On Properties of Generalized Bi-Γ-Ideals of Γ-Semirings

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Abstract—The notion of Γ-semirings was introduced by Murali Krishna Rao [10] as a generalization of the notion of Γ-rings as well as of semirings. We have known that the notion of Γ-semirings is a generalization of the notion of semirings. In this paper, extending Kaushik, Moin and Khan’s work, we generalize the notion of generalized bi-Γ-ideals of Γ-semirings and study some related properties of generalized bi-Γ-ideals.

Keywords—Γ-semiring, bi-Γ-ideal, generalized bi-Γ-ideal.

I. INTRODUCTION AND PRELIMINARIES


The concept of ideals for many types of Γ-semirings is the really interested and important thing in Γ-semirings. Therefore, we will introduce and study generalized bi-Γ-ideals of Γ-semirings in the same way as of bi-Γ-ideals of Γ-semirings which was studied by Kaushik, Moin and Khan [7].

To present the main results we first recall the definition of a Γ-semiring which is important here and discuss some elementary definitions that we use later.

Definition 1.1. [10] Let M and Γ be two additive commutative semigroups. Then M is called a Γ-semiring if there exists a mapping η : M × Γ → M (the image of (a, b, c) to be denoted by aab for all a, b, c ∈ M and α, β ∈ Γ) satisfying the following conditions:

1. a(ab + c) = aab + cac,
2. (a + b)c = ac + bc,
3. a(α + β)b = aαb + aβb,
4. aα(bβc) = (aαb)βc

for all a, b, c ∈ M and α, β ∈ Γ.

Let M be a Γ-semiring, A and B nonempty subsets of M, and Λ a nonempty subset of Γ. Then we define

\[ A + B := \{a + b | a ∈ A \text{ and } b ∈ B\} \]

and

\[ AΛB := \left\{\sum_{i=1}^{n} a_i \lambda_i b_i | n ∈ \mathbb{Z}_+ , a_i ∈ A , b_i ∈ B \text{ and } \lambda_i ∈ \Lambda \text{ for all } i\right\}.\]

If \(A = \{a\}\), then we also write \(a + B\) as \(aB\), and similarly if \(B = \{b\}\) or \(\Lambda = \{\lambda\}\).

Example 1.2. [6] Let \(Q\) be set of rational numbers. Let \((S, +)\) be the commutative semigroup of all \(2 \times 3\) matrices over \(Q\) and \((\Gamma, +)\) commutative semigroup of all \(3 \times 2\) matrices over \(Q\). Define \(W \circ Y\) usual matrix product of \(W, \alpha\) and \(Y\) for all \(W, Y ∈ S\) and for all \(α ∈ \Gamma\). Then \(S\) is a Γ-semiring but not a semiring.

Example 1.3. [6] Let \(N\) be the set of natural numbers and \(Γ = \{1, 2, 3\}.\) Then \((N, \max\) and \((\Gamma, \max\) are commutative semigroups. Define the mapping \(N × Γ → N\) by \(aαb = \min\{a, \alpha, b\}\) for all \(a, b ∈ N\) and \(\alpha ∈ Γ\). Then \(N\) is a Γ-semiring.

Example 1.4. [6] Let \(Q\) be set of rational numbers and \(Γ = N\) the set of natural numbers. Then \((Q, +)\) and \((N, +)\) are commutative semigroups. Define the mapping \(Q × Γ → Q\) by \(aαb\) usual product of \(a, \alpha, b\); for all \(a, b ∈ Q\) and \(α ∈ Γ\). Then \(Q\) is a Γ-semiring.

