.1_g)1_x)(y) = \sup\{(\Gamma.1_g)(k): k \in G \text{ and } kx = y\}$
$= \sup\{\Gamma(kg^{-1}): k \in G \text{ and } kg^{-1}gx = y\}$
$= \sup\{\Gamma(t): t \in G \text{ and } tgx = y\}$
$= (\Gamma 1_{gx})(y)$.

This completes the proof.

Proposition 3.3. For each filterbasis F in I^G and for $x \in X$.

$$\{\Gamma 1_x: \Gamma \in \mathrm{F}^{\sim}\} \subset \{\Psi 1_x : \Psi \in \mathrm{F}\}^{\sim} \subset I^X \tag{5}$$

Proof: Let $\Gamma \in \mathrm{F}^{\sim}$ Then for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in \mathrm{F}$ such that $\Gamma + \varepsilon \geq \Gamma_\varepsilon$. Then for all $y \in X$ we have

$\varepsilon + (\Gamma 1_x)(y) = \varepsilon + \sup\{\Gamma(g): gx = y\}$
$= \sup\{\varepsilon + \Gamma(g): gx = y\}$

$\geq sup \{\Gamma_\varepsilon(g): gx = y\}$
$= (\Gamma_\varepsilon 1_x)(y)$.

Thus, $\varepsilon + \Gamma 1_x \geq \Gamma_\varepsilon 1_x \in \{\Psi 1_x: \Psi \in \mathrm{F}\}$. Hence $\Gamma 1_x \in \{\Psi 1_x: \Psi \in \mathrm{F}\}^{\sim}$. This proves (5).

Proposition 3.4. The fuzzy closure operator on X defined in (2) does not depend on the particular choice of a fundamental system $\mathfrak{R}$ of e.

Proof: All fundamental systems $\mathfrak{R}$ of G at e have the same saturation $\mathfrak{R}^{\sim}$. Also, for each $x \in X$

$$\beta_x = \{\Gamma 1_x: \Gamma \in \mathfrak{R}\}$$
$$\subset \{\Gamma 1_x: \Gamma \in \mathfrak{R}^{\sim}\}$$
$$\subset \{\Gamma 1_x: \Gamma \in \mathfrak{R}\}^{\sim} = \beta_x^{\sim}.$$

As $\{\beta_x\}$, $\{\beta_x^{\sim}\}$ induce the same fuzzy closure operator on X, then the fuzzy closure operator defined in (2) is also given by

$$\bar{\eta}(x) = \inf_{\Gamma \in \tilde{\mathfrak{R}}} \sup_{g \in G} \Gamma(g) T \eta(gx) \tag{6}$$

Which is independent of the particular choice of a fundamental system $\mathfrak{R}$ of e.

The following definition is well phrased by virtue of Theorem 3.1, and Proposition 3.4;

Definition 3.2. Let G be a T-neighborhood group acting on a set X. A T-G-action-topology on X denoted by τ_X^{T-G} is introduced through its closure operator $\bar{\ }$, defined in (2).

Proposition 3.5. Let $\mathfrak{R}$ be a fundamental system at e of G, $\mu \in \mathfrak{R}$. Then

$$1_g . \mathfrak{R} . 1_{g^{-1}} \subset \mathfrak{R}^{\sim} \tag{7}$$

Proof: From condition (d) in Theorem 2.1, for all $\varepsilon > 0$ there exists $v_\varepsilon \in \mathfrak{R}$ such that
$v_\varepsilon - \varepsilon \leq 1_g . \mu . 1_{g^{-1}}$.
This proves that $1_g . \mu . 1_{g^{-1}} \in \mathfrak{R}^{\sim}$

Notion: In T-G-action topology
(1) We denote the T-neighborhood system at $x \in X$ by $\aleph(x)$.
(2) Let $\mathfrak{R}$ be the T-neighborhood system of G at e, $x \in X$. We denote $\mathfrak{R} 1_x$ by $Ç(x)$. Recall that $Ç^{\sim} = \aleph$; i.e $Ç(x)$ is a T-neighborhood basis at x for this space.