Example 1.5. [2] For consider the additively abelian groups \(Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}\) and \(Γ = \{2, 4, 6\}.\) Let \(η : Z_8 × Γ → Z_8, (y, α, s) = yαs.\) Then \(Z_8\) is a Γ-semiring.
Definition I.6. A nonempty subset \( A \) of a \( \Gamma \)-semiring \( M \) is called

1. a sub-\( \Gamma \)-semiring of \( M \) if \((A, +)\) is a subsemigroup of \((M, +)\) and \( a\gamma b \in A \) for all \( a, b \in A \) and \( \gamma \in \Gamma \).
2. a \( \Gamma \)-ideal of \( M \) if \((A, +)\) is a subsemigroup of \((M, +)\), and \( x\gamma a \in A \) and \( a\gamma x \in A \) for all \( a \in A, x \in M \) and \( \gamma \in \Gamma \).
3. a quasi-\( \Gamma \)-ideal of \( M \) if \( A \) is a sub-\( \Gamma \)-semiring of \( M \) and \( A \cap M \subseteq A \).
4. a bi-\( \Gamma \)-ideal of \( M \) if \( A \) is a sub-\( \Gamma \)-semiring of \( M \) and \( \Gamma M \cap \Gamma A \subseteq A \).
5. a generalized bi-\( \Gamma \)-ideal of \( M \) if \( \Lambda \Gamma M \Gamma A \subseteq A \).

Remark I.7. Let \( M \) be a \( \Gamma \)-semiring. We have the following:

1. Every quasi-\( \Gamma \)-ideal of \( M \) is a bi-\( \Gamma \)-ideal.
2. Every bi-\( \Gamma \)-ideal of \( M \) is a generalized bi-\( \Gamma \)-ideal.

Definition I.8. A \( \Gamma \)-semiring \( M \) is called a \( GB \)-simple \( \Gamma \)-semiring if \( M \) is the unique generalized bi-\( \Gamma \)-ideal of \( M \).

II. MAIN RESULTS

Before the characterizations of generalized bi-\( \Gamma \)-ideals of \( \Gamma \)-semirings for the main results, we give some auxiliary results which are necessary in what follows. By Lemma I.7 (2) and [7], we have the following lemma.

Lemma II.1. Let \( M \) be a \( \Gamma \)-semiring and \( a \in M \). Then \( a\Gamma M \) and \( M\Gamma a \) are generalized bi-\( \Gamma \)-ideals of \( M \).

Lemma II.2. Let \( M \) be a \( \Gamma \)-semiring and \( \{B_i \mid i \in I\} \) a nonempty family of generalized bi-\( \Gamma \)-ideals of \( M \) with \( \bigcap_{i \in I} B_i \neq \emptyset \). Then \( \bigcap_{i \in I} B_i \) is a generalized bi-\( \Gamma \)-ideal of \( M \).

Proof: For all \( i \in I \), we have

\[ \left( \bigcap_{i \in I} B_i \right) \Gamma M \left( \bigcap_{i \in I} B_i \right) \subseteq B_i \Gamma M \Gamma B_i \subseteq B_i. \]

Thus

\[ \left( \bigcap_{i \in I} B_i \right) \Gamma M \left( \bigcap_{i \in I} B_i \right) \subseteq \bigcap_{i \in I} B_i. \]

Hence \( \bigcap_{i \in I} B_i \) is a generalized bi-\( \Gamma \)-ideal of \( M \).

Lemma II.3. Let \( M \) be a \( \Gamma \)-semiring and \( \emptyset \neq A \subseteq M \). Then

\[ A \cup \Lambda \Gamma M \Gamma A \]

is the smallest generalized bi-\( \Gamma \)-ideal of \( M \) containing \( A \).

Proof: Let \( B = A \cup \Lambda \Gamma M \Gamma A \). Then \( A \subseteq B \). Therefore

\[ \begin{align*}
\Lambda \Gamma M \Gamma B &= (A \cup \Lambda \Gamma M \Gamma A) \Lambda \Gamma M \Gamma (A \cup \Lambda \Gamma M \Gamma A) \\
&\subseteq [A(\Gamma M) \cup (A \Gamma M)] \cup [\Lambda \Gamma M \Gamma (A \Gamma M)] \\
&\subseteq [A(\Gamma M) \cup (A \Gamma M)] \cup [\Lambda \Gamma M \Gamma (A \Gamma M)] \\
&\subseteq (A \Gamma M) \cup (A \Gamma M) \\
&= A \Gamma M \Gamma A \\
&\subseteq A \cup \Lambda \Gamma M \Gamma A.
\end{align*} \]

Thus \( B = A \cup \Lambda \Gamma M \Gamma A \) is a generalized bi-\( \Gamma \)-ideal of \( M \).

Hence \( B \) is the smallest generalized bi-\( \Gamma \)-ideal of \( M \) containing \( A \).

By Lemma II.3, let \( (A) \) be the smallest generalized bi-\( \Gamma \)-ideal of \( M \) containing \( \{a\} \) as \( \{a\} \).

Lemma II.4. Let \( T \) be a sub-\( \Gamma \)-semiring of a \( \Gamma \)-semiring \( M \), \( a \in M \) and \( (a\Gamma T \Gamma a) \cap T \neq \emptyset \). Then \( (a\Gamma T \Gamma a) \cap T \) is a generalized bi-\( \Gamma \)-ideal of \( T \).

Proof: Consider

\[ (a\Gamma T \Gamma a) \cap T \subseteq [(a\Gamma T \Gamma a) \cap T] \Gamma (a\Gamma T \Gamma a) \cap T \subseteq [(a\Gamma T \Gamma a) \cap T] \Gamma (a\Gamma T \Gamma a) \cap T \subseteq [(a\Gamma T \Gamma a) \cap T] \cap [T \Gamma (a\Gamma T \Gamma a) \cap T] \subseteq [(a\Gamma T \Gamma a) \cap T] \cap T \subseteq [a\Gamma T \Gamma a] \cap T. \]

Hence \( (a\Gamma T \Gamma a) \cap T \) is a generalized bi-\( \Gamma \)-ideal of \( T \).

Lemma II.5. Let \( M \) be a \( \Gamma \)-semiring and \( a \in M \). Then \( a\Gamma M \Gamma a \) is a generalized bi-\( \Gamma \)-ideal of \( M \).

Proof: Consider

\[ (a\Gamma M \Gamma a) \Gamma (a\Gamma M \Gamma a) \subseteq a \Gamma M \Gamma a. \]

Hence \( a\Gamma M \Gamma a \) is a generalized bi-\( \Gamma \)-ideal of \( M \).

Proposition II.6. Let \( M \) be a \( \Gamma \)-semiring and \( T \) a sub-\( \Gamma \)-semiring of \( M \). Then every subset of \( T \) containing \( MT \) is a sub-\( \Gamma \)-semiring of \( M \).
**Proof:** Let $A$ be a subset of $T$ such that $MTT \subseteq A$. Then
\[ \forall A \subseteq MT \subseteq A. \]
Hence $A$ is a sub-$\Gamma$-semiring of $M$.

**Proposition II.7.** Let $M$ be a $\Gamma$-semiring and $T$ a $\Gamma$-ideal of $M$. Then every subset of $T$ containing $MTT \cup TTM$ is a $\Gamma$-ideal of $M$.

**Proof:** Let $B$ be a subset of $T$ such that $MTT \cup TTM \subseteq B$. Then
\[ M \Gamma B \subseteq M \Gamma T \subseteq M \Gamma T T \subseteq B \]
and
\[ B \Gamma M \subseteq T T M \subseteq T T M \cap M \Gamma T \subseteq B. \]
Hence $B$ is a $\Gamma$-ideal of $M$.

**Proposition II.8.** Let $M$ be a $\Gamma$-semiring and $T$ a quasi-$\Gamma$-ideal of $M$. Then every subset of $T$ containing $TTM \cap TTT$ is a quasi-$\Gamma$-ideal of $M$.