Definition 3.3. Let $(X, ., t(\beta))$ be a T-neighborhood space, M *be* a non-empty set in X. Then $\mu \in I^X$ is said to be *a* T-neighborhood of M if μ is a T-neighborhood of all points x in M. It follows that the set of all T-neighborhoods of M (called the T-neighborhood system of M) is the set $\bigwedge_{x \in M} \aleph(x)$.

Proposition 3.6. Let $\Gamma \in I^G$, $g \in G$, $z \in X$ then
$1_g^{-1} (\Gamma 1_z) = (1_g^{-1} . \Gamma) 1_z$

Proof:
$1_{g^{-1}}(\Gamma 1_z)(y) = (\Gamma 1_z)(gy)$
$= sup \{\Gamma(h): h \in G, hz = gy\}$
$= sup \{\Gamma(gk): k \in G, kz = y\}$
$(1_{g^{-1}} . \Gamma) 1_z(y) = sup \{(1_{g^{-1}} . \Gamma)(k): kz = y\}$
$= sup \{\Gamma(gk): k \in G, kz = y\}$
Then
$1_{g^{-1}}(\Gamma 1_z) = (1_{g^{-1}} . \Gamma) 1_z$

Theorem 3.2. Under this T-neighborhood topology the functions $\{\hat{g}\}$ are homeomorphisms on X.

Proof: Without loss of generality, we take $\mathfrak{R}$ the whole T-neighborhood system at e. Then from Proposition 3.5, $1_g . \mathfrak{R} . 1_{g-1} \subset \mathfrak{R}$. Given $x \in X$, $g \in G$, $\mathfrak{R} 1_{gx}$ is a T-neighborhood basis at $\hat{g} x$. Let $\mu \in \mathfrak{R} 1_{gx}$ we have $\hat{g}^{-1}(\mu)(y) = \mu(gy) = 1_{g-1} \mu(y)$, then $\hat{g}^{-1}(\mu) = 1_{g-1} \mu \in 1_g^{-1} \mathfrak{R} 1_{gx}$, and from Proposition 3.6

$1_g^{-1} (\mathfrak{R} 1_{gx}) = (1_g^{-1} . \mathfrak{R}) 1_{gx}$,
$= (1_g^{-1} . \mathfrak{R} . 1_g) 1_x$ by Proposition 3.2
$\subset \mathfrak{R}^{\sim} 1_x$ by Proposition 3.5
$\subset \aleph(x)$.

i.e., $\hat{g}^{-1}(\mu)$ is a T- neighborhood of x. So by Theorem 5.1 in [5] $\hat{g}$ is continuous at x for all x, and hence it is continuous. Since $(g^{-1})^{\wedge} = (\hat{g})^{-1}$. Then $(\hat{g})^{-1}$ is also continuous. Thus $\hat{g}$ is a homeomorphism.

Proposition 3.7. For any symmetric T-neighborhood Δ of e, and any $M \subset X$; $x, z \in X$
$(\Delta 1_x)(z) = (\Delta 1_z)(x)$
$(\Delta 1_M)(x) = \sup_{y \in X} 1_M(y) \text{ T } (\Delta 1_x)(y)$.

Proposition 3.8. For any subset M of X and any T-neighborhood Γ of e, $\Gamma 1_M$ is a T-neighborhood of M, and

$$(1_M)^{-} \leq \Gamma 1_M \in I^X. \tag{8}$$

Proof: Since $\Gamma 1_M = \bigvee_{x \in M} \Gamma 1_x$, then $\Gamma 1_M$ is a T-neighborhood of all points of M, hence $\Gamma 1_M$ is a T-neighborhood of M.