**Proof:** Let $C$ be a subset of $T$ such that $TTM \cap TTT \subseteq C$. Then
\[ C \Gamma C \subseteq T T M \cap M \Gamma T \subseteq C \]
and
\[ C \Gamma T M \cap M \Gamma T \subseteq C. \]
Hence $C$ is a quasi-$\Gamma$-ideal of $M$.

**Proposition II.9.** Let $M$ be a $\Gamma$-semiring and $T$ a bi-$\Gamma$-ideal of $M$. Then every subset of $T$ containing $TTM \cap MTT$ and all of its images is a bi-$\Gamma$-ideal of $M$.

**Proof:** Let $D$ be a subset of $T$ such that $TTM \cap MTT \subseteq D$ and $DT \subseteq D$. Then
\[ DT \cap T \Gamma T \cap T \Gamma T \subseteq D. \]
Hence $D$ is a bi-$\Gamma$-ideal of $M$.

**Proposition II.10.** Let $M$ be a $\Gamma$-semiring and $T$ a generalized bi-$\Gamma$-ideal of $M$. Then every subset of $T$ containing $TTMT$ is a generalized bi-$\Gamma$-ideal of $M$.

**Proof:** Let $E$ be a subset of $T$ such that $TTMT \subseteq E$. Then
\[ ET \cap T \Gamma M \subseteq T \Gamma T \subseteq E. \]
Hence $E$ is a generalized bi-$\Gamma$-ideal of $M$.

**Theorem II.11.** Let $M$ be a $\Gamma$-semiring. Then the following statements are equivalent.

1. $M$ is a GB-simple $\Gamma$-semiring.
2. $a \Gamma M \Gamma a = M$ for all $a \in M$.
3. $(a) = M$ for all $a \in M$.

**Proof:** (1) $\Rightarrow$ (2) Assume that $M$ is a GB-simple $\Gamma$-semiring and $a \in M$. By Lemma II.5, we have $a \Gamma M \Gamma a$ is a generalized bi-$\Gamma$-ideal of $M$. Since $M$ is a GB-simple $\Gamma$-semiring, we have $a \Gamma M \Gamma a = M$.

(2) $\Rightarrow$ (3) Assume that $a \Gamma M \Gamma a = M$ for all $a \in M$ and let $a \in M$. Then, by (2), we have
\[ (a) = \{a\} \cup a \Gamma M \Gamma a = \{a\} \cup M = M. \]

(3) $\Rightarrow$ (1) Assume that $(a) = M$ for all $a \in M$, and let $A$ be a generalized bi-$\Gamma$-ideal of $M$ and $a \in A$. Then $(a) \subseteq A$. By assumption, we have
\[ M = (a) \subseteq A \subseteq M. \]
Thus $M = A$. Therefore $M$ is a GB-simple $\Gamma$-semiring.

**Lemma II.12.** Let $B$ be a generalized bi-$\Gamma$-ideal of a $\Gamma$-semiring $M$ and $T$ a sub-$\Gamma$-semiring of $M$. If $T$ is a GB-simple $\Gamma$-semiring such that $T \cap B \neq \emptyset$, then $T \subseteq B$.

**Proof:** Assume that $T$ is a GB-simple $\Gamma$-semiring such that $T \cap B \neq \emptyset$ and let $a \in T \cap B$. By Lemma II.3, we have $\{a\} \cup a \Gamma T \Gamma a$ is a generalized bi-$\Gamma$-ideal of $T$. Since $T$ is a GB-simple $\Gamma$-semiring, we have $\{a\} \cup a \Gamma T \Gamma a = T$. Thus
\[ T = \{a\} \cup a \Gamma T \Gamma a \subseteq B \cup B \Gamma M \Gamma B \subseteq B \cup B \subseteq B. \]
Hence $T \subseteq B$.

**Theorem II.13.** Let $M$ be a $\Gamma$-semiring, $B$ a generalized bi-$\Gamma$-ideal of $M$ and $\emptyset \neq A \subseteq M$. Then $B \Gamma A$ and $A \Gamma B$ are generalized bi-$\Gamma$-ideals of $M$.

**Proof:** Since $B$ is a generalized bi-$\Gamma$-ideal of $M$, we have
\[ (B \Gamma A) \Gamma M \Gamma (B \Gamma A) = (B \Gamma (A \Gamma M) \Gamma B) \Gamma A \subseteq (B \Gamma M \Gamma B) \Gamma A \subseteq B \Gamma A \]
and
\[ (A \Gamma B) \Gamma M \Gamma (A \Gamma B) = A \Gamma (B \Gamma (M \Gamma A) \Gamma B) \subseteq A \Gamma (B \Gamma M \Gamma B) \subseteq A \Gamma B. \]
Therefore $B \Gamma A$ and $A \Gamma B$ are generalized bi-$\Gamma$-ideals of $M$.

**Theorem II.14.** Let $M$ be a $\Gamma$-semiring and $B$ a bi-$\Gamma$-ideal of $M$. Then $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$ if and only if $B$ is a GB-simple $\Gamma$-semiring.

**Proof:** Assume that $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$. By assumption, $B$ is a $\Gamma$-semiring. Let $C$ be a generalized bi-$\Gamma$-ideal of $B$. Then
\[ C \subseteq B. \]
Since $B$ is a generalized bi-$\Gamma$-ideal of $M$ and by Theorem II.13, we have $C \subseteq B \Gamma C$ is a generalized bi-$\Gamma$-ideal of $M$. Since $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$, we get $C \subseteq B$. Thus, by (3), we have $B = C$. Hence $B$ is a GB-simple $\Gamma$-semiring.

Conversely, assume that $B$ is a GB-simple $\Gamma$-semiring. Let $C$ be a generalized bi-$\Gamma$-ideal of $M$ such that $C \subseteq B$. Then
\[ C \Gamma B \subseteq C \subseteq B. \]
Since $B$ is a generalized bi-$\Gamma$-ideal of $M$ and by Theorem II.13, we have $C \subseteq B \Gamma C$ is a generalized bi-$\Gamma$-ideal of $M$. Since $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$, we get $C \subseteq B$. Thus, by (3), we have $B = C$. Hence $B$ is a minimal generalized bi-$\Gamma$-ideal of $M$.

**Theorem II.15.** Let $M$ be a $\Gamma$-semiring having a proper generalized bi-$\Gamma$-ideal. Then every proper generalized bi-$\Gamma$-ideal of $M$ is minimal if and only if the intersection of any two distinct proper generalized bi-$\Gamma$-ideals is empty.
Proof: Assume that every proper generalized bi-$\Gamma$-ideal of $M$ is minimal and let $B_1$ and $B_2$ be two distinct proper generalized bi-$\Gamma$-ideals of $M$. By assumption, we have $B_1$ and $B_2$ are minimal. We shall show that $B_1 \cap B_2 = \emptyset$. Suppose that $B_1 \cap B_2 \neq \emptyset$. By Lemma II.2, we have $B_1 \cap B_2$ is a proper generalized bi-$\Gamma$-ideal of $M$. Since $B_1 \cap B_2 \subseteq B_1$ and $B_1 \cap B_2 \subseteq B_2$, we get $B_1 \cap B_2 = B_1$ and $B_1 \cap B_2 = B_2$, thus $B_1 = B_2$ which is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

Conversely, assume that the intersection of any two distinct proper generalized bi-$\Gamma$-ideals is empty. Let $B$ be a proper generalized bi-$\Gamma$-ideal of $M$ and $C$ a generalized bi-$\Gamma$-ideals of $M$ such that $C \subseteq B$. Suppose that $C \neq B$. Then $C$ is a proper generalized bi-$\Gamma$-ideal of $M$. Since $C \subseteq B$ and by assumption, we have $C = C \cap B = \emptyset$ which is a contradiction.

Therefore $C = B$, so $B$ is minimal.

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