Next, let Γ be a T-neighborhood of e. Then Γ contains a symmetric T-neighborhood Δ of e. For any $x \in X$

$$(1_M)^{-}(x) = \inf_{\lambda \in Ç} \sup_{y \in X} 1_M(y) T \lambda(y)$$

$$\leq \sup_{y \in X} 1_M(y)\ T\ \Delta\ 1_x(y)$$

$$= \sup_{y \in M} \Delta\ 1_x(y)$$

$$= \sup_{y \in M} \Delta\ 1_y(x) \quad \text{by Proposition 3.7}$$

$$= (\Delta\ 1_M)(x)$$

$$\leq (\Gamma\ 1_M)(x).$$ This proves (8).

Proposition 3.9. Let $\Re$ be a fundamental system of T-neighborhoods of e. For any subset M of X

$$(1_M)^{-} = \bigwedge_{\Gamma \in \Re} \Gamma 1_M$$

Proof: From Proposition 3.8, $(1_M)^{-} \leq \Gamma 1_M$ for every $\Gamma \in \Re$ .Then

$$(1_M)^{-} \leq \bigwedge_{\Gamma \in O} \Gamma 1_M.$$

Next we prove that

$$\bigwedge_{\Gamma \in \Re} \Gamma 1_M = \bigwedge_{\Gamma \in \Re^{\sim}} \Gamma 1_M.$$

Since $\Re \subset \Re^{\sim}$, then

$$\bigwedge_{\Gamma \in \Re} \Gamma 1_M \geq \bigwedge_{\Gamma \in \Re^{\sim}} \Gamma 1_M$$

Also, let $\Gamma \in \Re^{\sim}$, for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in \Re$ such that $\varepsilon + \Gamma \geq \Gamma_\varepsilon$,

$$\varepsilon + \Gamma 1_M \geq (\varepsilon + \Gamma) 1_M \geq \Gamma_\varepsilon 1_M \geq \bigwedge_{\Gamma \in \Re} \Gamma 1_M$$

Since this holds for all $\varepsilon > 0$, then

$$\Gamma 1_M \geq \bigwedge_{\Gamma \in \Re} \Gamma 1_M.$$

This inequality holds for all $\Gamma \in \Re^{\sim}$.
Consequently,

$$\bigwedge_{\Gamma \in \Re^{\sim}} \Gamma 1_M \geq \bigwedge_{\Gamma \in \Re} \Gamma 1_M$$

Hence, equality holds.
It is clear that if O is the set of symmetric elements in $\Re^{\sim}$ then.

$$\bigwedge_{\Gamma \in \Re} \Gamma 1_M = \bigwedge_{\Gamma \in \Re^{\sim}} \Gamma 1_M \leq \bigwedge_{\Delta \in O} \Delta 1_M$$

Conversely, let O is the set of symmetric elements in $\Re^{\sim}$. Then O is a fundamental system at e:

$$\left(\bigwedge_{\Delta \in O} \Delta 1_M\right)(x) = \inf_{\Delta \in O} (\Delta 1_M)(x)$$

$$= \inf_{\Delta \in O} \sup_{y \in X} 1_M(y)\ T\ (\Delta 1x)(y) \quad \text{by Proposition 3.7}$$

$$= (1_M)^{-}(x)$$

because the set $\{\Delta 1_x : \Delta \in O\}$ is a T-neighborhood basis at x.

Theorem 3.3. A T-G-action topology on X is a T-regular T-neighborhood topology.

Proof: Let $M \subset X$ and $x \in X$. We establish condition (N^4-T-regularity) of Theorem 3.2 in [6], which is equivalent to the T-regularity of X. For all $M \subset X$, $x \in X$ such that

$$Inf\ hgt(\rho\ T\ v: \rho \in Ç(M), v \in O1_x)$$

$$\leq \inf_{\Delta \in O} hgt(\Delta 1_M T \Delta 1_x)$$

$$\leq \inf_{\Delta \in O} sup\{((\Delta 1_M) \wedge \Delta 1_x)(y): y \in X\}$$

$$= \inf_{\Delta \in O} sup\{((\Delta 1_M)(y) \wedge (\Delta 1_x)(y)): y \in X\}$$

$$= \inf_{\Delta \in O} sup\{sup\{\Delta(h): h \in G, hy \in M\} \wedge sup\{\Delta(k): k \in G, y = kx\}: y \in X\}$$

$$= \inf_{\Delta \in O} \sup \{sup\{\Delta(h): h \in G, hy \in M\} \wedge sup\{\Delta(k): k \in G, y = kx\}: y \in orbit\ x\}$$

So. (call $y \in kx$)

$$Inf\ hgt(\rho\ T\ v: \rho \in Ç(M), v \in O1_x)$$

$$\leq \inf_{\Delta \in O} \sup \{sup\{\Delta(h): hkx \in M\} \wedge \Delta(k): k \in G\}$$

$$= \inf_{\Delta \in O} sup\{\Delta(h) \wedge \Delta(k): h, k \in G \text{ and } hkx \in M\}$$

$$= \inf_{\Delta \in O} sup\{(\Delta\Delta)(g): g \in G \text{ and } gx \in M\}$$

$$= \inf_{\Delta \in O} sup((\Delta\Delta)(1_M))(x)$$

But by Theorem 2.2 in [7], for every $\Delta \in O$, $\varepsilon \geq 0$ there exists $\Delta_1 \in O$ such that $\Delta_1 \Delta_1 \leq \Delta + \varepsilon$. Hence,

$$Inf\ hgt(\rho\ T\ v: \rho \in Ç(M), v \in O1_x)$$

$$\leq \inf_{\Delta \in O, \varepsilon > 0} ((\Delta + \varepsilon) 1_M)(x)$$

$$= \inf_{\Delta \in O} (\Gamma 1_M)(x)$$

$$= \bigwedge_{\Gamma \in O} (\Gamma 1_M)(x) = (1_M)^{-}(x) \quad \text{by Proposition 3.9.}$$

The opposite inequality is always valid.

Theorem 3.4. A T-G-action-topology τ_X^{T-G} coincides with the final T-neighborhood topology τ_f on X defined by the set of functions

$$\{\hat{x} : G \rightarrow X: x \in X\}, \hat{x}(g) = gx$$

Proof: For any $x \in X$, the function $\hat{x} : G \rightarrow (X, \tau_X^{T-G})$ is continuous, because for all $g \in G$ and for each neighborhood $\Gamma(1_{gx})$ in the fundamental system $\Re 1_{gx}$ of $\hat{x}(g) = gx$, where $\Gamma \in \Re$, we have

$$\hat{x}(\Gamma . 1_g)(y) = sup\{(\Gamma. 1_g)(h): h \in G, \hat{x}(h) = y\}$$

$$= sup\{(\Gamma.1_g)(h): h \in G, hx = y\}$$

$$= (\Gamma . 1_g)1_x(y)$$

then $\hat{x}(\Gamma.1_g) = (\Gamma.1_g)1_x = \Gamma 1_{gx}$ and $\Gamma.1_g$ is a T-neighborhood of g by Theorem 2.3 in [7]. Therefore

$\tau_X^{T-G} \subset \tau_f$ since τ_f is the finest T-neighborhood topology making all $\hat{x}$ continuous.

Next, let $x \in X$, μ a T-neighborhood of x in τ_f. Then $\hat{x}^{-1}(\mu)$ a T-neighborhood of e in G; i.e. $(\hat{x}^{-1}(\mu))1_x$ is a T-neighborhood of x in τ_X^{T-G}.

But $(\hat{x}^{-1}(\mu))1_x = \hat{x}(\hat{x}^{-1}(\mu)) = \mu \wedge 1_{range\hat{x}} \leq \mu$. This proves that μ is a T-neighborhood of x in τ_X^{T-G}. Then $\tau_f \subset \tau_X^{T-G}$. Hence, equality holds.

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